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On the number of segments needed in a piecewise linear approximation

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ABSTRACT

The introduction of high-speed circuits to realize an arithmetic function f as a piecewise linear approximation has created a need to understand how the number of segments depends on the interval $a \leq x \leq b$ and the desired approximation error ε. For the case of optimum non-uniform segments, we show that the number of segments is given as $s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}$, $(\varepsilon \to 0^+)$, where $c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}dx$. Experimental data shows that this approximation is close to the exact number of segments for a set of 14 benchmark functions. We also show that, if the segments have the same width (to reduce circuit complexity), then the number of segments is given by $s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}$, $(\varepsilon \to 0^+)$, where $c = \frac{(b-a)\sqrt{|f''|_{\max}}}{4}$.

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1. Introduction

A **numeric function generator** (**NFG**) is a logic circuit [1–6] that realizes an arithmetic function like $f(x) = \sin(\pi x)$ over some specified interval $a \leq x \leq b$. We consider the numeric function generator shown in Fig. 1. This architecture realizes a given function as a set of segments or pieces, where f is approximated in each segment by the linear equation $f(x) \approx c_1x + c_0$. The values of c_1 and c_0 are stored in the Coefficients Memory shown in Fig. 1 in a location whose address is specified by the Segment Index Encoder. In designing this circuit, one partitions the interval $a \leq x < b$ into segments, where the coefficients c_1 and c_0 are the same in each segment and approximate the function to within some specified error. It is known that the Segment Index Encoder is tractably realized [7].

This circuit is said to realize a **non-uniform segmentation** because, in general, the segments have different widths. Indeed, we will choose the segment widths as large as possible so that the approximation is not more than the approximation error away from the exact value. In this way, we produce a segmentation that has as few segments as possible.

In contrast, we also consider a **uniform segmentation**. In this case, all segments have equal width. If the segment widths are 2^m, then the segment index encoder can be removed and the higher order $n - m$ bits used to drive the address of the Segment Index Encoder, where n is the number of bits to encode x. In this case, the circuit is likely to be smaller and faster.

Up to this point, we have not had an analytical tool to predict the size of the Coefficients Memory as a function of the function realized, the domain, and the required approximation error. That is, our understanding of its size has only been through experimental results; i.e. specific implementations [7–11].

In this paper, we derive an expression for the number of segments (size of the Coefficients Memory) for both the non-uniform and the uniform case. We expect that, as the approximation error decreases, the number of segments needed to accommodate that improvement increases. Specifically we show that, for non-uniform segmentation, the number of

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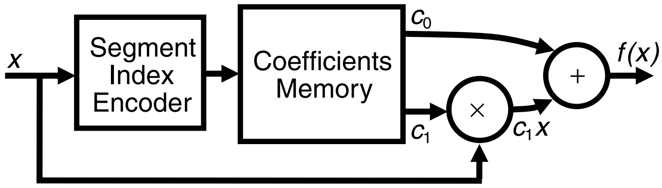

Fig. 1. Architecture of a numerical function generator using piecewise linear approximation and non-uniform segmentation.

segments is approximately $\frac{c}{\sqrt{\varepsilon}}$, where ε is the approximation error, and $c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}\mathrm{d}x$. It follows that, the number of segments also increases as the interval increases, and that the magnitude of the second derivative of the realized function has a significant influence on the number of segments. In the case of a uniform segmentation, we show that the number of segments is approximately $\frac{c}{\sqrt{\varepsilon}}$, where $c = \frac{(b-a)\sqrt{|f''|_{\max}}}{4}$.

2. Non-uniform approximation with unrestricted slope

Let f be a three times continuously differentiable function defined on the domain $[a, b]$. In the case of unrestricted slope, our algorithm proceeds by generating a segmentation $\{x_0, x_1, \dots, x_\sigma\}$ of $[a, b]$ with the property that, in each of the σ segments $[x_i, x_{i+1}]$, $i = 0, \dots, \sigma - 1$, the chord between x_i and x_{i+1} produces a linear approximation $(c_1x + c_0)$ to f within a previously specified approximation error ε:

$$|f(x) - c_1x - c_0| \le \varepsilon, \quad x \in [x_i, x_{i+1}].$$

In the example of the $\sin(\pi x)$ function, experimental results show that, for a specified approximation error ε, the segmentation algorithm, in the case of unrestricted slope, determines a number of segments s that is proportional to $1/\sqrt{\varepsilon}$. We now show that this is a general result for a large set of functions.

Specifically, we give an asymptotic approximation for the number of segments $s(\varepsilon)$ needed to approximate a given function $f(x)$ to within a given approximation error ε. We say that $t(\varepsilon)$ is an asymptotic approximation to $s(\varepsilon)$, expressed as $s(\varepsilon) \sim t(\varepsilon)$, if $\lim_{\varepsilon\to 0^+} s(\varepsilon)/t(\varepsilon) = 1$. In our use of this, $s(\varepsilon)$ is the exact number of segments, while $t(\varepsilon)$ is an approximation to $s(\varepsilon)$, which has a simple form. Intuitively, we expect the number of segments to increase as the approximation error ε decreases. We seek to determine this relationship, since it provides insight into how hardware complexity depends on the approximation error.

