Gamma ray transport in the cardiac region: an inverse problem

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Abstract

We study an inverse problem for gamma ray transport in the cardiac region $V = V_1 \cup V_2 \cup V_3 \subset \mathbb{R}^3$. The inner region $V_1$ contains a known gamma ray source, $V_2$ models the heart walls, whereas $V_3$ is the region external to the heart and is where the photon far field is measured. Given a finite number of values of the photon far field detected within $V_3$, we show that the thickness $h$ of the heart walls can be evaluated. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

In a recent paper [1], a gamma ray transport problem was studied in the (time dependent) cardiac region. The quasi-static solution was shown to be a very good approximation to the exact solution, because the speed of the heart walls is obviously much smaller than the speed of $\gamma$-photons. In [1] the direct problem was studied: there the distribution of radio pharmaceutical and target characteristics were known and the aim was to predict the far field both when the target was stationary and also when it was moving. However, of prime practical interest is

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the much more difficult inverse problem in which the target characteristics have
to be determined from the far field data. Here we investigate this problem with
particular attention being paid to the requirements of radionuclide imaging in
cardiology. We remark that procedures, similar to those considered in Sections
2–6, can be used to study inverse problems in astrophysics (where the cross sec-
tions within interstellar clouds must be evaluated, starting from the knowledge
of UV-photon densities detected by terrestrial astronomers [2]), in meteorology,
in glass manufacturing and in the modelling of radioactive properties of porous
insulating materials [3].

We also note that, in the particle transport literature, inverse problems usually
deal with the evaluation of some physical property of the host medium, starting
from the knowledge of the emergent particle distribution; see, for instance, [4–10]
(we only quote some of the most recent papers). An exception is the recent
article [11], where the entering distribution is determined from the knowledge
of the emerging one.

In the present paper, we show that a geometrical property of the host medium
(the cardiac region) can be evaluated starting from a measurement of the emergent
γ-photon density (far field), and this is done by using rather simple mathematical
tools.

2. The mathematical model

As in [1], we shall assume that the cardiac region can be represented by the
bounded medium \( V = V_1 \cup V_2 \cup V_3 \subset \mathbb{R}^3 \), shown in Fig. 1. The inner region \( V_1 \) is
bounded by the surface \( \Sigma_0 \) and contains the \( \gamma \)-ray source. The region \( V_2 \) models
the heart walls and is bounded by \( \Sigma_0 \) and by the surface \( \Sigma_h \), which is homothetic
to \( \Sigma_0 \). The region \( V_3 \), bounded by \( \Sigma_h \) and by the “external” surface \( \Sigma \), is where
the \( \gamma \)-photon far field is detected.

The regular closed surfaces \( \Sigma_0 , \Sigma_h \) and \( \Sigma \) are defined by

\[
\begin{align*}
\Sigma_0 : & \quad \varphi \left( \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right) = 0, \\
\Sigma_h : & \quad \varphi \left( \frac{x}{a+h}, \frac{y}{a+h}, \frac{z}{a+h} \right) = 0, \\
\Sigma : & \quad \psi(x, y, z) = 0,
\end{align*}
\]

(1)

where \( \varphi(p_1, p_2, p_3) \) and \( \psi(x, y, z) \) are given. We also assume that the bounded
and closed region \( V \subset \mathbb{R}^3 \) is convex.

Remark 1. In (1), \( a = a(t) \) is a given positive function of time and \( |\dot{a}| \) repre-
sents the speed of the heart walls, whereas \( h \) is proportional to the thickness of
the walls [1]. Since we use the quasi-static approximation to the photon transport
equation, we write \( a \) instead of \( a(t) \) because \( t \) is a parameter. In other words, at
each \( t \geq 0 \), the time dependent region \( V \) is seen as if were frozen in its instanta-
neous configuration and the photon density is evaluated by using a stationary-like
transport equation. Furthermore, in general, $h$ is also a function of time. For the moment we shall, as for $a(t)$, write $h$ instead of $h(t)$ and treat $t$ as a parameter. We shall return to this aspect in the final section.

In what follows, we shall assume that the total cross section $\sigma_i$, the scattering cross section $\sigma_{si}$ and the $\gamma$-ray source $q_i$ are given constants within each region $V_i$, $i = 1, 2, 3$. The positive function $a = a(t)$ is also given (with $|\dot{a}| \ll c = \text{speed of light}$), characterizing the quasi-static motion of the surfaces $\Sigma_0$ and $\Sigma_h$; see (1). On the other hand, the thickness $h$ of the heart walls is unknown. Thus, our inverse problem can be stated as follows: given the physical properties of host medium within $V$ (cross sections and sources) and assuming that the “shape” of $\Sigma_0$ and $\Sigma_h$ is known (they both have a defining equation of the form $\varphi(p_1, p_2, p_3) = 0$), find $h$ (the geometrical quantity that localizes $\Sigma_h$) from known, experimentally determined values of the photon field. More precisely, let $V_1$, $V_2$ and $V_3$ be the closed subsets of $\mathbb{R}^3$ bounded by the surfaces $\Sigma_0$, $\Sigma_0$ and $\Sigma_h$, $\Sigma_h$ and $\Sigma$, respectively; moreover, let $V_i$ be the interior of $V_i$, $i = 1, 2, 3$ (thus, for instance, $V_1 = V_1 \cup \Sigma_0$). Then, the functions $\sigma_s(x)$, $\sigma(x)$ and $q(x)$ are defined by

