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Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process

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1. INTRODUCTION

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for all x, y in C. It has been shown that if X is uniformly convex, then every nonexpansive mapping $T: C \to C$ has a fixed point (see [1], cf. also [7]). In 1974, Ishikawa [6] introduced a new iteration procedure for approximating fixed points of pseudo-contractive compact mappings in Hilbert spaces as follows.

$$x_{n+1} = t_n T(s_n T x_n + (1 - s_n) x_n) + (1 - t_n) x_n, \qquad n = 0, 1, 2, ..., \tag{I}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$ satisfying certain restrictions. Note that the normal Mann iteration procedure (cf. [8, 3, 5]),

$$x_{n+1} = t_n T x_n + (1 - t_n) x_n, \qquad n = 0, 1, 2, ...,$$

where $\{t_n\}$ is a sequence in $[0, 1]$, is a special case of the Ishikawa one (corresponding to the choice $s_n = 0$ for all $n \geqslant 0$). For comparison of the

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two iterative processes in the one-dimensional case, we refer the reader to [11].

It is the object of the present paper to show that if X is a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable, C is a bounded closed convex subset of X, and $T: C \to C$ is a nonexpansive mapping, then for any initial data x_0 in C the Ishikawa iterates $\{x_n\}$ defined by (I), where $\{t_n\}$ and $\{s_n\}$ are chosen so that $\sum_n t_n(1-t_n)$ diverges, $\sum_n s_n(1-t_n)$ converges, and $\overline{\lim}_n s_n < 1$, converge weakly to a fixed point of T. This generalizes a theorem of Reich [10].

2. Preliminaries and Lemmas

Recall that a Banach space X is said to satisfy Opial's condition [9] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x_0$ weakly implies $\overline{\lim}_n \|x_n - x_0\| < \overline{\lim}_n \|x_n - y\|$ for all y in X, $y \neq x_0$. It is known [9] that all l^p spaces for $1 < p < \infty$ enjoy this property. However, the L^p spaces do not, unless $p = 2$. It is also known [4] that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition. Recall also that X is said to have a Frechet differentiable norm if, for each $x \in S(X)$, the unit sphere of X, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists and is attained uniformly in $y \in S(X)$. In this case we have

$$\tfrac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \leqslant \tfrac{1}{2}\|x+h\|^2 \leqslant \tfrac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + g(\|h\|) \quad (2.1)$$

for all bounded x, h in X, where $J(x) = \partial \frac{1}{2}\|x\|^2$ is the Frechet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $x \in X$, $\langle \cdot, \cdot \rangle$ is the pairing between X and X^*, and $g(\cdot)$ is a function defined on $[0, \infty)$ such that $\lim_{t \downarrow 0} g(t)/t = 0$.

Suppose now that C is a bounded closed convex subset of a uniformly convex Banach space X and $T: C \to C$ is a nonexpansive mapping. To each integer $n \geqslant 0$, we write

$$T_n(x) = t_n T(s_n Tx + (1 - s_n)x) + (1 - t_n)x, \qquad x \in C. \quad (2.2)$$

Then $T_n: C \to C$ is also nonexpansive and the Ishikawa iterates $\{x_n\}$ defined by (I) can be written as

$$x_{n+1} = T_n x_n, \qquad n = 0, 1, 2, \ldots. \quad (2.3)$$

We note that $F(T_n) \supseteq F(T)$ for $n \geqslant 0$, where $F(T)$ denotes the set of fixed points of T.

LEMMA 1. *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of nonnegative numbers such that* $a_{n+1} \leqslant a_n + b_n$ *for all* $n \geqslant 1$. *If* $\sum_n b_n$ *converges, then* $\lim_n a_n$ *exists.*

Proof. For $n, m \geqslant 1$, we have

$$a_{n+m+1} \leqslant a_{n+m} + b_{n+m} \leqslant \cdots \leqslant a_n + \sum_{j=n}^{n+m} b_j.$$

Hence $\overline{\lim}_m a_m \leqslant a_n + \sum_{j=n}^{\infty} b_j$, which implies that $\overline{\lim}_m a_m \leqslant \underline{\lim}_n a_n$. This completes the proof.