In what follows, we divide the domain $[a, b]$ into two sets depending of the value of $f''(x)$. Let

$$A_\varepsilon = \{x \in [a, b] : |f''(x)| \le \sqrt{\varepsilon}\} \quad \text{and} \tag{1}$$

$$B_\varepsilon = \{x \in [a, b] : |f''(x)| > \sqrt{\varepsilon}\}, \tag{2}$$

where $\varepsilon > 0$. Note that (1) A_ε is a closed set; (2) B_ε is an open set, (3) $A_\varepsilon \bigcap B_\varepsilon = \emptyset$; and (4) $A_\varepsilon \bigcup B_\varepsilon = [a, b]$. Under the assumption that the length of A_ε tends to zero as ε tends to zero, and B_ε is a finite union of open intervals, we have

Theorem 1. *Consider a piecewise linear approximation of f on the domain $[a, b]$ that is accurate to within ε, using a piecewise linear segmentation. Let f be three times continuously differentiable on $[a, b]$. Then, $s(\varepsilon)$, the number of segments in an optimum segmentation of $[a, b]$, satisfies the following asymptotic approximation:*

$$s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}, \quad (\varepsilon \to 0^+), \tag{3}$$

where

$$c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}\mathrm{d}x. \tag{4}$$

Proof. See Appendix.

For example, if we take $f(x) = \sin(\pi x)$ on $[0, \frac{1}{2}]$, we find

$$s(\varepsilon) \sim \frac{\pi\int_0^{1/2}\sqrt{\sin \pi x}\mathrm{d}x}{4\sqrt{\varepsilon}}. \tag{5}$$

Using MAPLE to numerically evaluate the integral in (5) yields

$$s(\varepsilon) \sim \frac{0.2995}{\sqrt{\varepsilon}}.$$

The following corollary of Theorem 1 relaxes the conditions on the end points of the domain.

Table 1
Number of segments for non-uniform and uniform segmentations.

Function $f(x)$	Domain of x	Non-Uni $s \sim$	Uniform $s \sim$	Non-Uni/Uni $=$
2^x	$[0, 1)$	$\frac{0.2071}{\sqrt{\varepsilon}}$	$\frac{0.2451}{\sqrt{\varepsilon}} = \frac{\ln 2}{\sqrt{8}\sqrt{\varepsilon}}$	84%
$1/x$	$[1, 2)$	$\frac{0.2071}{\sqrt{\varepsilon}}$	$\frac{0.3536}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{8}\sqrt{\varepsilon}}$	59%
$\sqrt{x}$	$\left[\frac{1}{1024}, 2\right)$	$\frac{0.5062}{\sqrt{\varepsilon}}$	$\frac{45.2327}{\sqrt{\varepsilon}} = \frac{2-\frac{1}{1024}}{4\sqrt{\varepsilon}}\sqrt{\left\lvert\frac{-1}{4(\frac{1}{1024})^{3/2}}\right\rvert}$	1%
$1/\sqrt{x}$	$[1, 2)$	$\frac{0.1378}{\sqrt{\varepsilon}}$	$\frac{0.2165}{\sqrt{\varepsilon}} = \frac{\sqrt{3}}{8\sqrt{\varepsilon}}$	64%
$\log_2(x)$	$[1, 2)$	$\frac{0.2081}{\sqrt{\varepsilon}}$	$\frac{0.3003}{\sqrt{\varepsilon}} = \frac{1}{4\sqrt{\ln 2}\sqrt{\varepsilon}}$	69%
$\ln x$	$[1, 2)$	$\frac{0.1733}{\sqrt{\varepsilon}}$	$\frac{0.2500}{\sqrt{\varepsilon}} = \frac{1}{4\sqrt{\varepsilon}}$	69%
$\sin(\pi x)$	$\left[0, \frac{1}{2}\right)$	$\frac{0.2995}{\sqrt{\varepsilon}}$	$\frac{0.3927}{\sqrt{\varepsilon}} = \frac{\pi}{8\sqrt{\varepsilon}}$	76%
$\cos(\pi x)$	$\left(0, \frac{1}{2}\right)$	$\frac{0.2995}{\sqrt{\varepsilon}}$	$\frac{0.3927}{\sqrt{\varepsilon}} = \frac{\pi}{8\sqrt{\varepsilon}}$	76%
$\tan(\pi x)$	$\left[0, \frac{1}{4}\right)$	$\frac{0.2005}{\sqrt{\varepsilon}}$	$\frac{0.3927}{\sqrt{\varepsilon}} = \frac{\pi}{8\sqrt{\varepsilon}}$	51%
$\sqrt{-ln(x)}$	$\left[\frac{1}{1024}, \frac{1}{4}\right)$	$\frac{0.6489}{\sqrt{\varepsilon}}$	$\frac{26.7609}{\sqrt{\varepsilon}} = \frac{31\frac{7}{8}\sqrt{2(\ln 1024)^{-1/2}-(\ln 1024)^{-3/2}}}{\sqrt{\varepsilon}}$	2%
$\tan^2(\pi x) + 1$	$[0, \frac{1}{4})$	$\frac{0.4200}{\sqrt{\varepsilon}}$	$\frac{0.7854}{\sqrt{\varepsilon}} = \frac{\pi}{4\sqrt{\varepsilon}}$	53%
$-(x\log_2 x + (1-x)\log_2(1-x))$	$\left[\frac{1}{1024}, 1-\frac{1}{1024}\right]$	$\frac{0.9058}{\sqrt{\varepsilon}}$	$\frac{9.5949}{\varepsilon} = \frac{1-\frac{1}{512}}{4\sqrt{\varepsilon}}\sqrt{\left\lvert\frac{1}{\ln 2(\frac{1}{1024}-1)} - \frac{1}{\ln 2\,\frac{1}{1024}}\right\rvert}$	9%
$\frac{1}{1+e^{-x}}$	$[0, 1)$	$\frac{0.0550}{\sqrt{\varepsilon}}$	$\frac{0.0754}{\sqrt{\varepsilon}} = \frac{\sqrt{e^1+e^{-1}}}{e^1+2+e^{-1}}\frac{1}{4\sqrt{\varepsilon}}$	73%
$\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$	$[0, \sqrt{2})$	$\frac{0.1452}{\sqrt{\varepsilon}}$	$\frac{0.2233}{\sqrt{\varepsilon}} = \frac{\sqrt{2}}{4(2\pi)^{1/4}\sqrt{\varepsilon}}$	65%