\begin{align}
\sigma_s(x) &= \sigma_s > 0 \quad \forall x \in V, \\
\sigma(x) &= \sigma_1 \quad \forall x \in V_1., \\
\sigma(x) &= \sigma_2 \quad \forall x \in V_2., \\
\sigma(x) &= \sigma_3 = \sigma_1 < \sigma_2 \quad \forall x \in V_3, \\
q(x) &= q_1 > 0 \quad \forall x \in V_1., \\
q(x) &= q_0 = 0 \quad \forall x \in V_2, \cup V_3. 
\end{align}
Remark 2. Relations (2) imply that (a) the scattering cross section is assumed to be constant in the whole $V$, (b) the medium within $V_2$ has the largest value of the total cross section, and (c) there is no $\gamma$-ray source outside $V_1$.

We note that, in (2), $V$, $V_1$, $V_2$, and $V_3$ refer to the configuration of the host medium (the cardiac region), frozen at time $t$. Correspondingly, the quasi-static $\gamma$-photon transport equation has the form

$$-u \cdot \nabla n_h - \sigma(x)n_h + \frac{1}{4\pi \sigma_s} \int_S n_h(x, u') \, du' + q(x) = 0, \quad x \in V. \quad (3)$$

In (3), $n_h = n_h(x, u)$ is the photon number density; i.e., $n_h(x, u) \, dx \, du$ is the expected number of photons in the volume element $dx$ centered at $x$, whose velocity $v = cu$ has direction within the solid angle $du = \sin \theta \, d\theta \, d\varphi$. Further, $S$ is the surface of the unit sphere and $du' = \sin \theta' \, d\theta' \, d\varphi'$. Finally, the symbol $n_h$ is used to stress that Eq. (3) holds in the medium $V = V_1 \cup V_2 \cup V_3$, whose configuration is determined by the value of $h$.

Remark 3. We recall that the exact transport equation has the form

$$\frac{1}{c} \frac{\partial}{\partial t} N_h(x, u, t) = -u \cdot \nabla N_h - \sigma(x, t)N_h + \frac{1}{4\pi \sigma_s} \int_S N_h(x, u', t) \, du' + q(x, t),$$

where in fact $\sigma$ and $q$ depend on time because the surfaces $\Sigma_0$ and $\Sigma_h$, which bound $V_1$, $V_2$ and $V_3$, depend on time through $a = a(t)$. However, it was shown in [1] that the $L^1$-norm of the error $(N_h - n_h)$ is proportional to $|\dot{a}|/c \ll 1$.

3. The integral form of the quasi-static equation

We notice that Eq. (3), with $x - ru$ replacing $x$, can be written in the following form:

$$\frac{\partial}{\partial r} \left\{ n_h(x - ru, u) \exp \left[ -\int_0^r \sigma(x - r'u) \, dr' \right] \right\} + \exp \left[ -\int_0^r \sigma(x - r'u) \, dr' \right]$$

$$\times \left\{ \frac{1}{4\pi \sigma_s} \int_S n_h(x - ru, u') \, du' + q(x - ru) \right\} = 0. \quad (4)$$
Integration of (4), with respect to $r$ between 0 and $\bar{r}$, gives

$$n_h(x, u) = \bar{r} \int_0^\infty dr \exp\left[ -r \int_0^r \sigma(x - r'u) dr' \right]$$

$$\times \left\{ q(x - ru) + \frac{1}{4\pi \sigma_s} \int_S n_h(x - ru, u') du' \right\},$$

(5)

where $\bar{r} = \bar{r}(x, u)$ is such that $(x - \bar{r}u)$ belongs to the surface $\Sigma$, which bounds $V_3$ and the whole $V$; see Fig. 1. Note that $n_h(x - \bar{r}u, u) = 0$ because no photons enter the convex region $V$ from outside.

