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On special strong differential subordinations using multiplier transformation

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ABSTRACT

For functions belonging to the class $SI_m(\delta, \lambda, l, \zeta)$, $\delta \in [0, 1)$, $\lambda, l \geq 0$ and $m \in \mathbb{N} \cup \{0\}$, of analytic functions in $U \times \overline{U}$, which are investigated in this paper, the author derives several interesting strong differential subordination results. These strong differential subordinations are established by means of a special case of the extended multiplier transformations $I(m, \lambda, l) f(z, \zeta)$ namely

$$I(m, \lambda, l) f(z, \zeta) := z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda (j-1) + l}{l+1} \right)^m a_j(\zeta) z^j,$$

where $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$ and $f \in \mathcal{A}_\zeta^*$,

$$\mathcal{A}_\zeta^* = \left\{ f \in \mathcal{H}(U \times \overline{U}) : f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j, z \in U, \zeta \in \overline{U} \right\}.$$

A number of interesting consequences of some of these strong subordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_\zeta^* = \{f \in \mathcal{H}(U \times \overline{U}), f(z, \zeta) = z + a_2(\zeta) z^2 + \cdots, z \in U, \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\mathcal{H}^*[a, n, \zeta] = \{f \in \mathcal{H}(U \times \overline{U}), f(z, \zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \cdots, z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Definition 1.1. For $f \in \mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + \cdots, z \in U\}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(m, \lambda, l) f(z)$ is defined by the following infinite series:

$$I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda (j-1) + l + 1}{l+1} \right)^m a_j z^j.$$

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Remark 1.2. The operator $I(m, \lambda, l)$ was studied in [1–4].

For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [5], which is reduced to the Sălăgean differential operator [6] for $\lambda = 1$. The operator $I(m, 1, l)$ was studied by Cho and Srivastava [7] and Cho and Kim [8]. The operator $I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [9] and the operator $I(\alpha, \lambda, 0)$ was introduced by Acu and Owa [10]. Cătaş [11] has studied the operator $I_p(m, \lambda, l)$ which generalizes the operator $I(m, \lambda, l)$.

We also extend the multiplier transformation to the new class of analytic functions $\mathcal{A}_\zeta^*$ introduced in [12].

Definition 1.3. For $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in \mathcal{A}_\zeta^*$, $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, the operator $I(m, \lambda, l) f(z, \zeta)$ is defined by the following infinite series:

$$I(m, \lambda, l) f(z, \zeta) = z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_j(\zeta) z^j, \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 1.4. It follows from the above definition that

$$(l+1) I(m+1, \lambda, l) f(z, \zeta) = [l + 1 - \lambda] I(m, \lambda, l) f(z, \zeta) + \lambda z \left(I(m, \lambda, l) f(z, \zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U}.$$

Generalizing the notion of differential subordinations, Antonino and Romaguera have introduced in [13] the notion of strong differential subordinations, which was developed by Oros and Oros in [14,12].

Definition 1.5 (*[14]*). Let $f(z, \zeta)$, $H(z, \zeta)$ be analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be strongly subordinate to $H(z, \zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z, \zeta) = H(w(z), \zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z, \zeta) \prec\prec H(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Remark 1.6 (*[14]*).

(i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.5 is equivalent to $f(0, \zeta) = H(0, \zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.

(ii) If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma 1.7 (*[15]*). *Let $h(z, \zeta)$ be a convex function with $h(0, \zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}^*[a, n, \zeta]$ and*

$$p(z, \zeta) + \frac{1}{\gamma} z p_z'(z, \zeta) \prec\prec h(z, \zeta),$$

then

$$p(z, \zeta) \prec\prec g(z, \zeta) \prec\prec h(z, \zeta),$$

where $g(z, \zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t, \zeta) t^{\frac{\gamma}{n} - 1} dt$ is convex and it is the best dominant.