Corollary 1. *Let f be three times continuously differentiable on the open interval (a, b) and $|\sqrt{f''}|$ be improperly Riemann integrable*[1] *on the closed interval $[a, b]$, with integrable singularities at the endpoints a or b. Then, $s(\varepsilon)$, the number of segments in an optimum segmentation of $[a, b]$, satisfies the following asymptotic approximation:*

$$s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}, \quad (\varepsilon \to 0^+), \tag{6}$$

where

$$c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}\mathrm{d}x. \tag{7}$$

The significance of Corollary 1 is that we can obtain an asymptotic approximation to the number of segments even when the function has singularities at the endpoints of the interval over which the approximation occurs.

Table 1 shows the asymptotic approximations to the number of segments for 14 functions.[2] The third column labeled **Non-Uni** shows the number of segments, where the segments are as large as possible (and thus, as few as possible). The fourth column labeled **Uniform** will be discussed in Section 3. The fifth and rightmost column, labeled **Non-Uni/Uni**, shows the ratio of the number of non-uniform segments needed compared to the number of uniform segments needed as a percentage, when the specified approximation error ε is small. For example, for 2^x, non-uniform segmentation uses 84% of the segments needed by uniform segmentation. For $\sqrt{x}$, $\sqrt{-\ln(x)}$, and $-(x\log_2 x + (1-x)\log_2(1-x))$, the fifth column contains 1%, 2%, and 9%, which are much smaller than for any other function. We discuss the derivations for the number of segments needed for uniform segmentation in Section 3.

3. Uniform approximation with unrestricted slope

In this part, we consider two ways to determine the number of segments needed in the case when a completely free choice of slope is used with uniform segmentation. The first approach is a direct computation, which can be applied to all functions considered in this paper, and the second is an asymptotic approximation that applies to a majority of the functions.

1 The function f is improperly Riemann integrable on $[a, b]$ if f is Riemann integrable on every open subinterval (c, d) of $[a, b]$, and the function f becomes unbounded in the neighborhoods of a or b, and $\lim_{c\to a, d\to b}\int_c^d f(x)\mathrm{d}x$ exists.

2 To accommodate a fixed bit representation, for most functions considered in this paper, we choose an interval that is left-closed and right-open. For example, the interval $0 \le x < 1$ or $[0, 1)$, in the case of 8-bit precision, consists of 00000000, 00000001, ..., and 1111111, representing 0, $\frac{1}{256}$, ..., and $\frac{255}{256} < 1$.

The first approach is illustrated as follows. Consider a uniform segmentation of a function $f(x)$ from $x = a$ to $x = b$. Consider a segment beginning at $x = \alpha$ and ending at $x = \beta$, where $\alpha < \beta$ near a point in the domain $[a, b]$, where $|f''(x)|$ is maximum. It is at this point that the maximum error between the function and its linear piecewise approximation occurs. That is, if we choose the segment width to be small enough at this point so that the error is equal to the specified approximation error ε, then that small a width for all segments will be sufficient to achieve an error no greater than ε in all segments. Therefore, the number of segments s, to achieve an approximation error with uniform segmentation is