The integral equation (5) can be written as

$$n_h = Q_h + B_h u_h,$$

(6)

where

$$Q_h = Q_h(x, u) = \bar{r} \int_0^\infty dr \left\{ \exp\left[ -r \int_0^r \sigma(x - r'u) dr' \right] q(x - ru) \right\},$$

(7)

$$B_h f = (B_h f)(x, u)$$

$$= \frac{1}{4\pi \sigma_s} \int_0^\infty dr \left\{ \exp\left[ -r \int_0^r \sigma(x - r'u) dr' \right] \int_S f(x - ru, u') du' \right\}. \quad (8)$$

Note that the source term $Q_h$ and the integral operator $B_h$ depend on $h$ through $\sigma$ and $q$ (see definitions (2a)–(2c), in which the configuration of $V = V_1 \cup V_2 \cup V_3$ is determined by $h$). The integral equation (6) will be studied in the Banach space $X = L^1(V \times S)$, with the usual norm

$$\| f \| = \int_S du \int_V |f(x, u)| dx,$$

which, if $f(x, u)$ is a photon number density, gives the total number of photons in $V$.

Since $\sigma(y) \geq \sigma_1 = \sigma_3 \forall y \in V$, we have from (7)

$$|Q_h(x, u)| \leq \bar{r} \int_0^\infty dr \exp(-\sigma_1 r) q(x - ru) \leq q_1 \int_0^\infty \exp(-\sigma_1 r) dr \leq q_1/\sigma_1$$

and so

$$\|Q_h\| \leq 4\pi \text{mes}(V) q_1/\sigma_1. \quad (9a)$$

In an analogous way, (8) gives
\[(B_h f)(x, u) \leq \frac{1}{4\pi} \sigma_s \int_0^\delta dr \exp(-\sigma_1 r) \int_S |f(x - ru, u')| du' \]
\[
\leq \frac{1}{4\pi} \sigma_s \int_0^\delta dr \exp(-\sigma_1 r) \int_S |f(x - ru, u')| du' \]

because \( \bar{r} = \bar{r}(x, u) \leq \delta = \text{diameter of } V \forall (x, u) \in V \times S \). Hence, we obtain
\[
\int_V |(B_h f)(x, u)| dx \leq \frac{1}{4\pi} \sigma_s \|f\| \int_0^\delta \exp(-\sigma_1 r) dr \leq \frac{1}{4\pi} \sigma_s \|f\| \frac{1}{\sigma_1};
\]
this implies that
\[
\|B_h f\| \leq (\sigma_s / \sigma_1) \|f\| \quad \forall f \in X
\]
and so
\[
\|B_h\| \leq \sigma_s / \sigma_1. \tag{9b}
\]
Since \( \sigma_s / \sigma_1 < 1 \) because \( \sigma_1 = \sigma_s + \sigma_{c1} \), where \( \sigma_{c1} \) is the capture cross section, inequality (9b) shows that the bounded operator \( B_h \) is a strict contraction \( \forall h \in [h_m, h_M] \), where \( h_m \) an \( h_M \) are reasonable lower and upper bounds for the thickness \( h \). We conclude that the unique solution of Eq. (6) has the form
\[
n_h = (I - B_h)^{-1} Q_h, \tag{10}
\]
where \( \| (I - B_h)^{-1} \| \leq 1/(1 - \sigma_s / \sigma_1) = \sigma_1 / (\sigma_1 - \sigma_s) \).

**Remark 4.** Equation (6) could also be studied in the Banach space \( C(V \times S) \), with norm \( \|f\|_C = \sup \{|f(x, u)|, (x, u) \in V \times S \} \). However, in this case the cross section \( \sigma \) and the source \( q \) should be re-defined in such a way that they are continuous \( \forall x \in V \). This would imply the introduction of the two thin layers bounded by the surfaces \( \Sigma_{0-\varepsilon} \) and \( \Sigma_{0+\varepsilon} \) (through which \( \sigma(x) \) increases continuously from \( \sigma_1 \) to \( \sigma_2 \) and \( q(x) \) decreases continuously from \( q_1 \) to \( 0 \)) and by the surfaces \( \Sigma_{h-\varepsilon} \) and \( \Sigma_{h+\varepsilon} \) (through which \( \sigma(x) \) decreases continuously from \( \sigma_2 \) to \( \sigma_3 = \sigma_1 \)). Even if \( 0 < \varepsilon \ll h_m \), this would imply a rather cumbersome machinery, and the associated effort is not felt to be worthwhile.

Note that, if we use definitions (2b) and (2c), \( \sigma(x) \) and \( q(x) \) do not define continuous functions of \( x \in V \); correspondingly, it may happen that, for instance, \( Q \notin C(V \times S) \) (if some portion of \( \Sigma_0 \) coincides with a part of a plane).

**4. The source term \( Q_h \) and the operator \( B_h \)**

We shall now derive some properties of the source term \( Q_h \) and of the integral operator \( B_h \), which will be needed later on.
Fig. 2. The configuration $V = V_1 \cup V'_2 \cup V'_3$, where $\Sigma_h: \varphi(x/(a + h), y/(a + h), z/(a + h)) = 0$, $\Sigma'_h: \varphi(x/(a + h'), y/(a + h'), z/(a + h')) = 0$, with $h_m \leq h \leq h' \leq h_M$.

