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Multiplicative Perturbation of Nonlinear m-Accretive Operators

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Criteria are obtained for when an accretive product (i.e., composition) BA of nonlinear m-accretive operators A and B in a Banach space X will be itself m-accretive; and, in particular, when a monotone product of two maximal monotone operators in a Hilbert space will be maximal monotone. This extends the theory of multiplicative perturbation of infinitesimal generators of contraction semigroups to the nonlinear case. Also obtained as a biproduct are existence theorems for certain Hammerstein integral equations.

1. INTRODUCTION

In this paper we obtain criteria for when an accretive product (i.e., composition) BA of nonlinear m-accretive operators A and B in a Banach space X will be itself m-accretive; and, in particular, when a monotone product of two maximal monotone operators in a Hilbert space will be maximal monotone. The class of m-accretive operators arises in initial value problems as infinitesimal generators of contraction semigroups which describe the time-evolution of a system; for applications a perturbation theory has developed. For linear A and B, additive perturbation (i.e., given A m-accretive, for what B is $A + B$ m-accretive) has been studied for example in [13, 16, 18, 20, 24, 30, 31, 34]; linear multiplicative perturbation (i.e., given A m-accretive, for what B is BA m-accretive) has been studied, for example, in [10, 12, 14, 15, 18, 19, 36]. More recently, for nonlinear A and B, additive perturbation results have been obtained in [1–4, 9, 23, 28, 33];

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it is the purpose of this paper to extend the multiplicative perturbation theory to the nonlinear case.

Let X be a Banach space (over the reals, for simplicity), with strong dual X^*. We recall an operator (single or multivalued, with single or multivalued inverse) $A : D(A) \subset X \to 2^X$ (to denote that A is a subset of $X \times X$ (see, e.g., [2, 4, 8] for the standard notation) with $D(A)$ and $R(A)$ subsets of X) is said to be *accretive* if, for all $\alpha > 0$, $(I + \alpha A)^{-1}$ is single-valued and nonexpansive; if, in addition, $R(I + A) = X$, A is said to be *m-accretive.* Thus accretive and m-accretive are the same as monotonic and m-monotonic in Kato [21] and g-accretive and hypermaximal accretive in Browder [4]. Let $J : X \to 2^{X^*}$ be the (everywhere defined, single-valued iff X is smooth iff X^* is strictly convex) maximal duality map, namely,

$$J(x) = \{x^* \mid \| x^* \|^2 = \| x \|^2 = (x^*, x)\}.$$

By [21] A is accretive iff for each u, v in $D(A)$ and for each $x \in Au$, $y \in Av$ there is an x^* in $J(u - v)$ such that $(x^*, x - y) \geqslant 0$; we specify this in particular by saying A is *accretive* (φ), where φ is the function $\varphi(u, v, x \in Au, y \in Av) = x^*$. Similarly, we say that A is accretive (J) if for $x \in Au$, $y \in Av$, $(x^*, x - y) \geqslant 0$ for all x^* in $J(u - v)$.

We recall $B : D(B) \subset X \to 2^{X^*}$ is monotone if u in Bx and v in By implies $(u - v, x - y) \geqslant 0$, B is *maximal monotone* if B (as a graph) is not properly contained in a larger monotone graph, and B is coercive if for u in Bx one has $(u, x) \cdot \| x \|^{-1} \to \infty$ as $\| x \| \to \infty$. If X is a Hilbert space, monotone is the same as accretive, and by Minty [29] maximal monotone is the same as m-accretive. We also recall $B : X \to X$ (to denote that B is everywhere defined and single valued) is hemicontinuous if B is weakly continuous from line segments; also, that an operator $T : D(T) \subset X \to 2^Y$ is said to be *locally bounded* if $\forall x \in X$ there exists a neighborhood U such that $T(U \cap D(T))$ is bounded.

The proofs of the following four results will be given in Section 2.