$$s = \left\lceil \frac{b-a}{\beta-\alpha} \right\rceil. \tag{8}$$

Consider a piecewise linear approximation, $f_{pl}(x)$ to $f(x)$ of the form $f_{pl}(x) = (f(\beta) - f(\alpha))\frac{x-\alpha}{\beta-\alpha} + f(\alpha)$. The error due to the approximation can be viewed as $|f(x) - f_{pl}|$. Note that $|f(x) - f_{pl}|$ is 0 at $x = \alpha$ and at $x = \beta$. However, we will approximate $f(x)$ in the domain $[\alpha, \beta]$ by adding a constant to f_{pl} so that the maximum error in the domain $[\alpha, \beta]$ is no greater than $\frac{1}{2}|f(x) - f_{pl}|$. As a result, the error function $e(x)$ for the domain $[\alpha, \beta]$ is $\frac{1}{2}(|f(x) - f_{pl}|)$. Substituting for f_{pl}, yields

$$e(x) = \frac{1}{2}\left[[f(x) - f(\alpha)] - [f(\beta) - f(\alpha)]\frac{x-\alpha}{\beta-\alpha} \right]. \tag{9}$$

To illustrate, consider the function $f(x) = \sqrt{x}$ in the domain $[0, 2)$. $f''(x)$ becomes unbounded near $x = 0$. Thus, we choose $\alpha = 0$, and from (9), we have

$$e(x) = \frac{1}{2}\left[\sqrt{x} - \sqrt{\beta}\frac{x}{\beta} \right].$$

By differentiating $e(x)$ with respect to x, we find that the maximum error occurs at $x = \frac{\beta}{4}$. At this value, the maximum e, e_{max}, is $\frac{\sqrt{\beta}}{8}$. We choose this value to be ε. That is, $e_{max} = \varepsilon$, and, so $\frac{\sqrt{\beta}}{8} = \varepsilon$. Substituting this into (8) yields

$$s_{\sqrt{x}} = \left\lceil \frac{1}{32\varepsilon^2} \right\rceil. \tag{10}$$

Thus, as ε decreases (improves), the number of segments needed for a uniform segmentation increases as the inverse of the square of ε. This results from the fact that f'' becomes unbounded near the endpoint 0. A similar analysis can be applied to the entropy function, $-(x\log_2 x + (1-x)\log_2(1-x))$, yielding

$$s_{\text{entropy}} = \left\lceil \frac{1}{2e\ln 2\varepsilon} \right\rceil. \tag{11}$$

The number of segments needed for uniform segmentation are shown in Column 4 of Table 1, labeled **Uniform**.[3] All expressions have the form $\frac{c}{\sqrt{\varepsilon}}$, which are the same form as the asymptotic approximation for the number of segments required in a non-uniform segmentation. This includes the two functions, $\sqrt{x}$ and $-(x\log_2 x + (1-x)\log_2(1-x))$, where the number of segments is given as (10) and (11), when a singularity is included, as discussed above. The segmentation algorithm with which we compared the asymptotic approximation values had to exclude the singularities, and, for consistency, we chose an interval without singularities. The right column of Table 1, labeled **Non-Uni/Uni** shows the ratio of segments required in a non-uniform segmentation to the number of segments required in a uniform segmentation (expressed as a percentage). For some functions, like 2^x, $\sin(\pi x)$, and $\cos(\pi x)$, this is high, 84%, 76%, and 76%. For such functions, there is a small penalty for using a uniform segmentation. For other functions, like $\sqrt{-\ln(x)}$, this percentage is low, 9%, and the penalty is high. A similar statement is true of $\sqrt{x}$ and $-(x\log_2 x + (1-x)\log_2(1-x))$, where the percentage number of segments is 1% and 2%. It is interesting that, when the domain for the $\sqrt{x}$ function is reduced to $[\frac{1}{2}, 2]$, the number of segments is given by $s \sim \frac{0.3153}{\sqrt{\varepsilon}}$.

The second approach to determining the number of segments requires the second derivative $f''(x)$ to be bounded over the domain of approximation. We have

Theorem 2. *Consider a piecewise linear approximation of a function $f(x)$ on the domain $[a, b]$ with a specified approximation error ε or less using uniform segmentation. Let the absolute value of the second derivative $|f''(x)|$ of $f(x)$ on the domain $[a, b]$ be finite. Then, the number of segments s is*

$$s \sim \frac{c}{\sqrt{\varepsilon}}, \tag{12}$$

[3] Recall that, in order to take advantage of uniform segmentation, we must choose the number of segments to be the next higher power of 2.

Table 2
Comparing the number of estimated segments with the exact number for nonuniform segmentation.

