Weak Compactness and Uniform Convergence of Operators in Spaces of Bochner Integrable Functions

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1. INTRODUCTION

The classical characterization of weak compactness in the space $L^1(\mu)$ over a measure space $(X, \Sigma, \mu)$, states that a bounded set $K \subset L^1(\mu)$ is relatively weakly compact iff it is uniformly $\sigma$-additive (or, equivalently, uniformly integrable, or uniformly $\mu$-absolutely continuous, in case $\mu(X) < \infty$).

In this paper we give a new characterization of weak compactness in $L^1(\mu)$ which is completely different from the classical one. It is stated in terms of uniform weak convergence of certain "admissible" sequences of operators (Theorem 6).

It is interesting to make a comparative analysis of these two characterizations of weak compactness.

(a) Consider the space $L^1_b(\mu)$ of Bochner integrable functions, where $E$ is a Banach space such that both $E$ and its dual $E'$ have the Radon–Nikodym Property (RNP). Both characterizations can be extended for sets $K \subset L^1_b(\mu)$, (by adding to uniform $\sigma$-additivity or to uniform weak convergence, the following condition: (*) for every set $A \in \Sigma$, the set \{ $\int_A f \, d\mu; f \in K$ \} is relatively weakly compact in $E$).

But if $E'$ does not have the RNP, the classical weak compactness criterion states that uniform $\sigma$-additivity and condition (*) are only necessary, but not sufficient, for relative weak compactness of $K$; and it has been shown by examples in [1] that this result cannot be improved further.

In contrast, if $E'$ does not have the RNP, our result states that uniform weak convergence of operators and condition (*) are only sufficient, but not necessary for relative weak compactness of $K$. Again, it has been shown by examples in [2] that this result cannot be improved.

So, in a certain sense, the classical characterization and our characterization of weak compactness are complementary to each other.
(b) If we allow the Banach space $E$ to be arbitrary, the extended classical characterization states that uniform $\sigma$-additivity and condition (*) are necessary and sufficient for conditional compactness of the set $K \subseteq L^1_\mu(\mu)$ in the weaker topology $\sigma' = (L^1_\mu, L^\infty_\mu)$. Similarly, our result gives a characterization of conditional $\sigma'$-compactness of $K$, in terms of uniform $\sigma'$-convergence of operators (Theorem 6). So, for an arbitrary Banach space, both characterizations regain their full symmetry if we replace the weak topology by the $\sigma'$-topology. We mention that, in case $E'$ has the RNP, the weak topology and the $\sigma'$-topology coincide.

(c) Coming back to the space $L^1$, there is no "strong analog" of the classical characterization of weak compactness, that is, there is no characterization of strong compactness in terms of uniform $\sigma$-additivity.

In contrast, there is a characterization of strong compactness in any Banach space, in terms of uniform strong convergence of operators, namely the Phillips' lemma (see [13, IV.5.4]). Our characterization of weak compactness appears as the "weak analog" in $L^1(\mu)$ of Phillips' lemma. This shows that uniform convergence of operators is a very general and powerful tool, since it can be used to characterize compactness both in the strong topology and in the weak topology.

Theorem 6 of this paper unifies and greatly extends previous results in [2, 9, 4] where weak compactness was characterized in terms of uniform weak convergence of conditional expectations, respectively, of convolution operators or translation operators in case $X$ is a locally compact group and $\mu$ is a Haar measure. This latter case is the "weak analog" of the strong compactness criterion of Kolmogorov [17], Tamarkin [20], and Marcel Riesz [19].

Finally, we remark that Theorem 6 is stated for sequences of operators. This result is further improved for nets of operators, provided $E'$ has the RNP (Theorem 12).

An open question is whether the separability condition on $E'$ can be removed in Theorem 12.

In a forthcoming paper [22] we show that the answer is positive, in case the net of operators $(T_\phi)$ on $L^1_\mu(\mu)$ are "extensions" of operators $(S_\phi)$ on $L^1(\mu)$, in the sense that $T_\phi(x) = S_\phi(x)$ for $\phi \in L^1(\mu)$ and $x \in E$. In particular, this is the case for conditional expectations, for convolutions and for translations.

We shall use the following notations and terminology:

A subset $K$ of a topological vector space $H$, endowed with a topology $\tau$, is said to be relatively $\tau$-compact, if its closure in $\tau$ is $\tau$-compact; $K$ is said to be conditionally $\tau$-compact, if every sequence from $K$ contains a Cauchy subsequence for the topology $\tau$.

We shall denote by $(X, \Sigma, \mu)$ a measure space and by $E$ a Banach space
with dual $E'$. The norm of an element $z$ from $E$ or $E'$ will be denoted by $|z|$. We denote by $\Sigma$ the $\delta$-ring of sets $A \in \Sigma$ with $\mu(A) < \infty$.

For $1 \leq p < \infty$ and $1 < q \leq \infty$ such that $(1/p) + (1/q) = 1$, we shall denote by $\sigma'$ the topology $\sigma(L^p(E), L^q(E))$ on $L^p(E)$. If $E'$ has the RNP, and, in case $p = 1$, if $(X, \Sigma, \mu)$ is strictly localizable, then the topology $\sigma'$ coincides with the weak topology on $L^p(E)$.

2. CHARACTERIZATION OF WEAK AND $\sigma'$-COMPACTNESS IN TERMS OF UNIFORM $\sigma$-ADDITIVITY

In this section, which has an introductory character, we give characterizations of conditionally $\sigma'$-compact and relatively $\sigma'$-compact sets in the space $L^p(E)$, $1 \leq p < \infty$, which, in case $E$ is the scalar field, reduce to the classical characterization of relatively weakly compact sets in $L^p$. They are slight improvements of similar known theorems in which the RNP was imposed on $E$ and $E'$, and are given here only for the sake of completeness and for comparison with our characterization given in Section 3.