Lemma 1.8 (*[15]*). *Let $g(z, \zeta)$ be a convex function in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and let*

$$h(z, \zeta) = g(z, \zeta) + n\alpha z g_z'(z, \zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z, \zeta) = g(0, \zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots, \quad z \in U,\ \zeta \in \overline{U},$$

is holomorphic in $U \times \overline{U}$ and

$$p(z, \zeta) + \alpha z p_z'(z, \zeta) \prec\prec h(z, \zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$p(z, \zeta) \prec\prec g(z, \zeta)$$

and this result is sharp.

2. Main results

Definition 2.1. Let $\delta \in [0, 1)$, $\lambda, l \geq 0$ and $m \in \mathbb{N} \cup \{0\}$. A function $f(z, \zeta) \in \mathcal{A}_\zeta^*$ is said to be in the class $SI_m(\delta, \lambda, l, \zeta)$ if it satisfies the inequality

$$\operatorname{Re}(I(m, \lambda, l) f(z, \zeta))'_z > \delta, \quad z \in U, \ \zeta \in \overline{U}. \tag{1}$$

Theorem 2.2. *The set $SI_m(\delta, \lambda, l, \zeta)$ is convex.*

Proof. Let the function

$$f_j(z, \zeta) = z + \sum_{j=2}^{\infty} a_{jk}(\zeta) z^j, \quad k = 1, 2, \ z \in U, \ \zeta \in \overline{U},$$

be in the class $SI_m(\delta, \lambda, l, \zeta)$. It is sufficient to show that the function

$$h(z, \zeta) = \eta_1 f_1(z, \zeta) + \eta_2 f_2(z, \zeta)$$

is in the class $SI_m(\delta, \lambda, l, \zeta)$, with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since

$$h(z, \zeta) = z + \sum_{j=2}^{\infty} \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) z^j, \quad z \in U, \ \zeta \in \overline{U},$$

then

$$I(m, \lambda, l) h(z, \zeta) = z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) z^j, \quad z \in U, \ \zeta \in \overline{U}. \tag{2}$$

Differentiating (2) we obtain

$$(I(m, \lambda, l) h(z, \zeta))'_z = 1 + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m \left(\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)\right) j z^{j-1}, \quad z \in U, \ \zeta \in \overline{U}.$$

Hence

$$\begin{aligned} \operatorname{Re}(I(m, \lambda, l) h(z, \zeta))'_z = {} & 1 + \operatorname{Re}\left(\eta_1 \sum_{j=2}^{\infty} j \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m a_{j1}(\zeta) z^{j-1}\right) \\ & + \operatorname{Re}\left(\eta_2 \sum_{j=2}^{\infty} j \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m a_{j2}(\zeta) z^{j-1}\right). \end{aligned} \tag{3}$$

Taking into account that $f_1, f_2 \in SI_m(\delta, \lambda, l, \zeta)$ we deduce

$$\operatorname{Re}\left(\eta_k \sum_{j=2}^{\infty} j \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m a_{jk}(\zeta) z^{j-1}\right) > \eta_k(\delta - 1), \quad k = 1, 2. \tag{4}$$

Using (4) we get from (3)

$$\operatorname{Re}(I(m, \lambda, l) h(z, \zeta))'_z > 1 + \eta_1(\delta - 1) + \eta_2(\delta - 1), \quad z \in U, \ \zeta \in \overline{U},$$

that is

$$\operatorname{Re}(I(m, \lambda, l) h(z, \zeta))'_z > \delta, \quad z \in U, \ \zeta \in \overline{U},$$

which is equivalent that $SI_m(\delta, \lambda, l, \zeta)$ is convex. □

Theorem 2.3. *Let $g(z, \zeta)$ be a convex function such that $g(0, \zeta) = 1$ and let h be the function $h(z, \zeta) = g(z, \zeta) + \frac{1}{c+2} z g'_z(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$, $c > 0$. If $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in SI_m(\delta, \lambda, l, \zeta)$ and $F(z, \zeta) = I_c(f)(z, \zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t, \zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, then*

$$(I(m, \lambda, l) f(z, \zeta))'_z \prec\prec h(z, \zeta), \quad z \in U, \ \zeta \in \overline{U}, \tag{5}$$

implies

$$(I(m, \lambda, l) F(z, \zeta))'_z \prec\prec g(z, \zeta), \quad z \in U, \ \zeta \in \overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z,\zeta) = (c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{6}$$