Function $f(x)$	Interval x	Estimated/exact no. of segments			
		8	16	32	64
2^x	$[0, 1)$	5/5	75/75	19,195/19,191	1.26×10^9/-
$1/x$	$[1, 2)$	5/5	75/75	19,195/19,191	1.26×10^9/-
$\sqrt{x}$	$\left[\frac{1}{1024}, 2\right)$	11/12	183/184	46,917/46,715	3.07×10^9/-
$1/\sqrt{x}$	$[1, 2)$	3/4	50/50	12,770/12,769	8.37×10^8/-
$\log_2(x)$	$[1, 2)$	5/5	75/76	19,291/19,286	1.26×10^9/-
$\ln x$	$[1, 2)$	4/4	63/63	16,061/16,058	1.05×10^9/-
$\sin(\pi x)$	$\left[0, \frac{1}{2}\right)$	7/7	108/109	27,761/27,752	1.82×10^9/-
$\cos(\pi x)$	$\left(0, \frac{1}{2}\right)$	7/7	108/109	27,761/27,752	1.82×10^9/-
$\tan(\pi x)$	$\left[0, \frac{1}{4}\right)$	5/5	73/73	18,579/18,572	1.22×10^9/-
$\sqrt{-ln(x)}$	$\left[\frac{1}{1024}, \frac{1}{4}\right)$	15/15	235/235	60,142/59,627	3.94×10^9/-
$\tan^2(\pi x) + 1$	$\left[0, \frac{1}{4}\right)$	10/10	152/152	38,925/38,892	2.55×10^9/-
$-(x \log_2 x + (1-x)\log_2(1-x))$	$\left[\frac{1}{1024}, 1 - \frac{1}{1024}\right]$	20/21	328/328	83,953/83,740	5.50×10^9/-
$\frac{1}{1+e^{-x}}$	$[0, 1)$	1/2	20/20	5,101/5,101	3.34×10^8/-
$\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$	$[0, \sqrt{2})$	3/3	53/53	13,453/13,449	8.82×10^8/-

where

$$c = \frac{(b-a)\sqrt{|f''|_{\max}}}{4}, \tag{13}$$

where $|f''|_{\max}$ is the maximum of the absolute value of $f''(x)$ over the domain $[a, b]$.

Proof. See Appendix.

The right column of Table 1 shows the results of Theorem 2. Specifically, all functions in this table satisfy the restriction that $|f''(x)|$ is finite. For example, for $\sin(\pi x)$, $|f''(x)|_{\max} = \pi^2$. Therefore, for this function, (13) yields $s \sim \frac{\pi}{8\sqrt{\varepsilon}}$, which agrees with Table 1.

4. Comparison of the estimates with an exact segmentation algorithm

We compare the estimates of the number of segments as obtained by Theorem 1 to the actual number obtained by a segmentation algorithm that produces the minimum needed (non-uniform segmentation) [12]. Table 2 shows the result. Each table entry shows this data as "Estimated/Exact", where "Estimated" refers to the estimated number of segments as obtained by Theorem 1 and "Exact" is obtained by the algorithm of [12]. We were not able to obtain the exact values for 64 bit precision because of extreme computation times. The data shows that the estimates and exact values are close. Note that the same data for uniform segmentation is not so interesting because Theorem 2 specifies an exact number η of segments. In order to avoid using the segment index encoder, it is then necessary to use $2^{\lceil \log_2 \eta \rceil}$ segments, which is the next power of 2 equal to or greater than η.

5. Concluding remarks

As a result of our analysis, we have an understanding of how the hardware complexity, as measured by the number of segments, depends on the specified precision ε. Our results also show that, for some functions, it is reasonable to use uniform segmentation, thus eliminating the segment index encoder. For such functions, our results validate the past research on uniform segmentation. For example, with the $\sin(\pi x)$ and $\cos(\pi x)$ functions approximated using unrestricted slope, $\frac{0.3927}{\sqrt{\varepsilon}}$ segments are needed for uniform segmentation, while no more than $\frac{0.2995}{\sqrt{\varepsilon}}$ segments are needed for non-uniform segmentation. This is about 31% more segments. The penalty is substantial, but if memory is inexpensive, and speed is essential, this may be a welcome tradeoff.

Table 2 shows the estimates obtained are close to the number of segments as determined by an algorithm that obtains the exact minimum number of segments in a non-uniform segmentation [12]. Thus, we are now able to accurately estimate the computation time needed to compute a non-uniform segmentation. Although this time occurs only once during a design, it may be extremely long, and there may be a high premium on doing it once only. We are now able to distinguish between infeasible and difficult designs.

Table 3
Number of segments for non-uniform and uniform segmentation for four precisions, 8, 16, 32, and 64 bits.