**Theorem 1.** A set $K \subset L^p(E)$ is conditionally $\sigma'$-compact, if and only if:

1. $K$ is bounded.
2. The set $K(A) = \{ \int_A f d\mu; f \in K \}$ is conditionally weakly compact in $E$, for each $A \in \Sigma$;
3. the set $|K| = \{ \int |f| d\mu; f \in K \}$ is uniformly $\sigma$-additive.

A set $K \subset L^p(E)$, $1 < p < \infty$, is conditionally $\sigma'$-compact, if and only if it satisfies conditions (1) and (2).

**Proof.** The fact that conditions (1), (2), and (3)—respectively conditions (1) and (2)—imply that $K$ is conditionally $\sigma'$-compact in $L^p(E)$—respectively in $L^p(E)$ with $1 < p < \infty$—can be found, for example, in steps A and B of the proof of Theorem 1 in [4]. Assume now $K \subset L^p(E)$, $1 \leq p < \infty$, is conditionally $\sigma'$-compact. If $K$ is not bounded, there is a sequence $(f_n)$ from $K$ such that $\|f_n\|_p \to \infty$. Extracting a subsequence, if necessary, we can assume that $(f_n)$ is a Cauchy sequence for the $\sigma'$-topology.

Then for every $g \in L^q$, $1/p + 1/q = 1$, the sequence $\{ \int \langle f_n, g \rangle d\mu \}$ is bounded. By the Banach–Steinhaus theorem, $(f_n)$ is bounded and we reached a contradiction. This proves (1).

Condition 2 follows from the continuity of the mapping $f \to \int_A f d\mu$ from $L^p(E)$ endowed with the $\sigma'$-topology, into $E$ endowed with the weak topology.

Assume now $K \subset L^p(E)$ is conditionally $\sigma'$-compact and prove con-
dition (3). For every $g \in L^\infty_E(\mu)$, the set $\langle K, g \rangle = \{ \langle f, g \rangle : f \in K \}$ is relatively weakly compact in $L^1(\mu)$, therefore $\langle K, g \rangle$ is uniformly $\sigma$-additive. From Lemma 1a in [7] we deduce that $|K|$ is uniformly $\sigma$-additive (see also step H in the proof of the theorem 1 in [4]).

**Remark 1.** A slightly different proof of this theorem is given in (14, Theorem 4.9).

**Remark 2.** If $E$ is reflexive, condition (2) is superfluous.

**Remark 3.** One of the implications in Theorem 1 is valid for conditional or relative compactness in either the weak or the $\sigma'$-topology. More precisely:

If $K \subseteq L^p_E(\mu), 1 \leq p < \infty$ is conditionally (resp. relatively) compact in either the weak or the $\sigma'$-topology, then

1. $K$ is bounded;
2. $K(A)$ is conditionally (resp. relatively) weakly compact in $E$, for every $A \in \Sigma_f$;
3. $|K|$ is uniformly $\sigma$-additive, in case $K \subseteq L^1_E(\mu)$.

**Remark 4.** If $E'$ has the RNP and, in case $p = 1$, if $\mu$ is strictly localizable, then Theorem 1 gives necessary and sufficient conditions for $K$ to be conditionally weakly compact.

**Remark 5.** If we impose the RNP on $E$, we obtain a characterization of relatively $\sigma'$-compact sets:

**Theorem 2.** Assume $E$ has the RNP. A set $K \subseteq L^1_E(\mu)$ is relatively $\sigma'$-compact if and only if:

1. $K$ is bounded;
2. $K(A)$ is relatively weakly compact in $E$, for every $A \in \Sigma_f$;
3. $|K|$ is uniformly $\sigma$-additive.

A set $K \subseteq L^p_E(\mu), 1 < p < \infty$, is relatively $\sigma'$-compact, if and only if it satisfies conditions (1) and (2).

In particular, if both $E$ and $E'$ have the RNP and, in case $p = 1$, if $\mu$ is strictly localizable, Theorem 2 gives a characterization of relatively weakly compact sets.

Theorem 2 follows immediately from the following general lemma, which enables us to deduce relative $\sigma'$-compactness from conditional $\sigma'$-compactness.

**Lemma 3.** A set $K \subseteq L^p_E(\mu), 1 < p < \infty$ is relatively $\sigma'$-compact, if and only if:
(a) $K$ is conditionally $\sigma'$-compact;

(b) $K(A)$ is relatively weakly compact in $E$, for every $A \in \Sigma_f$;

(c) any measure $m : \Sigma_f \to E$ of the form $m(A) = \lim \int_A f_n \, d\mu$ (weak limit in $E$) for every $A \in \Sigma_f$ and some sequence $f_n \in K$, has the RNP with respect to $\mu$, on each set of finite $m$-variation.

The proof is the same as that of Lemma 4 in [5] (where $E'$ was assumed to have the RNP). If $E$ has the RNP, then condition (c) is evidently satisfied, and we obtain theorem 2.

3. CHARACTERIZATION OF WEAK OR $\sigma'$-COMPACTNESS IN TERMS OF UNIFORM CONVERGENCE OF SEQUENCES OF OPERATIONS

This section is the main part of the paper. Phillips' lemma (see [13, IV.5.4]) gives necessary and sufficient conditions of relative strong compactness in an arbitrary Banach space, in terms of uniform strong convergence of operators. A natural question is whether or not we can characterize weak compactness in terms of weak uniform convergence of operators. For the spaces $L^p$ we show in Theorem 6 below that the answer is affirmative: weak compactness can be characterized by uniform weak convergence of “admissible” sequences of operators. This result is stated, in fact, for the spaces $L^p_E$ and the topology $\sigma'$. At the same time, the uniform convergence of operators is used to characterize the uniform $\sigma$-additivity of sets in $L^1_E$.