LEMMA 2. *For each* $f \in F(T)$, $\lim_n \|x_n - f\|$ *exists.*

Proof. We have $\|x_{n+1} - f\| = \|T_n x_n - T_n f\| \leqslant \|x_n - f\|$; i.e., $\{\|x_n - f\|\}$ is nonincreasing and the lemma is proved.

LEMMA 3. *Suppose* $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$, $\sum_{n=0}^{\infty} s_n(1-t_n) < \infty$, *and* $\overline{\lim}_n s_n < 1$. *Then* $\lim_n \|Tx_n - x_n\| = 0$.

Proof. Set $y_n = s_n T x_n + (1-s_n) x_n$. Then $x_{n+1} = t_n T y_n + (1-t_n) x_n$. Let f be in $F(T)$. We may assume $\lim_n \|x_n - f\| \neq 0$. Then, noting that $\|y_n - f\| \leqslant \|x_n - f\|$, we obtain

$$\begin{aligned} \|x_{n+1} - f\| &= \|t_n(Ty_n - f) + (1-t_n)(x_n - f)\| \\ &\leqslant \|x_n - f\| \left[1 - 2t_n(1-t_n)\, \delta_X\left(\frac{\|Ty_n - f\|}{\|x_n - f\|}\right)\right], \end{aligned} \tag{2.4}$$

where δ_X is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|\tfrac{1}{2}(x+y)\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\}$$

for $0 \leqslant \varepsilon \leqslant 2$. Now it is readily seen from (2.4) that $\sum_{n=0}^{\infty} t_n(1-t_n)\, \delta_X(\|Ty_n - x_n\|/\|x_n - f\|)$ converges. But, since $\sum_{n=0}^{\infty} t_n(1-t_n)$ diverges, we have $\underline{\lim}_n \delta_X(\|Ty_n - x_n\|/\|x_n - f\|) = 0$ and thus

$$\underline{\lim}_n \|Ty_n - x_n\| = 0, \tag{2.5}$$

since δ_X is strictly increasing and continuous and $\lim_n \|x_n - f\| > 0$. Since

$$\begin{aligned} \|Tx_n - x_n\| &\leqslant \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\ &\leqslant \|x_n - y_n\| + \|Ty_n - x_n\| \\ &= s_n \|Tx_n - x_n\| + \|Ty_n - x_n\|, \end{aligned}$$

that is,

$$\|Tx_n - x_n\| \leqslant \frac{1}{1-s_n} \|Ty_n - x_n\|,$$

we have from (2.5) that

$$\varliminf_n \|Tx_n - x_n\| = 0. \tag{2.6}$$

Since

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\leqslant t_n \|Tx_{n+1} - Ty_n\| + (1-t_n) \|Tx_{n+1} - x_n\| \\ &\leqslant t_n \|x_{n+1} - y_n\| + (1-t_n) \\ &\quad \times (\|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\leqslant t_n(t_n \|Ty_n - y_n\| + (1-t_n) \|x_n - y_n\|) \\ &\quad + (1-t_n)(\|Tx_{n+1} - x_{n+1}\| + t_n\|Ty_n - x_n\|),\end{aligned}$$

we have

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\leqslant t_n \|Ty_n - y_n\| + (1-t_n)(\|Ty_n - x_n\| + \|x_n - y_n\|) \\ &\leqslant t_n(s_n \|Ty_n - Tx_n\| + (1-s_n) \|Ty_n - x_n\|) \\ &\quad + (1-t_n)(\|Ty_n - x_n\| + \|x_n - y_n\|) \\ &\leqslant (1 + t_n s_n - t_n) \|x_n - y_n\| + (1 - t_n s_n) \|Ty_n - x_n\| \\ &\leqslant s_n(1 + t_n s_n - t_n) \|x_n - Tx_n\| \\ &\quad + (1 - t_n s_n)(\|Ty_n - Tx_n\| + \|Tx_n - x_n\|) \\ &\leqslant (s_n(1 + t_n s_n - t_n) + (1 - t_n s_n)(1 + s_n)) \|x_n - Tx_n\| \\ &= (1 + 2s_n(1-t_n)) \|x_n - Tx_n\|.\end{aligned}$$

Since $\sum_n s_n(1-t_n)$ converges and $\{\|x_n - Tx_n\|\}$ is bounded, it follows from Lemma 1 that $\lim_n \|Tx_n - x_n\|$ exists and equals zero by (2.6).