THEOREM 1. *Let X be any Banach space, $A : D(A) \subset X \to X$, A m-accretive (φ), $B : X \to X$ such that $\epsilon B - I$ has (uniform) Lipschitz constant $k < 1$ for some $\epsilon > 0$. Then if BA is accretive (ψ) such that $\psi(u, v, BAu, BAv) = \varphi(u, v, Au, Av)$, it is m-accretive ($\psi$).*

THEOREM 2. *Let X be reflexive, $A : D(A) \subset X \to 2^{X^*}$, $B : D(B) \subset X^* \to 2^X$, both A and B maximal monotone, both of the following conditions holding:*

(1) *either* B^{-1} *or* A *is locally bounded;*

(2) *either* B *or* A^{-1} *is locally bounded.*

Then if BA *is accretive, it is m-accretive.*

THEOREM 3. *Let* X *be any Banach space,* $A : D(A) \subset X \to 2^X$, A *m-accretive* (J), $B : X \to X$ *of the form* $B = (C + \delta I)^{-1}$ *for some* $\delta > 0$ *and some* $C : X \to X$, C *uniformly Lipschitz and accretive. Then if* BA *is accretive, it is m-accretive. If* X^* *is uniformly convex,* C *need be only hemicontinuous and accretive.*

THEOREM 4. *Let* X^* *be uniformly convex,* $A : D(A) \subset X \to 2^X$, A *m-accretive,* $B : X \to X$, B *hemicontinuous, accretive, weakly closed,* B *and* B^{-1} *bounded, and for all* $\alpha > 0$, $B\alpha^{-1}(I - (I + \alpha A)^{-1}$ *is accretive. Then if* BA *is accretive, it is m-accretive.*

Briefly, additive perturbation results are utilized for the proofs of Theorems 1, 2, and 3, semigroup methods for the proof of Theorem 4.

2. PROOFS

The following additive perturbation result, to be used here in the proofs of Theorems 1 and 3, extends Crandall and Pazy [9, Theorem 4.2] and is similar to results of Kato [23] and Mermin [28]; here $| S | \equiv \inf\{\| s \|, s \in S\}$, for a nonempty set S.

LEMMA 1. *Let* A *be m-accretive* (φ) *in a Banach space* X, B *single valued,* $D(B) \supset D(A)$, *such that* $A + B$ *is accretive* (ψ) *such that* $\psi(u, v, x + Bu, y + Bv) = \varphi(u, v, x \in Au, y \in Av)$. *Suppose there exist constants* a *and* b, $b < 1$, *such that for* x_1, x_2 *in* $D(A)$,

$$\| Bx_1 - Bx_2 \| \leqslant a \| x_1 - x_2 \| + b \, | Ax_1 - Ax_2 |. \tag{2.1}$$

Then $A + B$ *is m-accretive* (ψ).

Proof. Writing $A + tB \subset t(A + B) + (1 - t) A$ one has by direct verification that for $0 \leqslant t \leqslant 1$, $A + tB$ is accretive. By use of the doubling lemma [16] it is sufficient to show $A + B$ m-accretive for $b < \frac{1}{2}$; i.e., in accordance with [16] this amounts to observing the two inequalities

$$\| \tfrac{2}{3}Bx_1 - \tfrac{2}{3}Bx_2 \| \leqslant \tfrac{2}{3}a \| x_1 - x_2 \| + \tfrac{2}{3}b \, | Ax_1 - Ax_2 |, \tag{2.2}$$

and, for $b < 3/4$,

$$\| \tfrac{1}{3}Bx_1 - \tfrac{1}{3}Bx_2 \| \leqslant \tfrac{2}{3}a \| x_1 - x_2 \| + \tfrac{2}{3}b \,|(A + \tfrac{2}{3}B)x_1 - (A + \tfrac{2}{3}B)x_2 |. \quad (2.3)$$

Considering then the case $b < \frac{1}{2}$, let $\lambda > 0$ be chosen (large) so that $a\lambda^{-1} + 2b < 1$; one then has

$$\begin{aligned} &\| B(A + \lambda)^{-1}y_1 - B(A + \lambda)^{-1}y_2 \| \\ &\quad \leqslant (a + b\lambda) \|(A + \lambda)^{-1}y_1 - (A + \lambda)^{-1}y_2 \| + b \| y_1 - y_2 \| \\ &\quad \leqslant (a\lambda^{-1} + 2b) \| y_1 - y_2 \|. \end{aligned} \quad (2.4)$$

Thus for each fixed y in X the map $C_y \equiv y - B(A + \lambda)^{-1}$, being a strict contraction, has a fixed point $x_y = C_y x_y$; hence

$$R(I + B(A + \lambda)^{-1}) = X,$$

and consequently $R(\lambda + A + B) = X$.