Particular cases of these results have been proved earlier in [4, Theorem 1] for admissible nets of conditional expectations, and in [9] for admissible nets of convolutions with an approximate unit, or of translation operators, on an $L^p_E$ space over a locally compact group endowed with the Haar measure.

Some properties are valid for operators between $L^p$-spaces over different measure spaces. For this reason, in the sequel we consider two measure spaces $(X, \Sigma, \mu)$ and $(X', \Sigma', \nu)$, two Banach spaces $E$ and $F$, a number $p \in [1, \infty)$ and linear operators from $L^p_E(\mu)$ into $L^p_F(\nu)$.

If $T : L^p_E(\mu) \to L^p_F(\nu)$ is a linear operator we denote

$$\|T\|_p = \sup \{ \|Tf\|_p : f \in L^1_E(\mu) \cap L^p_E(\mu), \|f\|_p \leq 1 \}$$

and

$$\|T\|_\infty = \sup \{ \|Tf\|_\infty : f \in L^1_E(\mu) \cap L^p_E(\mu), \|f\|_\infty \leq 1 \}.$$ 

If $T$ is linear and continuous and if $K \subset L^p_E(\mu)$ is conditionally $\sigma'$-compact, then $TK$ is conditionally $\sigma'$-compact in $L^p_F(\nu)$. 
It is also easy to check that if \( T: L^p_\mu \rightarrow L^p_\nu \) is linear and continuous and if the adjoint \( T^* \) satisfies \( T^* L^p_\mu(v) \subseteq L^p_\mu(v) \), then \( T \) is continuous for the \( \sigma' \) topologies on \( L^p_\mu(v) \) and \( L^p_\nu(v) \).

**Definition 4.** We say that a continuous linear operator \( T: L^p_\mu \rightarrow L^p_\nu \) has the finite measure property (FMP) on a class of sets \( R \subset \Sigma_f \), if for every set \( C \in R \) there is a set \( \phi(C, T) \in \Sigma_f \) such that, if \( f \in L^p_\mu \) vanishes \( \mu \)-a.e. outside \( C \), then \( Tf \) vanishes \( \nu \)-a.e. outside \( \phi(C, T) \).

Evidently, if \( \nu(X') < \infty \), then any operator \( T \) has the FMP on \( \Sigma_f \).

**Examples 1.** Let \( \pi \) be a finite family of disjoint sets from \( \Sigma_f \), and let \( E_\pi \) be the conditional expectation determined by \( \pi \). Then the operator \( E_\pi: L^p_\mu \rightarrow L^p_\mu \) has the FMP on \( \Sigma_f \); namely, if \( C \in \Sigma_f \), we can take \( \phi(C, E_\pi) \) to be the union of the sets \( A \in \pi \) with \( A \cap C \neq \emptyset \).

**Example 2.** Let \( G \) be a locally compact abelian group, \( \mu \) a Haar measure on \( G \), \( V \) a compact neighborhood of 0 in \( G \) and \( f \in L^1_\mu \cap L^\infty_\mu \) a function vanishing outside \( V \). Then the convolution operator \( T_\phi: L^p_\mu \rightarrow L^p_\mu \) defined by \( T_\phi f = f \ast \phi \) for \( f \in L^p_\mu \), has the FMP on the class of compact subsets of \( G \); namely, if \( C \subset G \) is compact, then we can take \( \phi(C, T_\phi) = C \cup V \).

**Example 3.** Let \( G \) and \( \mu \) be as in Example 2, and let \( h \in G \). The translation operator \( T_h: L^p_\mu \rightarrow L^p_\mu \) defined by \( (T_h f)(x) = f(x + h) \) for \( f \in L^p_\mu \) and \( x \in G \), has the FMP on the class \( B_f \) of Borel sets of finite measure; if \( C \subset B_f \), then we can take \( \phi(C, T_h) = C - h \).

**Definition 5.** Let \( I \) be a directed set, \( (T_x)_{x \in I} \) a net of continuous linear operators from \( L^p_\mu \) into \( L^p_\nu \), and \( T_\beta: L^p_\mu \rightarrow L^p_\nu \) a continuous linear operator.

We say that the net of operators \( (T_x) \) is adissible and has limit \( T_\beta \), if the following conditions are satisfied:

In case \( p = 1 \):

1. \( \sup_x \|T_x\|_1 < \infty \) and \( \sup_x \|T_x\|_\infty < \infty \);
2. for each \( x \in I \), the adjoint \( T^*_x \) maps \( L^1_\mu(v) \cap L^\infty_\mu(v) \) into \( L^1_\mu(v) \cap L^\infty_\mu(v) \) and for every \( g \in L^1_\mu(v) \cap L^\infty_\mu(v) \) we have \( \lim_x T^*_x g = T^*_\beta g \), strongly in \( L^1_\mu(v) \);
3. There is a class \( R \subset \Sigma_f \), generating \( \Sigma_f \), such that \( T \) and \( T_x \) have the FMP on \( R \), and such that, if for each \( C \in R \) we denote \( C_x = \phi(C, T_x) \), then \( \lim_x \phi_{C_x} = \phi_{C_\beta} \) strongly in \( L^1(v) \).

In case \( 1 < p < \infty \):
(1') \( \sup_x \|T_n\|_p < \infty \); 

(2') for each \( x \) we have \( T_x^* L_p^\mu(v) \subseteq L_p^\mu(\mu) \) and for every \( g \in L_p^\mu(v) \) we have \( \lim_x T_x^* g = T_\mu^* g \) strongly in \( L_p^\mu(\mu) \).