Differentiating (6), with respect to z, we have $(c+1)F(z,\zeta) + zF_z'(z,\zeta) = (c+2)f(z,\zeta)$ and

$$(c+1)I(m,\lambda,l)F(z,\zeta) + z\left(I(m,\lambda,l)F(z,\zeta)\right)_z' = (c+2)I(m,\lambda,l)f(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \tag{7}$$

Differentiating (7) with respect to z we have

$$\left(I(m,\lambda,l)F(z,\zeta)\right)_z' + \frac{1}{c+2}z\left(I(m,\lambda,l)F(z,\zeta)\right)_{z^2}'' = \left(I(m,\lambda,l)f(z,\zeta)\right)_z', \quad z\in U,\ \zeta\in\overline{U}. \tag{8}$$

Using (8), the strong differential subordination (5) becomes

$$\left(I(m,\lambda,l)F(z,\zeta)\right)_z' + \frac{1}{c+2}z\left(I(m,\lambda,l)F(z,\zeta)\right)_{z^2}'' \prec\prec g(z,\zeta) + \frac{1}{c+2}zg_z'(z,\zeta). \tag{9}$$

Denote

$$p(z,\zeta) = \left(I(m,\lambda,l)F(z,\zeta)\right)_z', \quad z\in U,\ \zeta\in\overline{U}. \tag{10}$$

Replacing (10) in (9) we obtain

$$p(z,\zeta) + \frac{1}{c+2}zp_z'(z,\zeta) \prec\prec g(z,\zeta) + \frac{1}{c+2}zg_z'(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.8 we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \text{ i.e.}$$
$$\left(I(m,\lambda,l)F(z,\zeta)\right)_z' \prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. □

Theorem 2.4. *Let $h(z,\zeta) = \frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$, $\delta\in[0,1)$ and $c>0$. If $\lambda, l\ge 0$, $m\in\mathbb{N}$ and I_c is given by Theorem 2.3, then*

$$I_c\left[SI_m(\delta,\lambda,l,\zeta)\right] \subset SI_m\left(\delta^*,\lambda,l,\zeta\right), \tag{11}$$

where $\delta^* = 2\delta-\zeta+2(c+2)(\zeta-\delta)\beta(c)$ and $\beta(x) = \int_0^1 \frac{t^{x+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.3 we get from the hypothesis of Theorem 2.4 that

$$p(z,\zeta) + \frac{1}{c+2}zp_z'(z,\zeta) \prec\prec h(z,\zeta),$$

where $p(z,\zeta)$ is defined in (10).

Using Lemma 1.7 we deduce that

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

that is

$$\left(I(m,\lambda,l)F(z,\zeta)\right)_z' \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where

$$g(z,\zeta) = \frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\frac{\zeta+(2\delta-\zeta)t}{1+t}dt = (2\delta-\zeta) + \frac{2(c+2)(\zeta-\delta)}{z^{c+2}}\int_0^z \frac{t^{c+1}}{1+t}dt.$$

Since g is convex and $g\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned}\operatorname{Re}\left(I(m,\lambda,l)F(z,\zeta)\right)_z' &\ge \min_{|z|=1}\operatorname{Re} g(z,\zeta) = \operatorname{Re} g(1,\zeta) = \delta^* \\ &= 2\delta-\zeta+2(c+2)(\zeta-\delta)\beta(c).\end{aligned} \tag{12}$$

From (12) we deduce the inclusion (11). □

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(I(m,\lambda,l)f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{13}$$

holds, then

$$\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. Consider $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}=\frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j}{z}=1+p_1(\zeta)z+p_2(\zeta)z^2+\cdots$, $z\in U$, $\zeta\in\overline{U}$.