Proof of Theorem 1. Write $\epsilon BA = A + (\epsilon B - I) A$; then

$$\|(\epsilon B - I) Ax_1 - (\epsilon B - I) Ax_2 \| \leqslant k \| Ax_1 - Ax_2 \|,$$

so that ϵBA (and hence BA) is m-accretive (ψ) by Lemma 1.

We note that it is sufficient that $D(B) \supset R(A)$ in Theorem 1; further multiplicative perturbation results similar to Theorem 1 for not everywhere defined nonlinear B could be obtained along the lines of [19].

COROLLARY 1. *Let X be a Banach space, A m-accretive and B as in Theorem* 1 (*or Lemma* 1), *BA* (*or $A + B$*) *accretive. Then BA* (*or $A + B$*) *is m-accretive under any of the following conditions*:

(i) *A is m-accretive* (J);

(ii) *A is linear*;

(iii) *X is smooth.*

Proof. In each case, A is m-accretive (φ) for all φ. The linear cases (ii) were obtained previously in [13] and [14].

Proof of Theorem 2. First we recall the following result due to Rockafellar [32]:

If X is reflexive and $T : D(T) \subset X \to 2^{X^*}$ is maximal monotone, then $R(T) = X^*$ if and only if T^{-1} is locally bounded. (*)

Since an operator is maximal monotone iff its inverse is maximal monotone, A^{-1} and B are both maximal monotone; therefore by (*) condition (2) is equivalent to: either $D(B) = X^*$ or $D(A^{-1}) = X^*$. Consequently, by Rockafellar [33] (for an alternate proof, see [2]), $A^{-1} + B$ is maximal monotone; if $(A^{-1} + B)^{-1}$ is shown to be locally bounded, then by (*) one has $R(A^{-1} + B) = X$, and therefore $R(I + BA) = X$.

Consider any $x_n + z_n = y_n \to y$, $x_n \in A^{-1}f_n$, $z_n \in Bf_n$; then $y_n \in (I + BA)\, x_n$. Since $(I + BA)^{-1}$ is nonexpansive, $\{x_n\}$ is a Cauchy sequence with a limit point x, and $z_n \to y - x$. Considering now condition (1), suppose B^{-1} is locally bounded, let U be a neighborhood of $y - x$ with $B^{-1}(U)$ bounded; then $f_n \in B^{-1}(U)$ for large n and hence $\{f_n\}$ is bounded. Suppose instead that A is locally bounded, let U be a neighborhood of x with $A(U)$ bounded; then $f_n \in A(U)$ for large n and hence $\{f_n\}$ is bounded. Consequently $(A^{-1} + B)^{-1}$ is locally bounded.

Remark 1. For linear A and B results similar to Theorem 2 were obtained for example in [18, Theorem 3.3], [19, Corollary 3.8, Theorem 3.9]; further nonlinear right multiplication results *per sé* and simultaneous nonlinear right and left multiplication results could be obtained along those lines. The result (*) of [32] used above is analogous to the fact (e.g., [19, Lemma 4.2]) that for linear m-accretive A in a Banach space, $R(A) = X$ iff A^{-1} is bounded.

Remark 2. Condition (2) can clearly be weakened (and still satisfy the domain condition of [33], so that Theorem 2 remains valid) to:

(2′) $\{\text{int } D(B)\} \cap R(A)$ nonempty, or $\{\text{int } R(A)\} \cap D(B)$ nonempty.