Let $X_+ = \{g: g \in X, g(x, u) \geq 0 \text{ at almost all } (x, u) \in V \times S\}$ be the closed positive cone of the Banach space $X = L^1(V \times S)$; see Section 3. If $f \in X_+$, we shall write $f \geq 0$; in an analogous way, $f \geq g$ will mean that $f - g \geq 0$. Furthermore, if $f \in X_+$ and a subset $\Omega_f \subset V \times S$ exists such that $\operatorname{mes}(\Omega_f) > 0$ and $f(x, u) > 0$ at almost all $(x, u) \in \Omega_f$, we shall define $f > 0$. In what follows, we shall also use the symbol $\gamma_{x, u}$ to denote the (half) straight line \{y: y = x - ru, r \geq 0\}, passing through $x$ and parallel to the unit vector $u$.

With this preparation we can establish the following lemmas.

**Lemma 1.** (i) $Q_h > 0$, (ii) $Q_h - Q_{h'} > 0$ if $h < h'$, (iii) $\|Q_h - Q_{h'}\| \to 0$ as $h' \to h$.

**Proof.** (i) That $Q_h \in X_+$ follows directly from (7) and (2c). On the other hand, since $\sigma(y) \leq \sigma_2$ at almost all $y \in V$, (7) gives

$$Q_h(x, u) \geq \int_0^\infty \exp(-\sigma_2 r)q(x - ru) \, dr.$$  

Hence, $Q_h(x, u) > 0$ if $(x, u)$ is such that $\gamma_{x, u}$ crosses $V_1$, because $q(x - ru) = q_1 > 0 \, \forall x - ru \in V_1$. Thus, in particular, $Q_h(x, u) > 0 \, \forall (x, u) \in V_1 \times S$.

(ii) Let $\sigma'(y)$ and $q'(y)$ be the total cross section and the source term corresponding to the configuration $V = V_1 \cup V'_2 \cup V'_3$, see Fig. 2.
Then (2b) and (2c) imply that \( \sigma'(y) \geq \sigma(y) \) and \( q'(y) = q(y) \) at almost all \( y \in V \). As a consequence, we have

\[
Q_h(x, u) - Q_{h'}(x, u)
= \int_0^\tilde{r} dr \exp \left[ - \int_0^r \sigma(x - r'u) dr' \right] \times \left\{ 1 - \exp \left[ - \int_0^r \left( \sigma'(x - r'u) - \sigma(x - r'u) \right) dr' \right] \right\} q(x - ru) \\
\geq \int_0^\tilde{r} dr \exp(-\sigma_2 r) \left\{ 1 - \exp \left[ - \int_0^r \left( \sigma'(x - r'u) - \sigma(x - r'u) \right) dr' \right] \right\} q(x - ru)
\]

and so \( Q_h - Q_{h'} \in X_+ \).

Consider now \((\hat{x}, \hat{u})\) with \(\hat{x}\) belonging to the region bounded by \(\Sigma_{h_M}^\dagger\) and \(\Sigma\), and with \(\hat{u}\) such that \(\gamma_{\hat{x}, \hat{u}}\) crosses \(V_1\); see Fig. 2. If \(\hat{x} - \tilde{r}' \hat{u}\) and \(\hat{x} - \tilde{r}'' \hat{u}\) (\(\tilde{r}'' > \tilde{r}'\)) are the points of intersection of \(\gamma_{\hat{x}, \hat{u}}\) with \(\Sigma_0\), we obtain

\[
Q_h(\hat{x}, \hat{u}) - Q_{h'}(\hat{x}, \hat{u}) \\
\geq q_1 \int_{\tilde{r}'}^\tilde{r} dr \exp(-\sigma_2 r) \left\{ 1 - \exp \left[ - \int_0^r \left( \sigma'(\hat{x} - r' \hat{u}) - \sigma(\hat{x} - r' \hat{u}) \right) dr' \right] \right\}
\]

because \(q(y) = 0\) if \(y \notin V_1\). On the other hand, if \(r \in (\tilde{r}', \tilde{r}'')\), we have

\[
\int_0^{\tilde{r}'} \left( \sigma'(\hat{x} - r' \hat{u}) - \sigma(\hat{x} - r' \hat{u}) \right) dr' = \int_0^{\tilde{r}'} \left( \sigma' - \sigma \right) dr' + \int_0^{\tilde{r}'} \left( \sigma' - \sigma \right) dr' \\
= \int_0^{\tilde{r}'} \left( \sigma'(\hat{x} - r' \hat{u}) - \sigma(\hat{x} - r' \hat{u}) \right) dr' > 0
\]

because \(\sigma'(y) = \sigma(y) = \sigma_1 \quad \forall y \in V_1\), and \(\sigma'(y) = \sigma_2 \geq \sigma_3 = \sigma(y)\) if \(y\) belongs to the region bounded by \(\Sigma_h\) and \(\Sigma_{h'}\). Thus, \(Q_h(\hat{x}, \hat{u}) - Q_{h'}(\hat{x}, \hat{u}) > 0\).