LEMMA 4. *Suppose in addition that X has a Frechet differentiable norm. Then for every f_1, f_2 in $F(T)$ and $0 < t < 1$, $\lim_n \|tx_n + (1 - f_1) - f_2\|$ exists.*

Proof. Set $S_{n,m} = T_{n+m-1} T_{n+m-2} \cdots T_{n+1} T_n$. Here T_n is defined as in (2.2). Then $S_{n,m}$ is nonexpansive and $x_{n+m} = S_{n,m} x_n$. We also set

$$a_n = a_n(t) = \|tx_n + (1-t) f_1 - f_2\|$$

and

$$d_{n,m} = \|S_{n,m}(tx_n + (1-t)f_1 - (tx_{n+m} + (1-t)f_1)\|.$$

By a result of Bruck [2], there exists a strictly increasing continuous function $g\colon [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} &g(\|S(tx + (1-t)y) - (tSx + (1-t)Sy)\|) \\ &\quad \leqslant \|x - y\| - \|Sx - Sy\| \end{aligned}$$

for all nonexpansive $S\colon C \to X$, x, $y \in C$, and $0 \leqslant t \leqslant 1$. It then follows that

$$\begin{aligned} g(d_{n,m}) &\leqslant \|x_n - f_1\| - \|S_{n,m}x_n - S_{n,m}f_1\| \\ &= \|x_n - f_1\| - \|x_{n+m} - f_1\|. \end{aligned}$$

Since $\lim_n \|x_n - f_1\|$ exists, we conclude that

$$\lim_{n,m \to \infty} d_{n,m} = 0. \tag{2.7}$$

From (2.7) and the fact that $a_{n+m} = \|tx_{n+m} + (1-t)f_1 - f_2\| \leqslant d_{n+m} + \|S_{n,m}(tx_n + (1-t)f_1 - f_2)\| \leqslant d_{n,m} + a_n$, it follows that

$$\overline{\lim_n}\, a_n \leqslant \lim_{n,m \to \infty} d_{n,m} + \underline{\lim_n}\, a_n = \underline{\lim_n}\, a_n.$$

This completes the proof.

3. Convergence of the Ishikawa Iteration Process

In this section we prove the weak and strong convergence of the Ishikawa iteration process in a uniformly convex Banach space.

THEOREM 1. *Let X be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable, C be a bounded closed convex subset of X, and $T\colon C \to C$ a nonexpansive mapping. Then for any initial guess x_0 in C, the Ishikawa iteration process $\{x_n\}$ defined by* (I), *with the restrictions that $\sum_{n=0}^{\infty} t_n(1-t_n)$ diverges, $\sum_{n=0}^{\infty} s_n(1-t_n)$ converges, and $\overline{\lim}_n\, s_n$ is less than one, converges weakly to a fixed point of T.*

Proof. By Browder [1], we know that if X is uniformly convex, then T has a fixed point and $I - T$ is demiclosed at the origin; i.e., for any sequence $\{y_n\}$ in C, the conditions $y_n \to y$ weakly and $y_n - Ty_n \to 0$ strongly imply

$y - Ty = 0$. It thus follows from Lemma 3 that $\omega_w(x_n) \subset F(T)$. (Here $\omega_w(x_n)$ denotes the weak ω-lim set of the sequence $\{x_n\}$, i.e., the set $\{u \in X : u = \text{weak-lim}_{k \to \infty} x_{n_k}$ for some $n_k \uparrow \infty$.) To show that $\{x_n\}$ converges weakly to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. To this end, we first suppose that X satisfies Opial's condition and suppose $p \neq q$ are in $\omega_w(x_n)$. Then $p = \text{weak-lim}_{k \to \infty} x_{n_k}$ and $q = \text{weak-lim}_{j \to \infty} x_{m_j}$ for some $n_k \uparrow \infty$ and $m_j \uparrow \infty$. By Lemma 2 and Opial's condition of X, we then have

$$\begin{aligned} \lim_n \|x_n - p\| &= \lim_k \|x_{n_k} - p\| < \lim_k \|x_{n_k} - q\| \\ &= \lim_j \|x_{m_j} - q\| < \lim_j \|x_{m_j} - p\| \\ &= \lim_n \|x_n - p\|, \end{aligned}$$

arriving at a contradiction. This proves the theorem in the case in which X satisfies Opial's condition. We now assume that X has a Frechet differentiable norm. Substituting $f_1 - f_2$ and $t(x_n - f_1)$ for x and h, respectively, in (2.1), where $f_1, f_2 \in F(T)$ and $0 < t < 1$, we obtain

$$\begin{aligned} &\tfrac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle \\ &\quad \leqslant \tfrac{1}{2}\|tx_n + (1-t) f_1 - f_2\|^2 \\ &\quad \leqslant \tfrac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle + g(t\,\|x_n - f_1\|). \end{aligned}$$