The following partially linear version of Theorem 2 also holds; replace (1) and (2) by

(1″) $\forall y \in X$, $\{y - R(B)\} \cap \{\text{int } D(A)\}$ is nonempty, or $\forall y \in X$, $\{y - D(A)\} \cap \{\text{int } R(B)\}$ is nonempty, and

(2″) either B is linear and bounded, or A is linear and A^{-1} is bounded.

To verify this, for any y in X let $C_y(x) = -B^{-1}(y - x)$, $D(C_y) = \{y - R(B)\}$; since A and C_y are maximal monotone, $A + C_y$ is maximal monotone by [33] and (1″). By (*) it suffices to show that $(A + C_y)^{-1}(N)$ is bounded for N any bounded set. Let

$$z \in \{(A + C_y)x\} \cap N;$$

then $B^{-1}(y - x) \cap \{Ax - z\}$ is nonempty, so $y \in \{x + B(ax - z)\}$. If B is linear, $(I + BA)\,x$ contains $y + Bz$, and hence

$$x \in (I + BA)^{-1}(y + B(N)),$$

which is bounded since $(I + BA)^{-1}$, B, and N are bounded. If A is linear with bounded inverse, by (*) A is onto, $z = Aw$ for some w, $y - w \in (I + BA)(x - w)$, and consequently

$$x \in (I + BA)^{-1}(y - A^{-1}(N)) + A^{-1}(N),$$

which is bounded.

Remark 3. We observe in this context that Browder, de Figueiredo, Gupta [5, Theorem 1] state[1] that $R[I + BA] = X$ under the same conditions (X reflexive, A and B maximal monotone) as in Theorem 2 above, with the additional assumption that A is single-valued, everywhere defined, (so that, by (*), A is locally bounded, so that (1) is satisfied), coercive (so that A^{-1} is locally bounded, i.e., (2) is satisfied), hemicontinuous, and monotone (so that A is maximal monotone) but without BA assumed to be accretive. Thus Theorem 2, and indirectly, the other results of this paper, provide existence theorems for Hamerstein integral equations as in [5]. In particular, to avoid the assumption of BA accretive, let us replace in Theorem 2 the conditions (1), (2), and BA accretive by

(1‴) B^{-1} locally bounded and

(2‴) A coercive;

then $R[I + BA] = X$, as follows. As in Remark 2 above, $A + C_y$ is maximal monotone (because $D(C_y) = X$), and by (*), $R(A + C_y) = X^*$ if $A + C_y$ is coercive, which is the case by $\| x \|^{-1}((A + C_y)\,x, x) \geqslant \| x \|^{-1}(Ax, x) - \| B^{-1}(y)\|$. Consequently there exists an x such that $0 \in (A + C_y)\,x$, so that $y \in (I + BA)\,x$. This (‴) result thus complements that of [5], since there it was assumed that $D(A) = X$, whereas here we assumed that $R(B) = X$; and in the case that BA is accretive, Theorem 2 contains and generalizes the result of [5].

[1] The proof of [5, Theorem 1] contains a small discrepancy; e.g., a correct version of [5, Theorem 1] is: A and B maximal monotone, $D(A) = X$, $A_0(x) \equiv A(x - x_0)$ coercive for all x_0. The verification proceeds as in [5], modified as follows: select $x_1 \in R(B)$, let $x_0 = x_1 - y$, let $C_y(x) = -B^{-1}(x_1 - x)$, let $A_y(x) = A(x - x_0)$, observe that $0 \in D(A_y + C_y)$, $A_y + C_y$ is maximal monotone by [33] and $A_y + C_y$ is coercive by $((A_y + C_y)(x), x) \geqslant (A(x - x_0) - B^{-1}(x_1), x)$; thus by [3] $0 \in R(A_y + C_y)$, and consequently $y \in R(I + BA)$.

The proof of Theorem 2, since the condition BA is accretive entered only to assure that $\{(I + BA)^{-1}y_n\}$ was Cauchy when $\{y_n\}$ was Cauchy, actually provides the following more general statement concerning the range of $I + BA$.

THEOREM 2^0. *In Theorem 2, replace the condition BA accretive by the condition $(I + BA)^{-1}$ continuous; then $R[I + BA] = X$.*

Let us note in particular the following special case, of possible interest in the theory of noncompact Hamerstein equations.