(iii) Let \(h < h'\) (the case \(h > h'\) is analogous); then definition (7) gives

\[
0 \leq Q_h(x, u) - Q_{h'}(x, u) \\
\leq \int_0^{\tilde{r}} dr \exp(-\sigma_1 r) \left\{ 1 - \exp \left[ - \int_0^r \left( \sigma'(x - r'u) - \sigma(x - r'u) \right) dr' \right] \right\} q(x - ru).
\]
$AB + CD = \text{length of the “crossing” of the region bounded by } \Sigma_h \text{ and } \Sigma_{h'}, \text{ determined by the straight line } \gamma_{x,u}. \text{ It may happen that either } AB = 0, \text{ or } CD = 0, \text{ or } AB = CD = 0.$

Since $\sigma'(y) - \sigma(y) = \sigma_2 - \sigma_1$ if $y$ belongs to the region bounded by $\Sigma_h$ and $\Sigma_{h'}$, and $\sigma'(y) - \sigma(y) = 0$ otherwise, then we have

$$0 \leq \int_0^r \left( \sigma' (\hat{x} - r' \hat{u}) - \sigma (\hat{x} - r' \hat{u}) \right) dr' \leq (\sigma_2 - \sigma_1) \left[ AB + CD \right]$$

$$\leq (\sigma_2 - \sigma_1) l(h, h'), \quad (x, u) \in V \times S$$

where

$$l(h, h') = \sup \left\{ \left( \frac{AB + CD}{2} \right), \ (x, u) \in V \times S \right\}$$

in the “largest crossing” of the region bounded by $\Sigma_h$ and $\Sigma_{h'}$; see Fig. 3. Note that $l(h, h') \leq \text{diameter of the region bounded by } \Sigma_{h_M}$ and that

$$\lim_{h' \to h} l(h, h') = 0, \quad (12)$$

provided that the family of homothetic surfaces $\{ \Sigma_h: h \in [h_m, h_M] \}$ is regular enough.

We obtain from (11) that

$$0 \leq Q_h(x, u) - Q_{h'}(x, u)$$

$$\leq \int_0^r dr \exp(-\sigma_1 r) \left[ 1 - \exp[-(\sigma_2 - \sigma_1) l(h, h')] \right] q(x - ru)$$
\[(\sigma_2 - \sigma_1) l(h, h') \int_0^\delta \exp(-\sigma_1 r) q(x - ru) \, dr \]

\[\leq (\sigma_2 - \sigma_1) l(h, h') \int_0^\delta \exp(-\sigma_1 r) q(x - ru) \, dr,\]

where we recall that \(\delta\) is the diameter of \(V\). Thus,

\[\int_S d\nu \int_V \left[ Q_h(x, u) - Q_{h'}(x, u) \right] \, dx \]

\[\leq (\sigma_2 - \sigma_1) l(h, h') \|q\| \int_0^\delta \exp(-\sigma_1 r) \, dr\]

and so

\[\|Q_h - Q_{h'}\| \leq \frac{\sigma_2 - \sigma_1}{\sigma_1} l(h, h') 4\pi q_1 \text{mes}(V_1)\]

because \(\|q\| = 4\pi q_1 \text{mes}(V_1)\). Hence, (iii) is proved because of (12).

\textbf{Lemma 2.} (i) \(B_h f > 0 \ \forall f > 0\); (ii) \(B_h f - B_{h'} f > 0\) if \(h < h'\) and \(f > 0\); (iii) \(\|B_h - B_{h'}\| \to 0\) as \(h' \to h\).

\textbf{Proof.} (i) That \(B_h f \in X_+ \ \forall f \in X_+\) follows directly from definition (8). Assume that \(f \in X_+\) and \(f(x, u') > 0\) at a.a. \((x, u') \in \Omega_f = V_f \times U_f\), with \(V_f \subseteq V, U_f \subseteq S\) and \(\text{mes}(V_f) > 0, \text{mes}(U_f) > 0\). As a consequence, we have

\[p(y) = \frac{1}{4\pi \sigma_s} \int_S f(y, u') \, du' \geq \frac{1}{4\pi \sigma_s} \int_{U_f} f(y, u') \, du' > 0\]

at a.a. \((y, w) \in V_f \times S\). Since (8) gives

\[(B_h f)(x, u) = \int_0^\delta \, dr \exp \left( - \int_0^r \sigma(x - ru) \, dr' \right) p(x - ru) \]

\[\geq \int_0^\delta \, dr \exp(-\sigma_2 r) p(x - ru),\]

we conclude that, for each \(x \in V\), \((B_h f)(x, u) > 0 \ \forall u \in U_x\); see Fig. 4. Hence \((B_h f)(x, u) > 0\) at a.a. \((x, u) \in \Omega = \bigcup_{x \in V} \{x\} \times U_x\).