Remark. If \( v(X') < \infty \), then condition (3) is superfluous, taking \( \phi(C, T_x) = X' \).

Example 1. Let \( (\Sigma_x) \) be a directed family of sub-\( \delta \)-rings of \( \Sigma_x \) and \( \Sigma_\mu = \bigvee_a \Sigma_a \). Let \( E_a \) and \( E_\mu \) be the conditional expectations corresponding to \( \Sigma_x \) and \( \Sigma_\mu \), respectively, [11]. Then the net of operators \( (E_a) \) from \( L^\mu_p(\mu) \) into \( L^\mu_p(\mu) \) is admissible with limit \( E_\mu \), for \( 1 < p < \infty \).

For \( p = 1 \), the net \( (E_a) \) is admissible and has limit \( E_\mu \), if and only if condition (3) is satisfied; in particular, condition (3) is satisfied in case \( \mu(X) < \infty \), or in case each \( E_a \) is generated by a finite partition \( \pi_a \).

Example 2. Let \( G \) be a locally compact abelian group, \( \mu \) a Haar measure on \( G \) and \( (u_\nu) \) an approximate unit, where \( \nu \) runs over a base of relatively compact neighborhoods of \( 0 \) in \( G \), ordered downwards by inclusion.

For each \( \nu \) let \( T_\nu \) be the convolution operator from \( L^\mu_p(\mu) \) into itself, defined by \( T_\nu f = f * u_\nu \) for \( f \in L^\mu_p(\mu) \). Then the family \( (T_\nu) \) of operators is admissible and has limit \( I \), the identity operator in \( L^\mu_p(\mu) \) (see [8]).

Example 3. Let \( G \) and \( \mu \) be as in Example 2, and let \( (h_\nu) \) be a net of elements of \( G \) converging to \( 0 \) in \( G \). For each \( \nu \) let \( T^\nu \) be the translation operator from \( L^\mu_p(\mu) \) into itself, defined by \( (T^\nu f)(x) = f(x + h_\nu) \) for \( f \in L^\mu_p(\mu) \) and \( x \in G \). Then the net \( (T^\nu) \) of operators is admissible and has limit the identity operator \( I \) (see [8]).

The main result of this paper is the following characterization of conditionally \( \sigma' \)-compact sets in \( L^\mu_p(\mu) \) and of uniformly \( \sigma \)-additive sets in \( L^\mu_p(\mu) \), by means of uniform convergence in the \( \sigma' \)-topology of an admissible sequence of operators. This is the "weak analog" in the spaces \( L^\mu_p(\mu) \), of the Phillips lemma.

Theorem 6. Let \( (T_n) \) be an admissible sequence of continuous linear operators from \( L^\mu_p(\mu) \) into itself, with limit \( I \), the identity operator on \( L^\mu_p(\mu) \).

(A) A set \( K \subseteq L^\mu_p(\mu) \) is conditionally \( \sigma' \)-compact iff each \( T_n K \) is conditionally \( \sigma' \)-compact, and

\[
\lim_n T_n f = f
\]

in \( L^\mu_p(\mu) \), for the \( \sigma' \)-topology, uniformly for \( f \in K \).
(A') A set $K \subset L^p_\mu$ is conditionally weakly compact if each $T_n K$ is conditionally weakly compact and

$$\lim_{n} T_n f = f$$

weakly in $L^p_\mu$ uniformly for $f \in K$.

(B) Let $p = 1$ and $K \subset L^1_\mu$ be bounded. The set $|K| = \{|f|; f \in K\}$ is uniformly $\sigma$-additive, iff each $|T_n K|$ is uniformly $\sigma$-additive, and

$$\lim_{n} T_n f = f$$

in $L^1_\mu$, for the $\sigma'$-topology, uniformly for $f \in K$.

Remark. The theorem remains valid for a net $(T_\alpha)$, having a cofinal subsequence $(T_{\alpha_n})$.

In case there is no cofinal sequence, the above theorem can still be stated, under additional conditions, for subsequences $(T_{\alpha_n})$ (see Theorem 12).

The proof of the theorem will be done in several steps which will be stated as separate lemmas, having their own interest. The first implication in part (B) of the Theorem 6 follows from

**Lemma 7.** Let $(T_\alpha)$ be an admissible net of operators from $L^1_\mu$ into $L^1_\nu$, with limit $T_\beta$, and let $K \subset L^1_\mu$ be a bounded set, such that $|K|$ is uniformly $\sigma$-additive. (In particular the assumption on $K$ is satisfied if $K$ is conditionally $\sigma'$-compact).

Then each $|T_\alpha K|$ is uniformly $\sigma$-additive, and

$$\lim_{\alpha} T_\alpha f = T_\beta f$$

in $L^1_\nu$, for the $\sigma'$-topology, uniformly for $f \in K$.

**Proof.** Let $R$ be the class of sets mentioned in condition (3) of Definition 5 of admissible nets.

Let $\epsilon > 0$. There is a set $C \in R$ such that

$$\int_{\chi \cap C} |f| \, d\mu < \epsilon \quad \text{for all } f \in K.$$

For each $\alpha$, let $C_\alpha = \phi(T_\alpha, C) \in \Sigma'_f$ be the set corresponding to $C$ and $T_\alpha$ in Definition 5.