Function $f(x)$	Interval x	Non-Uniform				Uniform			
		8	16	32	64	8	16	32	64
2^x	$[0, 1)$	5	75	19,195	1.26×10^9	6	89	22,717	1.49×10^9
$1/x$	$[1, 2)$	5	75	19,195	1.26×10^9	8	128	32,773	2.15×10^9
$\sqrt{x}$	$[\frac{1}{1024}, 2)$	12	183	46,917	3.07×10^9	8206	5.38×10^8	2.31×10^{18}	4.26×10^{37}
$1/\sqrt{x}$	$[1, 2)$	3	50	12,770	8.37×10^8	5	79	20,066	1.32×10^9
$\log_2(x)$	$[1, 2)$	5	75	19,291	1.26×10^9	7	109	27,833	1.82×10^9
$\ln x$	$[1, 2)$	4	63	16,061	1.05×10^9	6	91	23,171	1.52×10^9
$\sin(\pi x)$	$\left[0, \frac{1}{2}\right)$	7	108	27,761	1.82×10^9	9	143	36,397	2.39×10^9
$\cos(\pi x)$	$\left(0, \frac{1}{2}\right)$	7	108	27,761	1.82×10^9	9	143	36,397	2.39×10^9
$\tan(\pi x)$	$\left[0, \frac{1}{4}\right)$	5	73	18,579	1.22×10^9	9	143	36,397	2.39×10^9
$\sqrt{-ln(x)}$	$\left[\frac{1}{1024}, \frac{1}{4}\right)$	15	235	60,142	3.94×10^9	157	2507	641,600	4.20×10^{10}
$\tan^2(\pi x) + 1$	$\left[0, \frac{1}{4}\right)$	10	152	38,925	2.55×10^9	18	285	72,793	4.77×10^9
$-(x\log_2 x+(1-x)\log_2(1-x))$	$\left[\frac{1}{1024}, 1-\frac{1}{1024}\right]$	20	328	83,953	5.50×10^9	136	34,787	2.28×10^9	9.79×10^{18}
$\frac{1}{1+e^{-x}}$	$[0, 1)$	1	20	5101	3.34×10^8	2	28	6989	4.58×10^8
$\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$	$[0, \sqrt{2})$	3	53	13,453	8.82×10^8	6	81	20,696	1.36×10^9

Table 3 shows the number of segments needed in the case of non-uniform and uniform segmentation for functions approximated using unrestricted slope as calculated in Sections 4 and 5. Table 3 shows the number of segments needed for four precisions, 8, 16, 32, and 64 bits.[4]

From Table 3, we can make conclusions about the feasibility of realizing the various functions. Specifically, we can see that for 64 bit precision, very large memory size is needed in all cases. For 32 bit precision, both uniform and non-uniform segmentation yield feasible realizations, except for $\sqrt{x}$ and $-(x\log_2 x + (1-x)\log_2(1-x))$. Also, the memory required to realize $\sqrt{-\ln(x)}$ is quite large compared to that required for non-uniform segmentation. For 16 bit precision, all realizations of the functions are feasible, except for $\sqrt{x}$ using uniform segmentation. For 8 bit precision, all realizations are feasible using either non-uniform and uniform segmentation. In general, for many functions, uniform segmentation is good, especially when the cost of memory is low.

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Appendix

In the following theorem, we make the following reasonable assumptions.

Assumption 1. As $\varepsilon \to 0^+$, the *measure* of $A_\varepsilon \to 0$.

Assumption 2. The open set B_ε is a finite union of open intervals.

Given these assumptions, we now state

Theorem 1. *Consider a piecewise linear approximation of f on the domain $[a, b]$ that is accurate to within ε, using a piecewise linear segmentation. Let f be three times continuously differentiable on $[a, b]$. Then, $s(\varepsilon)$, the number of segments in an optimum segmentation of $[a, b]$, satisfies the following asymptotic approximation:*

$$s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}, \quad (\varepsilon \to 0^+),$$

4 Assuming that the most significant bit is the coefficient of 2^{-1}, we choose the error, ε, to be one-half of the value of the least significant bit. For example, for 8 bit precision, we choose the error to be 2^{-9}. We substitute this for ε in the equations for the number of segments for non-uniform segmentation and uniform segmentation.

where

$$c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}\mathrm{d}x.$$

Proof. Given $\varepsilon > 0$, divide the domain $[a, b]$ into segments with end points $\{x_0, x_1, \ldots, x_s\}$, where $x_0 = a$ and $x_s = b$. Assume $\{x_0, x_1, \ldots, x_s\}$ has the fewest segments such that all segments have an approximation error no greater than ε. Thus, for any segment, if we set

$$L_i(x) = f(x) - \left[\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) + f(x_i)\right], \tag{14}$$

then

$$|L_i(x)| \le 2\varepsilon,$$

for $x_i \le x \le x_{i+1}$, where $i = 0, 1, \ldots, s-1$. Without loss of generality, since the segmentation is optimal, we can assume that, for all but perhaps one segment, there exists an x_i^* in (x_i, x_{i+1}), for which $|L_i(x_i^*)| = 2\varepsilon$, for $i = 0, 1, \ldots, s-1$. Now let

$$x_{\mathrm{mid}}^{(i)} = \frac{x_i + x_{i+1}}{2}$$

and

$$\Delta_i = x_{i+1} - x_i$$

be respectively, the midpoint and length of the segment $[x_i, x_{i+1}]$ so that

$$x_i = x_{\mathrm{mid}}^{(i)} - \frac{\Delta_i}{2}, \tag{15}$$

$$x_{i+1} = x_{\mathrm{mid}}^{(i)} + \frac{\Delta_i}{2}, \tag{16}$$

and, for $x \in [x_i, x_{i+1}]$,

$$x = x_{\mathrm{mid}}^{(i)} - \alpha_i(x)\frac{\Delta_i}{2},$$

where $-1 \le \alpha_i(x) \le 1$.