(ii) We have from (8)
For each \( x \in V \), \( (Bhf)(x, u) > 0 \forall u \in U_x \), where \( U_x = S \) if \( x \) belongs to the interior of \( V_f \).

\[
(B_h f - B_{h'} f)(x, u) = \int_0^\tilde{r} dr \exp \left[ - \int_0^r \sigma(x - r'u) dr' \right] \times \left\{ 1 - \exp \left[ - \int_0^r (\sigma'(x - r'u) - \sigma(x - r'u)) dr' \right] \right\} p(x - ru)
\]

\[
\geq \int_0^\tilde{r} dr \exp(-\sigma_2 r) \left\{ 1 - \exp \left[ - \int_0^r (\sigma'(x - r'u) - \sigma(x - r'u)) dr' \right] \right\} \times p(x - ru),
\]

where we recall that \( p(y) > 0 \) at a.a. \( (y, w) \in V_f \times S \), see (i) of this lemma. Then, as in (ii) of Lemma 1, it is easy to see that \( (B_h f - B_{h'} f)(\hat{x}, \hat{u}) > 0 \) if \( \hat{x} \) belongs to the region bounded by \( \Sigma_h M \) and \( \Sigma \), and \( \hat{u} \) is such that \( \gamma_{\hat{x}, \hat{u}} \) crosses the interior of \( V_f \) and the region bounded by \( \Sigma_h \) and \( \Sigma_{h'} \).

(iii) Assume that \( h < h' \) (the case \( h > h' \) is analogous). Then, \( \sigma'(y) \geq \sigma(y) \) at a.a. \( y \in V \) and (8) gives

\[
| (B_h f - B_{h'} f)(x, u) | \leq \frac{1}{4\pi} \sigma_s \int_0^{\tilde{r}} dr \exp(-\sigma_1 r)
\]
\[ \int_{S} |f(x - ru, u')| \, du'. \]

As in (iii) of Lemma 1, we then obtain
\[ \| (B_h f - B_{h'} f)(x, u) \| \leq \frac{1}{4\pi} \sigma_s (\sigma_2 - \sigma_1) l(h, h') \int_0^\delta dr \exp(-\sigma_1 r) \int_{S} |f(x - ru, u')| \, du' \]
and so
\[ \| B_h f - B_{h'} f \| \leq \sigma_s \frac{\sigma_2 - \sigma_1}{\sigma_1} l(h, h') \| f \| \forall f \in X. \]

**Lemma 3.**

(i) \( (I - B_h)^{-1} f > 0 \forall f > 0, \)

(ii) \( (I - B_h)^{-1} f - (I - B_{h'})^{-1} f > 0 \forall f > 0 \) and \( h < h' \),

(iii) \( \| (I - B_h)^{-1} - (I - B_{h'})^{-1} \| \to 0 \) as \( h' \to h. \)

**Proof.** (i) follows from (i) of Lemma 2 because \( (I - B_h)^{-1} f = f + B_h f + B_h^2 f + \cdots. \)

(ii) Since
\[ (I - B_h)^{-1} f - (I - B_{h'})^{-1} f = (I - B_h)^{-1} (B_h - B_{h'}) (I - B_{h'})^{-1} f, \]

(ii) follows from (i) of this lemma and from (ii) of Lemma 2. Note that we also have
\[ (I - B_h)^{-1} (B_h - B_{h'}) (I - B_{h'})^{-1} f \geq (B_h - B_{h'}) f \forall f \in X_+. \]

(iii) We have
\[ \| (I - B_h)^{-1} - (I - B_{h'})^{-1} \| \leq \| (I - B_h)^{-1} \| \| B_h - B_{h'} \| (I - B_{h'})^{-1} \| \leq \frac{\sigma_1}{\sigma_1 - \sigma_s} \frac{\sigma_2 - \sigma_1}{\sigma_1} l(h, h') \frac{\sigma_1}{\sigma_1 - \sigma_s}, \]
where we have used the inequality found in (iii) of Lemma 2 and (9b).

5. Relation between \( n_h \) and \( h \)

Assume that \( \sigma_1, \sigma_2, \sigma_3, q_1, a \) and \( \varphi(p_1, p_2, p_3) \) are given and let \( h \in [h_m, h_M]. \) Then, (10) can be interpreted as follows:
\[ n_h = K(h), \] (13)
where
\[ K(h) = (I - B_h)^{-1}Q_h. \] (14)

We remark that \( K \) is a nonlinear operator with domain \( D(K) = [h_m, h_M] \subset \mathbb{R}^1 \) and range \( R(K) = \{ n: n \in X_+, n = K(h), h \in [h_m, h_M] \} \subset X_+ \). The properties of \( K \) follow from the corresponding ones, proved for \( Q_h, B_h \) and \( (I - B_h)^{-1} \) in Section 4.