By Lemma 3, we derive that

$$\begin{aligned} &\tfrac{1}{2}\|f_1 - f_2\|^2 + t \cdot \overline{\lim_n}\, \langle x_n - f_1, J(f_1 - f_2)\rangle \\ &\quad \leqslant \lim_n \tfrac{1}{2}\|tx_n + (1-t) f_1 - f_2\|^2 \\ &\quad \leqslant \tfrac{1}{2}\|f_1 - f_2\|^2 + t \cdot \underline{\lim_n}\, \langle x_n - f_1, J(f_1 - f_2)\rangle + o(t). \end{aligned}$$

Hence

$$\overline{\lim_n}\, \langle x_n - f_1, J(f_1 - f_2)\rangle \leqslant \underline{\lim_n}\, \langle x_n - f_1, J(f_1 - f_2)\rangle + o(t)/t.$$

On letting $t \to 0^+$, we see that $\lim_n \langle x_n - f_1, J(f_1 - f_2)\rangle$ exists. In particular, this implies that

$$\langle p - q, J(f_1 - f_2)\rangle = 0 \tag{3.1}$$

for all p, q in $\omega_w(x_n)$ and f_1, f_2 in $F(T)$. Since $\omega_w(x_n) \subset F(T)$ for any p, q in $\omega_w(x_n)$, by replacing f_1, f_2 in (3.1) by p, q, respectively, we obtain

$$\|p-q\|^2 = \langle p-q, J(p-q) \rangle = 0.$$

This shows that $\omega_w(x_n)$ must be singleton.

Remark. Theorem 1 above generalizes Theorem 2 of Reich [10] which corresponds to the choice $s_n = 0$ for all $n \geqslant 0$.

Next we briefly discuss the strong convergence of the Ishikawa iteration scheme.

THEOREM 2. *Suppose that X is a uniformly convex Banach space and T, C and $\{x_n\}$ are as in Theorem 1. Suppose also that the range of C under T is contained in a compact subset of X. Then the Ishikawa iterates $\{x_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemma 3 and the precompactness of $T(C)$, we see that $\{x_n\}$ admits a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denote by z. Then, again by Lemma 3, we have $z = Tz$; namely, z is a fixed point of T. Since $\|x_n - z\|$ is decreasing by Lemma 2, z is actually the strong limit of the sequence $\{x_n\}$ itself.

Recall that a mapping $T: C \to C$ is said to satisfy Condition A ([12]) if there exists a nondecreasing function f: $[0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geqslant f(d(x, F(T))$ for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

THEOREM 3. *Let X be a uniformly convex Banach space and let T, C, and $\{x_n\}$ be as in Theorem 1. If T satisfies Condition* A, *then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Condition A, we have

$$\|x_n - Tx_n\| \geqslant f(d(x_n, F(T))$$

fort all $n \geqslant 0$. Since $\{d(x_n, F(T))\}$ is decreasing by Lemma 2, it follows from Lemma 3 that

$$\lim_n d(x_n, F(T)) = 0.$$

We can thus choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - p_k\| < 2^{-k}$$

for all integers $k \geqslant 1$ and some sequence $\{p_k\}$ in $F(T)$. Again by Lemma 2, we have $\|x_{n_{k+1}} - p_k\| \leqslant \|x_{n_k} - p_k\| < 2^{-k}$, and hence

$$\begin{aligned}\|p_{k+1} - p_k\| &\leqslant \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &\leqslant 2^{-(k+1)} + 2^{-k} < 2^{-k+1},\end{aligned}$$

which shows that $\{p_k\}$ is Cauchy and therefore converges strongly to a point p in $F(T)$ since $F(T)$ is closed. Now it is readily seen that $\{x_{n_k}\}$ and hence $\{x_n\}$ itself by Lemma 2 converges strongly to p.

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