Since $K\phi_C$ is uniformly integrable, there exists $\lambda > 0$ such that

$$\int_{C \cap \{|f| > \lambda\}} |f| \, d\mu < \epsilon \quad \text{for all } f \in K.$$
Let $A \in \Sigma'$ and $g \in L^p_v(\nu)$ and let $T$ denote $T_\beta$ or 0. For each $x$ we have

$$\int_A \langle T_x f - T f, g \rangle \, dv$$

$$= \int_A \langle T_x (f\phi_{X\cap C}) - T(f\phi_{X\cap C}), g \rangle \, dv$$

$$+ \int_A \langle T_x (f\phi_C) - T(f\phi_C), g \rangle \, dv. \quad (1)$$

For the first term of the sum we have

$$\left| \int_A \langle T_x (f\phi_{X\cap C}) - T(f\phi_{X\cap C}), g \rangle \, dv \right|$$

$$\leq (M_1 + \|T\|) \|g\| \|f\phi_{X\cap C}\|_1 \leq (M_1 + \|T\|) \|g\| \varepsilon,$$  \quad (2)

where $M_1 = \sup_x \|T_x\|_1$. For the second term we have

$$\int_A \langle T_x (f\phi_C) - T(f\phi_C), g \rangle \, dv$$

$$= \int_A \langle f\phi_C, T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta}) \rangle \, d\mu$$

$$= \int \mathbb{I}_{|f| > \lambda} \langle f\phi_C, T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta}) \rangle \, d\mu$$

$$+ \int \mathbb{I}_{|f| \leq \lambda} \langle f\phi_C, T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta}) \rangle \, d\mu. \quad (3)$$

We have further

$$\left| \int \mathbb{I}_{|f| > \lambda} \langle f\phi_C, T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta}) \rangle \, d\mu \right|$$

$$\leq (\|T_x^*\|_\infty + \|T^*\|_\infty) \|g\| \int \mathbb{I}_{|f| > \lambda} |f| \, d\mu$$

$$\leq (\|T_x\|_1 + \|T\|_1) \|g\| \varepsilon \leq (M_1 + \|T\|_1) \|g\| \varepsilon. \quad (4)$$

Also we have

$$D = \left| \int \mathbb{I}_{|f| \leq \lambda} \langle f\phi_C, T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta}) \rangle \, d\mu \right|$$

$$\leq \lambda \|T_x^*(g\phi_{A\cap C_x}) - T^*(g\phi_{A\cap C_\beta})\|_1. \quad (5)$$
Now we take $T = T_\beta$ and $A = X'$ and we get

$$D \leq \lambda \left\| T_\beta^*(g\phi_{C_\beta}) - T_{\beta}^*(g\phi_{C_\beta}) \right\|_1 + \lambda M_\infty \left\| g \right\|_\infty v(C_\beta A C_\beta),$$

where $M_\infty = \sup \| g \|$. Let $\alpha_\varepsilon$ be such that, by conditions (2) and (3) of Definition 5, we have for $\alpha \geq \alpha_\varepsilon$,

$$\| T_\beta^*(g\phi_{C_\beta}) - T_{\beta}^*(g\phi_{C_\beta}) \|_1 < \varepsilon / \lambda \quad \text{and} \quad v(C_\beta A C_\beta) < \varepsilon / \lambda.$$

Then from (5) we deduce for $\alpha \geq \alpha_\varepsilon$,

$$\left| \int_{\{ |f| \leq \lambda \}} \left\langle f\phi_{C}, T_\beta^*(g\phi_{C_\beta}) - T_{\beta}^*(g\phi_{C_\beta}) \right\rangle \, d\mu \right| < \epsilon + M_\infty \left\| g \right\|_\infty \epsilon. \quad (6)$$

Using (4) and (6), we obtain from (3),

$$\left| \int \left\langle T_\beta(f\phi_{C}) - T_{\beta}(f\phi_{C}), g \right\rangle \, dv \right| \leq (M_1 \left\| g \right\|_\infty + \left\| T_{\beta} \right\|_1 \left\| g \right\|_\infty + 1 + M_\infty \left\| g \right\|_\infty) \varepsilon, \quad (7)$$

and from (1) and (2) we deduce finally

$$\left| \int \left\langle T_\beta f, g \right\rangle \, dv - \int \left\langle T_{\beta} f, g \right\rangle \, dv \right| \leq (2M_1 \left\| g \right\|_\infty + 2 \left\| T_{\beta} \right\|_1 \left\| g \right\|_\infty + 1 + M_\infty \left\| g \right\|_\infty) \varepsilon$$

for all $\alpha \geq \alpha_\varepsilon$ and all $f \in K$. This means that

$$\lim_\alpha \int \left\langle T_\beta f, g \right\rangle \, dv = \int \left\langle T_{\beta} f, g \right\rangle \, dv$$

uniformly for $f \in K$. Since $g \in L_\infty(v)$ was arbitrary, this proves the last assertion of the lemma.

We take now $T = 0$ in the above computations. Let $\alpha$ be arbitrary and prove that $|T_\beta K|$ is uniformly $\sigma$-additive. Let $(A_n)$ be a decreasing sequence from $\Sigma'$ with empty intersection. Since $v(C_\beta) < \infty$, we have $\lim_n v(C_\beta \cap A_n) = 0$.

There is then $n_\varepsilon$ such that for any $n \geq n_\varepsilon$ and for $A = A_n$ we have $v(A \cap C_\beta) = \varepsilon / \lambda$.