**Lemma 4.** (i) \( K(h) - K(h') > 0 \) if \( h < h' \), (ii) \( \| K(h) - K(h') \| \to 0 \) as \( h' \to h \).

**Proof.** (i) We have from (14)
\[
K(h) - K(h') = (I - B_h)^{-1}(Q_h - Q_h') + [(I - B_h)^{-1} - (I - B_h')^{-1}]Q_h'.
\]
Hence, (i) follows from (i) and (ii) of Lemma 1 and from (i) and (ii) of Lemma 3.

(ii) Since
\[
\| K(h) - K(h') \| \leq \frac{\sigma_1}{\sigma_1 - \sigma_s} \| Q_h - Q_h' \| + \| (I - B_h)^{-1} - (I - B_h')^{-1} \| \| Q_h' \|,
\]
we obtain
\[
\| K(h) - K(h') \| \leq kl(h, h'), \quad k = 4\pi q_1 \text{mes}(V)\sigma_1(\sigma_2 - \sigma_1)/(\sigma_1 - \sigma_2)^2.
\]
See (iii) of Lemma 1, (iii) of Lemma 3, and (9a). \( \square \)

Inequality (i) of Lemma 4 shows that, if \( h < h' \), \( n_h = K(h) \) and \( n_{h'} = K(h') \) are two distinct elements of \( X_+ \). Thus, we can state the following theorem:

**Theorem 1.** The nonlinear operator \( K^{-1} \) exists and \( D(K^{-1}) = R(K) = \{ n: n \in X_+, n = K(h), h \in [h_m, h_M] \} \), \( R(K^{-1}) = [h_m, h_M] \).

We conclude that, given any \( n \in D(K^{-1}) \), Eq. (13) leads to
\[ h = K^{-1}(n). \] (15)
Relation (15) shows that a knowledge of the photon field implies that the geometrical quantity \( h \) can be evaluated.

6. Evaluation of \( h \)

Let \( \hat{n} \) be the result of experimental measurements (in a sense that will be specified later on). Then, \( \hat{n} \in R(K) \), i.e., a suitable \( \hat{h} \in [h_m, h_M] = D(K) \) must
exist such that \( \hat{n} = K(\hat{h}) \). However, since the explicit form of \( K^{-1} \) is not known, some “numerical” procedure must be devised to evaluate \( \hat{h} = K^{-1}(\hat{n}) \).

Assume that we can find \( h^-_i \) and \( h^+_i \) (with \( h^-_m \leq h^-_1 < h^+_1 \leq h_M) \), such that \( K(h^+_1) < \hat{n} < K(h^-_1) \), i.e., \( \hat{n} - K(h^+_1) > 0 \) and \( K(h^-_1) - \hat{n} > 0 \).

**Remark 5.** We recall that \( K(h) \) is a strictly decreasing function of \( h \) because of (i) of Lemma 4.

Take then \( h = (h^+_1 + h^-_1)/2 \) and suppose that, for instance, \( K(h) > \hat{n} \); correspondingly, put \( h^+_2 = (h^+_1 + h^-_1)/2 \) and \( h^-_2 = h^+_1 \). Thus, \( h_m \leq h^-_1 < h^-_2 < h^+_2 = h^+_1 \leq h_M \) and \( K(h^+_1) = K(h^+_2) < \hat{n} < K(h^-_2) < K(h^-_1) \).

Further, let \( h = (h^+_1 + h^-_2)/2 \) and assume that, for instance, \( K(h) < \hat{n} \); correspondingly, put \( h^+_3 = (h^+_2 + h^-_2)/2 \) and \( h^-_3 = h^-_2 \). Thus, \( h_m \leq h^-_1 < h^-_2 < h^-_3 < h^+_3 < h^+_2 = h^+_1 \leq h_M \) and \( K(h^+_1) = K(h^+_2) < K(h^+_3) < \hat{n} < K(h^-_3) = K(h^-_2) < K(h^-_1) \).

By this procedure, we construct the four monotonic sequences \( \{h^-_j\} \), \( \{h^+_j\} \), \( \{K(h^-_j)\} \) and \( \{K(h^+_j)\} \). If \( h^+_1 - h^-_1 = d \), then we have that \( h^+_j - h^-_j = d/2^{j-1} \to 0 \) as \( j \to \infty \) and so a suitable \( \tilde{h} \) exists, such that \( \tilde{h} = \lim_{j \to \infty} h^-_j = \lim_{j \to \infty} h^+_j \) (with \( h_m \leq h^-_j \leq \tilde{h} \leq h^+_j \leq h_M \)).