COROLLARY 2. *Let X be reflexive, let K and F both be multivalued, nonlinear, maximal monotone, $D(K) = X$, $D(F) = X^*$. If $(I + KF)^{-1}$ is continuous, the equation*

$$(I + KF)u = f$$

possesses a solution for every f in X.

Proof of Theorem 3. Given y in X, we seek an x such that $y \in (I + BA)\,x$. Let $C_y(x) = -C(y - x)$ for all x in X; it follows that $A + C_y$ is m-accretive by Corollary 1 (of Lemma 1), or by [23, Corollary 10.3] when X^* is uniformly convex and C is hemicontinuous and accretive. Consequently there exists an x such that $Ax + C_y(x) + \delta x$ contains δy, i.e., $(C + \delta)(y - x) \in Ax$, so that this is the required x.

Remark 4. We note that Theorem 3 yields $R[I + BA] = X$ without BA being accretive. When X is a Hilbert space Theorem 3 contains for example the linear result of [14]. The condition on B in Theorem 3 is essentially that B be "strongly coaccretive" (see [4, section 3]; also see [14, Corollary 4, and the following Remark there]).

Proof of Theorem 4. Given y in X it will follow that $y \in (I + BA)\,x$ for some x if we have for some x_0 a solution of $(d/dt)\,x(t) \in y - x(t) - BAx(t)$, $x(0) = x_0$. For if so, let

$$\frac{d}{dt}\,x(t) = y - x(t) - Bz(t), \qquad z(t) \text{ in } Ax(t), \qquad x_n = x(t_n), \qquad z_n = z(t_n),$$

$t_n \to \infty$ as $n \to \infty$; since $\|(d/dt)\,x(t)\|$ decreases exponentially there exists $x \in X$ such that $x_n \to x$. The sequence z_n is bounded because B^{-1} is bounded. Since X^* is uniformly convex and A is m-accretive, one has (e.g., see [21]) $x \in D(A)$ and $z_n \rightharpoonup z$ for some z in Ax; since

Bz_n is bounded and B is weakly closed, $Bz_n \rightharpoonup Bz$. It then follows from $x_n + Bz_n \to y$ that $x + Bz = y$.

Consider x_0 in $D(A)$ and (using the techniques of [4, Theorem 9.23]) for $\epsilon > 0$ set $A_\epsilon = \epsilon^{-1}(I - (I + \epsilon A)^{-1})$; note that A_ϵ is accretive and Lipschitzian. By [22], B is demicontinuous; thus BA_ϵ is demicontinuous and accretive, so that by [4, 23] a solution x_ϵ of $(d/dt)\, x_\epsilon(t) = y - x_\epsilon(t) - BA_\epsilon x_\epsilon(t)$ exists, such that $x_\epsilon(0) = x_0$. By accretivity,

$$\left\| \frac{d}{dt} x_\epsilon(t) \right\| \leqslant \left\| \frac{d}{dt} x_\epsilon(0) \right\| \leqslant \| y \| + \| x_0 \| + \| BA_\epsilon x_0 \|, \tag{2.5}$$

and since A is accretive, $\| A_\epsilon x_0 \| \leqslant \| z_0 \|$ for any z_0 in Ax_0. Since B is bounded, $\| BA_\epsilon x_0 \|$ is bounded independently of ϵ; consequently $\| (d/dt)\, x_\epsilon(t) \|$ is bounded for $t \geqslant 0$ independently of ϵ.