From (5) we deduce that

$$D \leq \lambda \left\| T_\beta^*(g\phi_{A \cap C_\beta}) \right\|_1 \leq \lambda \left\| T_\beta^* \right\|_1 \left\| g\phi_{A \cap C_\beta} \right\|_1$$

$$\leq \lambda \left\| T_\beta \right\|_\infty \left\| g \right\|_\infty v(A \cap C_\beta) \leq \left\| T_\beta \right\|_\infty \left\| g \right\|_\infty \varepsilon. \quad (6')$$
Using (4) and (6') we obtain from (3),
\[ \left| \int_A \langle T_x(f\phi_C), g \rangle \, dv \right| \leq (\| T_x \|_1 + \| T_x \|_\infty) \| g \|_\infty \varepsilon; \] (7')
and from (1) and (2) we deduce
\[ \left| \int_A \langle T_x f, g \rangle \, dv \right| \leq (2 \| T_x \|_1 + \| T_x \|_\infty) \| g \|_\infty \varepsilon, \]
if \( A = A_n \) and \( n \geq n_\varepsilon \) for all \( f \in K \).

This means that the set \( \langle T_x K, g \rangle = \{ \langle T_x f, g \rangle ; f \in K \} \) is uniformly \( \sigma \)-additive. Since \( g \) was arbitrary in \( L^\infty_F(v) \), from Lemma 1a in [7] we deduce that \( |T_x K| \) is also uniformly \( \sigma \)-additive and this proves the lemma.

Taking \( T_x = T \) for all \( x \) we get the following

**Proposition 8.** Let \( T : L^1_L(\mu) \to L^1_L(v) \) be a continuous linear operation satisfying the following three conditions:

1. \( \| T \|_1 < \infty \) and \( \| T \|_\infty < \infty \).
2. The adjoint \( T^* \) of \( T \) maps \( L^1_F(v) \cap L^\infty_F(v) \) into \( L^1_L(\mu) \cap L^\infty_L(\mu) \).
3. There exists a ring \( R \subset \Sigma \) generating \( \Sigma \) such that \( T \) has the FMP on \( R \). (This condition is superfluous if \( v(X') < \infty \)).

If a set \( K \subset L^1_L(\mu) \) is bounded and if \( |K| \) is uniformly \( \sigma \)-additive, then \( |TK| \) is also bounded and uniformly \( \sigma \)-additive.

The second implication in part (B) of Theorem 6 follows from Lemma 1b in [7].

The first implication in part (A) of Theorem 6 follows from the above Lemma 7 for \( p = 1 \), and from the following lemma for \( 1 < p < \infty \):

**Lemma 9.** Let \( 1 < p < \infty \) and \( (T_x) \) be an admissible net of operators from \( L^p_L(\mu) \) into \( L^p_L(v) \), with limit \( T_\beta \).

If \( K \subset L^p_L(\mu) \) is a bounded set (in particular if \( K \) is conditionally \( \sigma' \)-compact), then
\[ \lim_{x} T_x f = T_\beta f \]
in \( L^p_L(v) \) for the \( \sigma' \)-topology, uniformly for \( f \in K \).
Proof. Let \( g \in L_p^\infty(v) \). Then for every \( f \in K \) we have

\[
\left| \int \langle T_\alpha f, g \rangle dv - \int \langle T_\beta f, g \rangle dv \right| = \int \left| \langle f, T_\alpha^* g - T_\beta^* g \rangle \right| d\mu = M_\rho \| T_\alpha^* g - T_\beta^* \|_q,
\]

where \( M_\rho = \sup \{ \| f \|_\rho ; f \in K \} \). We use then condition (2') of Definition 5.

The second implication in part (A) and part (A') of Theorem 6 follow directly from the following

**Lemma 10.** Let \( M, N \) be two vector spaces in duality, \( K \) a set and \( f_n, f : K \rightarrow M \) functions, with \( n \in N \). If

1. for each \( n \), the range \( f_n(K) \) is conditionally \( \sigma(M, N) \)-compact;
2. \( \lim_n f_n(x) = f(x) \) in \( M \), for the \( \sigma(M, N) \) topology, uniformly for \( x \in K \);

then the range \( f(K) \) is also conditionally \( \sigma(M, N) \) compact.

**Proof.** Let \( (x_k) \) be a sequence from \( K \). Since each set \( f_n(K) \) is conditionally \( \sigma(M, N) \)-compact, by a diagonal process we can extract a subsequence \( (y_k) \) of \( (x_k) \) such that, for each \( n \), the sequence \( (f_n(y_k))_k \) is \( \sigma(M, N) \)-Cauchy. Then \( \lim_k \langle f_n(y_k), z \rangle \) exists for each \( z \in N \). From hypothesis (2) we deduce that \( \lim_k \langle f(y_k), z \rangle \) exists for each \( z \in N \), hence \( (f(y_k))_k \) is \( \sigma(M, N) \)-Cauchy.

**Remark.** If we assume that all ranges \( f_n(K) \) are relatively \( \sigma(M, N) \)-compact, it does not follow, necessarily, that \( f(K) \) is relatively \( \sigma(M, N) \) compact (see [4]).

However, in the particular case where \( M = L_p^\infty(v) \) and \( N = L_p^\infty(v) \), using Lemma 3, under additional assumptions, we can ensure relative compactness in Lemma 10 and Theorem 6.

**Theorem 11.** Let \( (T_n) \) be an admissible sequence of operators from \( L_p^\infty(\mu) \) into itself, with limit I. Assume \( E \) has the RNP.

A set \( K \subset L_p^\infty(\mu) \) is relatively \( \sigma' \)-compact if and only if

1. each \( T_n K \) is relatively \( \sigma' \)-compact;
2. \( \lim_n T_n f = f \) in \( L_p^\infty(\mu) \) for the \( \sigma' \) topology, uniformly for \( f \in K \);
3. for each set \( A \in \Sigma_f \), the set \( K(A) = \{ \int_A f \, d\mu ; f \in K \} \) is relatively weakly compact in \( E \).