Apply Taylor's approximation to the terms in L_i in (14). This yields, after some algebra,

$$L_i(x) = \frac{1}{2}f''(x_{\mathrm{mid}}^{(i)})\frac{\Delta_i^2}{4}(\alpha_i^2 - 1) + O(\Delta_i^3).$$

Hence,

$$|L_i(x)| = \frac{\Delta_i^2}{8}|f''(x_{\mathrm{mid}}^{(i)})|\left(|\alpha_i^2 - 1| + O(\Delta_i)\right),$$

for $i = 0, 1, \ldots s-1$ and $x_i \le x \le x_{i+1}$. Since $-1 \le \alpha_i(x) \le 1$,

$$\max_{\alpha_i(x)\in[-1,1]} |\alpha_i(x)^2 - 1| = 1.$$

Therefore,

$$2\varepsilon = \max_{[x_i, x_{i+1}]} |L_i(x)| = \frac{\Delta_i^2}{8}|f''(x_{\mathrm{mid}}^{(i)})|\,(1 + O(\Delta_i)). \tag{17}$$

Take the square root of both sides and sum over i from 0 to $s-1$:

$$\sum_{i=0}^{s-1}\sqrt{2\varepsilon} = \sum_{i=0}^{s-1}\frac{\Delta_i}{\sqrt{8}}\sqrt{|f''(x_{\mathrm{mid}}^{(i)})|}\,(1 + O(\Delta_i)).$$

We now recognize that $x_{\mathrm{mid}}^{(i)}$ must lie in either the set A_ε or the set B_ε, where

$$A_\varepsilon = \{x \in [a, b] : |f''(x)| \le \sqrt{\varepsilon}\} \quad \text{and} \tag{18}$$

$$B_\varepsilon = \{x \in [a, b] : |f''(x)| > \sqrt{\varepsilon}\}, \tag{19}$$

for $\varepsilon > 0$.

Thus, we split the sum accordingly:

$$\sum_{i=0}^{s-1}\sqrt{2\varepsilon} = \sum_{x_{\text{mid}}^{(i)}\in A_\varepsilon} \frac{\Delta_i}{\sqrt{8}}\sqrt{|f''(x_{\text{mid}}^{(i)})|}\,(1 + O(\Delta_i)) + \sum_{x_{\text{mid}}^{(i)}\in B_\varepsilon} \frac{\Delta_i}{\sqrt{8}}\sqrt{|f''(x_{\text{mid}}^{(i)})|}\,(1 + O(\Delta_i)) \,. \tag{20}$$

The first sum on the right in (20) is small, $O(\varepsilon^{1/4})$. Also, since

$$|f''(x_{\text{mid}}^{(i)})| > \sqrt{\varepsilon}$$

for $x_{\text{mid}}^i \in B_\varepsilon$, (17) implies

$$2\varepsilon \geq \frac{\Delta_i^2}{8}|f''(x_{\text{mid}}^{(i)})|(1 + O(\Delta_i)) > \frac{\Delta_i^2}{8}\sqrt{\varepsilon}(1 + O(\Delta_i)). \tag{21}$$

It follows from (21) that $\Delta_i^2 = O(\sqrt{\varepsilon})$, and so $\Delta_i \to 0$ as $\varepsilon \to 0^+$. Now, (20) can be written as

$$s(\varepsilon)\sqrt{2\varepsilon} = O(\varepsilon^{1/4}) + \sum_{x_{\text{mid}}^{(i)}\in B_\varepsilon} \frac{\Delta_i}{\sqrt{8}}\sqrt{|f''(x_{\text{mid}}^{(i)})|}\,(1 + O(\Delta_i)) \,. \tag{22}$$

Since $\Delta_i \to 0$ as $\varepsilon \to 0^+$, $A_\varepsilon \bigcup B_\varepsilon = [a, b]$, and the measure of A_ε tends to 0 as $\varepsilon \to 0^+$, it follows that the number of terms in the sum in (22) goes to infinity as $\varepsilon \to 0^+$, and so, by our previous assumptions, we can approximate the sum in (22) as a Riemann integral:

$$\sum_{x_{\text{mid}}^{(i)}\in B_\varepsilon} \frac{\Delta_i}{\sqrt{8}}\sqrt{|f''(x_{\text{mid}}^{(i)})|}\,(1 + O(\Delta_i)) = \frac{1}{\sqrt{8}}\int_{B_\varepsilon}\sqrt{|f''(x)|}\mathrm{d}x \quad (1 + o(1)).$$

Thus,

$$s(\varepsilon)\sqrt{2\varepsilon} = O(\varepsilon^{1/4}) + \frac{1}{\sqrt{8}}\int_{B_\varepsilon}\sqrt{|f''(x)|}\mathrm{d}x \quad (1 + o(1))$$

and

$$s(\varepsilon) = O(\varepsilon^{-1/4}) + \frac{1}{4\sqrt{\varepsilon}}\int_{B_\varepsilon}\sqrt{|f''(x)|}\mathrm{d}x(1 + o(1)).$$