For arbitrary but fixed $t_0 > 0$, by the above there is a constant k_0 such that $\| x_\epsilon(t) \| \leqslant k_0$ for $t \leqslant t_0$ and $\epsilon > 0$; since $y - x_\epsilon(t) - BA_\epsilon x_\epsilon(t)$ is bounded independently of ϵ, there is therefore a constant k_1 such that $\| BA_\epsilon x_\epsilon(t) \| \leqslant k_1$ for $t \leqslant t_0$ and $\epsilon > 0$. Further, let $v_\epsilon(t) = (I + \epsilon A)^{-1}\, x_\epsilon(t)$; since B^{-1} is bounded and $B\epsilon^{-1}(x_\epsilon(t) - v_\epsilon(t)) = BA_\epsilon x_\epsilon(t)$, there is a constant k_2 such that $\| x_\epsilon(t) - v_\epsilon(t) \| \leqslant \epsilon k_2$ for $t \leqslant t_0$ and $\epsilon > 0$. Finally, recall that (e.g., see [4, 21]) since X^* is uniformly convex there is a function $f : R^+ \to R^+$, $f(s) \to 0$ as $s \to 0$, such that for $\| x \| \leqslant k_0$, $\| y \| \leqslant k_0$ one has $\| Jx - Jy \| \leqslant f(s)$ whenever $\| x - y \| \leqslant s$.

Hence for arbitrary positive ϵ and δ we have from the above, using the accretivity of BA appropriately as in [4, proof of Theorem 9.23], that

$$\begin{aligned} &\frac{d}{dt} \| x_\epsilon(t) - x_\delta(t) \|^2 \\ &\qquad \leqslant -2(J(x_\epsilon(t) - x_\delta(t)), BA_\epsilon x_\epsilon(t) - BA_\delta x_\delta(t)) \\ &\qquad \leqslant -2(J(x_\epsilon(t) - x_\delta(t)) - J(v_\epsilon(t) - v_\delta(t)), BA_\epsilon x_\epsilon(t) - BA_\delta x_\delta(t)) \\ &\qquad \leqslant 4k_1 f(k_2(\epsilon + \delta)). \end{aligned} \tag{2.6}$$

It follows from (2.6) and $x_\epsilon(0) - x_\delta(0) = 0$ that for $t \leqslant t_0$, $\| x_\epsilon(t) - x_\delta(t) \|^2 \leqslant 4t_0 k_1 f(k_2(\epsilon + \delta))$; consequently as $\epsilon \to 0$, x_ϵ converges strongly, uniformly for t in $[0, t_0]$, to a continuous function $x : [0, t_0] \to X$.

Considering now a given t in $[0, t_0]$ and $\epsilon_n \to 0$ as $n \to \infty$, we have $x_{\epsilon_n}(t) \to x(t)$ and $v_{\epsilon_n}(t) \to x(t)$. Since $BA_{\epsilon_n} x_{\epsilon_n}(t)$ and $A_{\epsilon_n} x_{\epsilon_n}(t)$ are bounded in the reflexive space C, there exists a subsequence, denoted ϵ_n again, and u and w in X, such that the former sequence converges weakly to u and the latter converges weakly to w. Because B is weakly closed, $u = Bw$, and since A is demiclosed (e.g., see [4 or 23]), w is in $Ax(t)$. Proceeding as, for example, in [4, Theorem 9.23] we obtain $x(t)$ weakly continuously differentiable with $x(0) = x_0$ and $(d/dt)\, x(t) \in y - x(t) - BAx(t)$.

Remark 5. Accretivity of B was used in the above only to conclude in combination with the hemicontinuity of B that B was demicontinuous, so the latter (demicontinuity) would suffice for the proof. On the other hand, if X is a Hilbert space and both B and BA are accretive, then for $\alpha > 0$, $B\alpha^{-1}(I - (I + \alpha A)^{-1})$ is accretive; i.e., letting $u = (I + \alpha A)^{-1}x$, $w = (I + \alpha A)^{-1}y$, then

$$\begin{aligned}(B\alpha^{-1}(x - u) - B\alpha^{-1}(y - w), x - y)\\ = (B\alpha^{-1}(x - u) - B\alpha^{-1}(y - w), (x - u) - (y - w))\\ + (B\alpha^{-1}(x - u) - B\alpha^{-1}(y - w), u - w),\end{aligned} \tag{2.7}$$

the first term being nonnegative by B accretive, the second term being nonnegative by $B\alpha^{-1}(x - u) \in BAu$ and $B\alpha^{-1}(y - w) \in BAw$. In this case Theorem 4 (with B strongly accretive) therefore contains (as did Theorems 1, 2, 3) the basic linear result of [14].

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