**Remark.** If \( (X, \Sigma, \mu) \) is a separable measure space, the net \( (\pi) \) of all finite partitions over a countable ring generating \( \Sigma \) has a cofinal sequence
Then the sequence \((E_{n_k})\) of conditional expectations is an admissible sequence of operators on \(L^p_x(\mu)\), with limit \(I\).

Then we can apply Theorems 6 and 11 and obtain characterizations of conditional and relative \(\sigma\)-compactness in \(L^p_x(\mu)\), in terms of uniform convergence of \((E_{n_k})\) in the \(\sigma\)-topology.

But if \((X, \Sigma, \mu)\) is not separable, the net \((E_{x})\) of conditional expectations generated by finite partitions is admissible, with limit \(I\), but does not have a cofinal sequence.

Similar considerations can be made for convolutions with an approximate unit \((u_{i_n})\) and for translation operators, in an abelian topological group \(G\), having a countable base \((V_n)\) of neighborhoods of \(0\). But if \(G\) does not have any countable base of neighborhoods of \(0\), Theorems 6 and 11 cannot be applied.

In these cases we can apply the results of the next section.

4. CHARACTERIZATION OF WEAK COMPACTNESS IN TERMS OF UNIFORM CONVERGENCE OF NETS OF OPERATORS

In this section we extend Theorems 6 and 11 for admissible nets of operators, under a certain restriction on \(E\). In particular, Theorem 12 is valid for \(E = R\) or \(E = C\).

**Theorem 12.** Let \((T_{x})\) be an admissible net of continuous linear operators from \(L^p_x(\mu)\) into itself, with limit \(I\), the identity operator on \(L^p_x(\mu)\). Assume every separable subspace of \(E\) has a separable dual (i.e., \(E'\) has the RNP).

\[ T_{x_0} f = f \]

in \(L^p_x(\mu)\), for the weak topology, uniformly for \(f \in K_0\).

\[ T_{x_0} f = f \]

weakly in \(L^p_x(\mu)\), uniformly for \(f \in K_0\).

**Proof.** Assume first the conditions of the theorem satisfied and let \(K_0 \subset K\) be separable. We apply then Theorem 6 to deduce that \(K_0\) is con-
conditionally $\sigma'$-compact, respectively that $|K_0|$ is uniformly $\sigma$-additive. There exists a separable subspace $E_0 \subset E$ and a countably generated $\sigma$-algebra $\Sigma_0 \subset \Sigma$ such that $K_0 \subset L^p_{E_0}(\Sigma_0, \mu)$. By hypothesis, $E'_0$ is separable, therefore $E'_0$ has the RNP. It follows that on $L^p_{E_0}(\Sigma_0, \mu)$ the $\sigma'$-topology is equivalent with the weak topology; hence $K_0$ is conditionally weakly compact. The conclusion follows from the fact that $K$ is conditionally weakly compact (or conditionally $\sigma'$-compact) iff every separable subset of $K$ has the same property; similarly, $|K|$ is uniformly $\sigma$-additive iff for every separable subset $K_0$ of $K$, the set $|K|$ is uniformly $\sigma$-additive.

Conversely, let $K \subset L^p_E$ be conditionally weakly compact, or, in case $p = 1$, such that $|K|$ is uniformly $\sigma$-additive. Let $K_0$ be a separable subset of $K$.

We can find a separable subspace $E_0$, a $\sigma$-finite set $X_0 \in \Sigma$ and a countably generated $\sigma$-algebra $\Sigma_0$ of $X_0$ such that $K_0 \subset L^p_{E_0}(\Sigma_0, \mu)$ and all the functions of $K_0$ vanish $\mu$-a.e. outside $X_0$. The conclusion follows then from Theorem 6 and from the following lemma (choosing $\Sigma_0$ and $F_0$ arbitrarily).

**Lemma 13.** Let $(T_x)$ be an admissible net of continuous linear operators from $L^p_{E}(X, \Sigma, \mu)$ into $L^p_{x'}(\Sigma', \nu)$ with limit $T_\beta$.
Assume that any separable subspace of $F$ has a separable dual.

For any countably generated $\sigma$-algebras $\Sigma_0 \subset \Sigma$ and $\Sigma'_0 \subset \Sigma'$, for any separable subspaces $E_0 \subset E$ and $F_0 \subset F$, there exist:

1. countably generated $\sigma$-algebras $\Sigma_\infty$, $\Sigma'_\infty$ satisfying
   \[ \Sigma_0 \subset \Sigma_\infty \subset \Sigma, \quad \Sigma'_0 \subset \Sigma'_\infty \subset \Sigma'; \]
2. separable subspaces $E_\infty$, $F_\infty$ satisfying
   \[ E_0 \subset E_\infty \subset E, \quad F_0 \subset F_\infty \subset F; \]
3. an increasing sequence $(\alpha_n)$ such that the restrictions of $T_{x_n}$ to $L^p_{E_E}(X, \Sigma_\infty, \mu)$ form an admissible sequence of operators from $L^p_{F_E}(X, \Sigma'_\infty, \mu)$ into $L^p_{F_F}(X', \Sigma'_\infty, \nu)$, with limit $T_\beta$ restricted to $L^p_{F_E}(X, \Sigma_\infty, \mu)$.

**Proof.** The proof is done in several steps:

(a) Let $R$ be a countable ring of sets of finite measure generating $\Sigma_0$. For each set $C \in R$, the limit $\lim_n \phi_{C_n} = \phi_{C_0}$ exists strongly in $L^1(\mu)$, according to condition (3) in Definition 5, where $C_0 = \phi(T_x, C)$. By a diagonal process, we can find a sequence $(\alpha_{0,n})_{n \geq 0}$ with $\alpha_{0,0} = \beta$, such that for any sequence $x_n \geq x_{0,n}$ the limit $\lim_x \phi_{C_{x_n}} = \phi_{C_0}$ exists strongly in $L^1(\mu)$.