Let $I(m,\lambda,l)f(z,\zeta)=zp(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Differentiating with respect to z we obtain $\left(I(m,\lambda,l)f(z,\zeta)\right)'_z=p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then (13) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.8, we have

$$p(z,\zeta)\prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \quad \text{i.e.}\quad \frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \quad \square$$

Theorem 2.6. *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(I(m,\lambda,l)f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{14}$$

holds, then

$$\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$ is convex and it is the best dominant.

Proof. With notation $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$, $p(z,\zeta)+zp'(z,\zeta)=\left(I(m,\lambda,l)f(z,\zeta)\right)'_z$.

We have $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Since $p(z,\zeta)\in\mathcal{H}^*[1,1,\zeta]$, using Lemma 1.7, for $n=1$ and $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. $\square$

Corollary 2.7. *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, $0\leq\beta<1$. If $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and verifies the strong differential subordination*

$$\left(I(m,\lambda,l)f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{15}$$

then

$$\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

where g is given by $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z\in U$, $\zeta\in\overline{U}$. The function g is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.6 and considering $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}$, the strong differential subordination (15) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}, \quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.7 for $n=1$ and $\gamma=1$, we have $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\begin{aligned}\frac{I(m,\lambda,l)f(z,\zeta)}{z}\prec\prec g(z,\zeta)&=\frac{1}{z}\int_0^z h(t,\zeta)\,dt\\ &=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z), \quad z\in U,\ \zeta\in\overline{U}. \quad \square\end{aligned}$$

Theorem 2.8. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{16}$$

holds, then

$$\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. For $f\in\mathcal{A}^*_\zeta$, $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ we have $I(m,\lambda,l)f(z,\zeta)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Consider $p(z,\zeta)=\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}=\frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}a_j(\zeta)z^j}{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}a_j(\zeta)z^j}=\frac{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}a_j(\zeta)z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}a_j(\zeta)z^{j-1}}$.

We have $p'_z(z,\zeta)=\frac{(I(m+1,\lambda,l)f(z,\zeta))'_z}{I(m,\lambda,l)f(z,\zeta)}-p(z,\zeta)\cdot\frac{(I(m,\lambda,l)f(z,\zeta))'_z}{I(m,\lambda,l)f(z,\zeta)}$. Then $p(z,\zeta)+zp'_z(z,\zeta)=\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z$.

Relation (16) becomes $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and by using Lemma 1.8 we obtain $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. □

Theorem 2.9. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{17}$$

holds, then

$$[I(m,\lambda,l)f(z,\zeta)]'_z\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

This result is sharp.

Proof. Let

$$p(z,\zeta)=(I(m,\lambda,l)f(z,\zeta))'_z = 1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m ja_j(\zeta)z^{j-1}=1+p_1(\zeta)z+p_2(\zeta)z^2+\cdots. \tag{18}$$

We obtain $p(z,\zeta)+z\cdot p'_z(z,\zeta)=I(m,\lambda,l)f(z,\zeta)+z(I(m,\lambda,l)f(z,\zeta))'_z=I(m,\lambda,l)f(z,\zeta)+\frac{(l+1)I(m+1,\lambda,l)f(z,\zeta)-(l+1-\lambda)I(m,\lambda,l)f(z,\zeta)}{\lambda}=\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$.

Using the notation in (18), the strong differential subordination becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta).$$

By using Lemma 1.8, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \text{i.e.}$$
$$(I(m,\lambda,l)f(z,\zeta))'_z\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. □

Theorem 2.10. *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{19}$$

holds, then

$$(I(m,\lambda,l)f(z,\zeta))'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$ is convex and it is the best dominant.

Proof. For $f \in \mathcal{A}_\zeta^*, f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$ we have $I(m,\lambda,l) f(z,\zeta) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^j,\ z \in U, \zeta \in \overline{U}$.

Consider $p(z,\zeta) = (I(m,\lambda,l) f(z,\zeta))'_z = 1 + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^{j-1} \in \mathcal{H}^*[1,1,\zeta]$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = \frac{l+1}{\lambda} I(m+1,\lambda,l) f(z,\zeta) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z,\zeta),\ z \in U, \zeta \in \overline{U}$.