On the other hand, (ii) of Lemma 4 gives

\[
\| K(h^+_j) - K(h^-_j) \| \leq kl(h^+_j, h^-_j), \quad \| K(h^-_j) - K(h^+_j) \| \leq kl(h^-_j, h^+_j)
\]

\( \forall i, j = 1, 2, \ldots \)

and so \( \{K(h^+_j)\} \) and \( \{K(h^-_j)\} \) are Cauchy sequences in \( X \). It follows that

\[
\lim_{j \to \infty} K(h^+_j) = n^+, \quad \lim_{j \to \infty} K(h^-_j) = n^-,
\]

where \( n^+ \) and \( n^- \) are suitable elements of \( X \). However, \( n^+ = n^- = \tilde{n} \) because

\[
\| K(h^+_j) - K(h^-_j) \| \leq kl(h^+_j, h^-_j) \to 0 \quad \text{as} \quad j \to \infty
\]

and so

\[
\lim_{j \to \infty} K(h^+_j) = \lim_{j \to \infty} K(h^-_j) = \tilde{n}.
\]

We conclude that \( K(\tilde{h}) = \tilde{n} \) because

\[
\lim_{j \to \infty} \| K(\tilde{h}) - K(h^+_j) \| \leq k \lim_{j \to \infty} l(\tilde{h}, h^+_j) = 0.
\]

Finally, consider the given experimental photon density \( \hat{n} \); since

\[
0 < K(h^+_j) < \hat{n} < K(h^-_j) \quad \forall j = 1, 2, \ldots
\]

we have
\[
\|\hat{n} - \tilde{n}\| = \|\hat{n} - K(\tilde{h})\| \leq \|\hat{n} - K(h_j^+)\| + \|K(h_j^+) - K(\tilde{h})\| \\
\leq \|K(h_j^-) - K(h_j^+)\| + \|K(h_j^+) - K(\tilde{h})\| \\
\leq k[l(h_j^-, h_j^+) + l(h_j^+, \tilde{h})] \to 0
\]

as \( t \to \infty \).

We conclude that \( \hat{n} = \tilde{n} \); as a consequence, \( K(\tilde{h}) = \hat{n} \) and so \( \tilde{h} = h \), where \( \hat{h} \) is the value of \( h \) we were looking for.

7. Concluding remarks

The procedure considered in Section 6 is a successive approximation method to solve the equation \( K(\tilde{h}) = \hat{n} \), where \( \tilde{h} \) is the unknown and \( \hat{n} \) is a given element of the Banach space \( X = L^1(V \times S) \) (i.e., \( \hat{n}(x, u) \) is known at a.a. \((x, u) \in V \times S\)).

However, in practice, only a finite number of values of the photon density are measured at \((x_j, u_j)\), \( j = 1, 2, \ldots, J \), where \( x_j \) belongs to the region bounded by \( \Sigma_{h_M} \) and \( \Sigma \) ("far field" measurements), and \( u_j \) is such that \( \gamma_{x_j, u_j} \) crosses the region \( V_1 \). From these \( J \) measurements we can evaluate (by some suitable interpolation method) the photon density \( n(x, u) \) at a.a. \((x, u) \in \hat{\Omega} = \tilde{V} \times \tilde{U} \), where \( \text{mes}(\hat{\Omega}) \) is positive and small. It follows that a suitable \( h \in [h_m, h_M] \) must exist, such that \( K(\tilde{h})(\hat{x}, \hat{u}) = n(\hat{x}, \hat{u}) \) at a.a. \((\hat{x}, \hat{u}) \in \hat{\Omega} \) (\( \tilde{h} \) is unique, see the proofs of (ii) of Lemma 1 and (ii) of Lemma 2).

From this we can start the successive approximation procedure considered in Section 6, leading to the evaluation of \( \hat{h} \). Of course, the first step will be to find \( h_1^- \) and \( h_1^+ \) such that \( K(h_1^+)(\hat{x}, \hat{u}) < n(\hat{x}, \hat{u}) < K(h_1^-)(\hat{x}, \hat{u}) \) at a.a. \((\hat{x}, \hat{u}) \in \hat{\Omega} \).

We mentioned earlier that in fact \( h = h(t) \). To accommodate this, we notice that measurements could be made at a number of fixed times \( t_m, m = 1, 2, \ldots, M \), and the above strategy employed to compute \( h(t_m) \), the thickness of the heart walls at these various instants. A plot of the \( M \) values \( h(t_m) \) together with a curve fitting procedure will then yield an indication of the time variation of \( h(t) \) (see [12] for a more detailed discussion).

We finally remark that, from a "very practical" point of view, the procedure of Section 6 should work even if just one value \( n(x_0, u_0) \) of the photon density is known at some \((x_0, u_0) \in \hat{\Omega} \). In this case, \( h_1^- \) and \( h_1^+ \) must be chosen so that \( K(h_1^+)(x_0, u_0) < n(x_0, u_0) < K(h_1^-)(x_0, u_0) \) and the successive approximation method must be re-interpreted in a pointwise way. This should lead to a simple and efficient machinery to find the value \( \tilde{h} \) for which \( K(\tilde{h})(x_0, u_0) = n(x_0, u_0) \).

References