(b) The set \{ $T_{x_n} L^1_{E_0}(\Sigma_0, \mu); n \geq 0$ \} is separable in $L^1(\Sigma', \nu)$. Enlarging $\Sigma'_0$ and $F_0$, if necessary, we can assume that

\[ T_{x_0} L^1_{E_0}(\Sigma_0, \mu) \subset L^1_{F_0}(\Sigma'_0, \nu) \quad \text{for} \quad n \geq 0. \]
(c) Then $L^1_{\mathcal{F}_0}(\Sigma'_0, \nu) \cap L^\infty_{\mathcal{F}_0}(\Sigma'_0, \nu)$ is separable for the $L^1_{\mathcal{F}}(\nu)$ norm. There exists a sequence $(\alpha_{1,n})_{n \geq 0}$ with $\alpha_{1,0} = \beta$, such that for any sequence $\alpha_n \geq \alpha_{1,n}$ we have

$$\lim_{n} T^\star_{\alpha_n} g \to T^\star_{\beta} g \quad \text{strongly in } L^1_{\mathcal{F}}(\mu).$$

for each $g \in L^1_{\mathcal{F}_0}(\Sigma'_0, \nu) \cap L^\infty_{\mathcal{F}_0}(\Sigma'_0, \nu)$.

(d) The set $\{ T^\star_{\alpha_n} L^1_{\mathcal{F}_0}(\Sigma'_0, \nu) \cap L^\infty_{\mathcal{F}_0}(\Sigma'_0, \nu); i = 0, 1, n \geq 0 \}$ is separable in $L^1_{\mathcal{F}}(\Sigma, \mu)$. There exist a countably generated $\sigma$-algebra $\Sigma_1$ with $\Sigma_0 \subset \Sigma_1 \subset \Sigma$, and a separable subspace $E_1 \subset E$ with $E_0 \subset E_1$, such that

$$T^\star_{\alpha_n} L^1_{\mathcal{F}_0}(\Sigma'_0, \nu) \cap L^\infty_{\mathcal{F}_0}(\Sigma'_0, \nu) \subset L^1_{E_1}(\Sigma_1, \mu).$$

(e) By induction, we can find: an increasing sequence $(E_n)_{n \geq 0}$ of separable subspaces of $E$; an increasing sequence $(F_n)_{n \geq 0}$ of separable subspaces of $F$; an increasing sequence $(\Sigma'_n)_{n \geq 0}$ of countably generated sub $\sigma$-algebras of $\Sigma$; an increasing sequence $(\Sigma'_n)_{n \geq 0}$ of countably generated sub $\sigma$-algebras of $\Sigma'$; for each $k \geq 0$, an increasing sequence $(\alpha_{k,n})_{n \geq 0}$ with $\alpha_{k,0} = \beta$ and $\alpha_{k,n} \leq \alpha_{k+1,n}$, such that

$$T^\star_{\alpha_n} L^1_{E_k}(\Sigma'_k, \mu) \subset L^1_{F_k}(\Sigma'_k, \nu) \quad \text{for } 0 \leq i \leq k, \ n \geq 0;$$

$$\lim_{n} T^\star_{\alpha_{i+1,n}} g = T^\star_{\beta} g \quad \text{strongly in } L^1_{E_k}(\mu)$$

for each $g \in L^1_{E_k}(\Sigma'_k, \nu) \cap L^\infty_{E_k}(\Sigma'_k, \nu)$;

$$T^\star_{\alpha_n} L^1_{F_k}(\Sigma'_k, \nu) \cap L^\infty_{F_k}(\Sigma'_k, \nu) \subset L^1_{E_{k+1}}(\Sigma_{k+1}, \mu)$$

for $0 \leq i \leq k + 1, \ n \geq 0$.

(f) Let $E_{\infty}$ be the separable subspace of $E$ generated by all $E_n$, $F_{\infty}$ the separable subspace of $F$ generated by all $F_n$, $\Sigma_{\infty} = \bigvee_n \Sigma_n$ and $\Sigma'_n = \bigvee_n \Sigma'_n$ and let $\alpha_n = \alpha_{n,n}$. Then $(T_{\alpha_n})$ is an admissible sequence from $L^1_{E_{\infty}}(\Sigma_{\infty}, \mu)$ into $L^1_{F_k}(\Sigma'_k, \nu)$ with limit $T^\star_{\beta}$.

(g) If $1 < p < \infty$ we start with $E_0$ and $\Sigma_0$ and chose $(\alpha_{0,n})_{n \geq 0}$ with $\alpha_{0,0} = \beta$ and $\alpha_{0,n}$ arbitrary for $n \geq 1$. Then we continue steps (a)–(e) to find sequences $(E_n)$, $(F_n)$, $(\Sigma_n)$, $(\Sigma'_n)$, and $(\alpha_{k,n})$ as above such that

$$T^\star_{\alpha_n} L^p_{E_k}(\Sigma'_k, \mu) \subset L^p_{F_k}(\Sigma'_k, \nu) \quad \text{for } 0 \leq i \leq k, \ n \geq 0;$$

$$\lim_{n} T^\star_{\alpha_{i+1,n}} g = T^\star_{\beta} g \quad \text{strongly in } L^p_{E_k}(\mu),$$

for each $g \in L^p_{E_k}(\Sigma'_k, \nu)$, and

$$T^\star_{\alpha_n} L^q_{F_k}(\Sigma'_k, \nu) \subset L^q_{E_{k+1}}(\Sigma_{k+1}, \mu)$$

for $0 \leq i \leq k + 1, \ n \geq 0$. We continue then the proof as in step (f).
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