Then $\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z,\zeta) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z,\zeta) \prec\prec h(z,\zeta), z \in U, \zeta \in \overline{U}$, becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta), z \in U, \zeta \in \overline{U}$. By using Lemma 1.7, for $n = 1$ and $\gamma = 1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), z \in U, \zeta \in \overline{U}$, i.e. $(I(m,\lambda,l) f(z,\zeta))'_z \prec\prec g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. □

Corollary 2.11. *Let $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta) z}{1+z}$ be a convex function in $U \times \overline{U}$, $0 \le \beta < 1$. If $\alpha \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and verifies the strong differential subordination*

$$\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z,\zeta) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{20}$$

then

$$(I(m,\lambda,l) f(z,\zeta))'_z \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where g is given by $g(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z),\ z \in U,\ \zeta \in \overline{U}$. The function g is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta) = (I(m,\lambda,l) f(z,\zeta))'_z$, the strong differential subordination (20) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta + (2\beta - \zeta) z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.7 for $n = 1$ and $\gamma = 1$, we have $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),\ z \in U, \zeta \in \overline{U}$, i.e.

$$\begin{aligned}(I(m,\lambda,l) f(z,\zeta))'_z \prec\prec g(z,\zeta) &= \frac{1}{z}\int_0^z h(t,\zeta)\,dt \\ &= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t} dt \\ &= 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z), \quad z \in U,\ \zeta \in \overline{U}. \quad \Box\end{aligned}$$

References

[1] A. Alb Lupaş, A special comprehensive class of analytic functions defined by multiplier transformation, Journal of Computational Analysis and Applications 12 (2) (2010) 387–395.
[2] A. Alb Lupaş, On special differential superordinations using a generalized Sălăgean operator and Ruscheweyh derivative, Computers and Mathematics with Applications 61 (2011) 1048–1058. doi:10.1016/j.camwa.2010.12.055.
[3] A. Alb Lupaş, On special differential superordinations using multiplier transformation, Journal of Computational Analysis and Applications 13 (1) (2011) 121–126.
[4] A. Alb Lupaş, Certain special differential superordinations using multiplier transformation, International Journal of Open Problems in Complex Analysis 3 (2) (2011) 50–60.
[5] F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Indian Journal of Mathematics and Mathematical Sciences (25–28) (2004) 1429–1436.
[6] G.St. Sălăgean, Subclasses of univalent functions, in: Lecture Notes in Math., 1013, Springer Verlag, Berlin, 1983, pp. 362–372.
[7] N.E. Cho, H.M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Mathematical and Computer Modelling 37 (1–2) (2003) 39–49.
[8] N.E. Cho, T.H. Kim, Multiplier transformations and strongly close-to-convex functions, Bulletin—The Korean Mathematical Society 40 (3) (2003) 399–410.
[9] B.A. Uralegaddi, C. Somanatha, Certain classes of univalent functions, in: Current Topics in Analytic Function Theory, World Sci. Publishing, River Edge, NJ, 1992, pp. 371–374.
[10] M. Acu, S. Owa, Note on a class of starlike functions, Regional Institute of Medical Sciences, Kyoto (2006).
[11] A. Cătaş, On certain class of p-valent functions defined by new multiplier transformations, in: Proceedings Book of the International Symposium on Geometric Function Theory and Applications, TC Istanbul Kultur University, Turkey, August 20–24, 2007, pp. 241–250.
[12] G.I. Oros, On a new strong differential subordination (in press).
[13] J.A. Antonino, S. Romaguera, Strong differential subordination to Briot-Bouquet differential equations, Journal of Differential Equations 114 (1994) 101–105.
[14] G.I. Oros, Gh. Oros, Strong differential subordination, Turkish Journal of Mathematics 33 (2009) 249–257.
[15] A. Alb Lupaş, G.I. Oros, Gh. Oros, On special strong differential subordinations using Sălăgean and Ruscheweyh operators, Journal of Computational Analysis and Applications, vol. 14 (2012) (in press).