Since the measure of the set $A_\varepsilon \to 0$, as $\varepsilon \to 0^+$, a combination of the above results gives

$$s(\varepsilon) \sim \frac{c}{\sqrt{\varepsilon}}, \quad (\varepsilon \to 0^+),$$

where

$$c = \frac{1}{4}\int_a^b \sqrt{|f''(x)|}\mathrm{d}x. \quad \blacksquare$$

Theorem 2. *Consider a piecewise linear approximation of a function $f(x)$ on the domain $[a, b]$ with a specified approximation error ε or less using uniform segmentation. Let the absolute value of the second derivative $|f''(x)|$ of $f(x)$ on the domain $[a, b]$ be bounded. Then, the number of segments s is*

$$s \sim \frac{c}{\sqrt{\varepsilon}}, \quad (\varepsilon \to 0^+), \tag{23}$$

where

$$c = \frac{(b-a)\sqrt{|f''|_{\max}}}{4}, \tag{24}$$

where $|f''|_{\max}$ is the maximum of the absolute value of $f''(x)$ over the domain $[a, b]$.

Proof. For any segment in a uniform segmentation of $f(x)$, the difference between the exact value of f and its linear piecewise approximation is

$$L_i(x) = f(x) - \left[\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) + f(x_i)\right]. \tag{25}$$

We require that

$$|L_i(x)| \le 2\varepsilon. \tag{26}$$

By a process similar to that used in Theorem 1,

$$L_i(x) = \frac{1}{2} f''(x_{\mathrm{mid}}^{(i)}) \frac{\Delta_i^2}{4} (\alpha_i^2(x) - 1) + O(\Delta_i^2). \tag{27}$$

Let $|f''|$ assume its maximum in $[x_i, x_{i+1}]$ at x_i^*. Since $x_{\mathrm{mid}}^{(i)} = x_i^* + O(\Delta_i)$, we have

$$|f''(x_{\mathrm{mid}}^{(i)})| = |f''(x_i^*)| + O(\Delta_i). \tag{28}$$

(26) and (27) imply that

$$\frac{1}{2} \left[\max_{x \in [x_i, x_{i+1}]} |f''(x)| \right] \frac{\Delta_i^2}{4} |\alpha_i^2(x) - 1| + O(\Delta_i^3) \le 2\varepsilon. \tag{29}$$

(29) can be written as

$$\frac{1}{16\varepsilon} \left[\max_{x \in [x_i, x_{i+1}]} |f''(x)| \right] |\alpha_i^2(x) - 1| + O\left(\frac{\Delta_i}{\varepsilon} \right) \le \frac{1}{\Delta_i^2}. \tag{30}$$

Now, as in Theorem 1, $\frac{\Delta_i}{\varepsilon} = O(\varepsilon^{-1/2})$, so that (30) can be written as

$$\Delta_i^2 \le \frac{1}{\frac{1}{16\varepsilon} \left[\max_{x \in [x_i, x_{i+1}]} |f''(x)| \right] |\alpha_i^2(x) - 1| + O\left(\varepsilon^{-1/2}\right)}$$

or

$$\Delta_i^2 \le \frac{16\varepsilon}{\left[\max_{x \in [x_i, x_{i+1}]} |f''(x)| \right] |\alpha_i^2(x) - 1| + O\left(\varepsilon^{1/2}\right)}. \tag{31}$$

Now, the right side of (31) is an upper bound for Δ_i^2, where $\Delta_i, i = 0, 1, \ldots, s-1$ are s segments covering $[a, b]$, in each of which $|L_i(x)| \le 2\varepsilon, x_i \le x \le x_{i+1}$. We want a uniform segmentation of $[a, b]$ with the *minimum* Δ_i. So, we choose $\alpha_i = 0$ in (31), replace $\left[\max_{x \in [x_i, x_{i+1}]} |f''(x)|\right]$ by the maximum of $|f''(x)|$ over the entire domain, and take the square root. We use the equality sign in (31) to conclude that

$$\Delta_i = \frac{4\sqrt{\varepsilon}}{\sqrt{\max_{x \in [a,b]} |f''(x)|}(1 + O(\varepsilon^{1/2}))}.$$

Thus,

$$s(\varepsilon) = \frac{b-a}{\Delta_i} = \frac{d(\varepsilon)}{\sqrt{\varepsilon}}, \tag{32}$$

where

$$d(\varepsilon) = \frac{\sqrt{\max_{x \in [a,b]} |f''(x)|}(b-a)\,(1 + O(\varepsilon^{1/2}))}{4}.$$

Since

$$\lim_{\varepsilon \to 0^+} d(\varepsilon) = d = \frac{\sqrt{\max_{x \in [a,b]} |f''(x)|}(b-a)}{4},$$

we see that (32) implies that, for a uniform optimal segmentation

$$s(\varepsilon) \sim \frac{d}{\sqrt{\varepsilon}} \quad (\varepsilon \to 0^+). \quad \blacksquare \tag{33}$$

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