Polynomial algorithm for sharp upper bound of rainbow connection number of maximal outerplanar graphs

Xing-Chao Deng a, *, Kai-Nan Xiang b, Baoyindureng Wu c

a Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin City, 300071, PR China
b School of Mathematical Sciences, LPMC, Nankai University, Tianjin City, 300071, PR China
c College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang 830046, PR China

A R T I C L E   I N F O
Article history:
Received 1 February 2011
Received in revised form 1 August 2011
Accepted 4 August 2011

Keywords:
Rainbow connection number
Rainbow coloring
Maximal outerplanar graph
Maximal cardinality search

A B S T R A C T
For a finite simple edge-colored connected graph G (the coloring may not be proper), a rainbow path in G is a path without two edges colored the same; G is rainbow connected if for any two vertices of G, there is a rainbow path connecting them. Rainbow connection number, rc(G), of G is the minimum number of colors needed to color its edges such that G is rainbow connected. Chakraborty et al. (2011) [5] proved that computing rc(G) is NP-hard and deciding if rc(G) = 2 is NP-complete. When edges of G are colored with fixed number k of colors, Kratochvil [6] proposed a question: what is the complexity of deciding whether G is rainbow connected? Is this an FPT problem? In this paper, we prove that any maximal outerplanar graph is k rainbow connected for suitably large k and can be given a rainbow coloring in polynomial time.

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1. Introduction

All finite graphs considered in this paper are simple and connected. For notations and terminologies not defined here, see West [1]. Let G be a nontrivial finite simple connected graph on which is assigned a coloring c : E(G) → {1, 2, . . . , n}, n ∈ N, of the edges of G, where adjacent edges may be colored by the same color. A rainbow path in G is a path without two edges being colored with same color. If for any two vertices of G, there is a rainbow path connecting them, then G is called rainbow connected and the coloring c is called a rainbow coloring. Obviously, any G has a trivial rainbow coloring. Chartrand et al. [2] defined the rainbow connection number rc(G) of graph G as the smallest number of colors that are needed in order to make G rainbow connected. Let diam(G) be the diameter of G and m the size of G. Then diam(G) ≤ rc(G) ≤ m.

From [2], rainbow connection number of any complete graph is 1 and that of a tree is its size; rc(Wn) = 1 if n = 3, rc(Wn) = 2 if 4 ≤ n ≤ 6, rc(Wn) = 3 if n ≥ 7. Here Wn = Cn ∨ K1, the join of Cn and K1, is a wheel. Chandran et al. [3] studied the relation between rainbow connection numbers and connected dominating sets, and proved the following results: (i) For any bridgeless chordal graph G, rc(G) ≤ 3rad(G), where rad(G) is the radius of G. Moreover, there is a bridgeless chordal graph G with rc(G) = 3rad(G). (ii) For any unit interval graph G with δ(G) ≥ 2, rc(G) = diam(G).

Recall an outerplanar graph is a planar graph which has a plane embedding with all vertices placed on the boundary of a face, usually taken to be the exterior one. A maximal outerplanar graph (MOP) is an outerplanar graph which cannot be added any line without losing outerplanarity.

By [4], an MOP can be recursively defined as follows: (a) The graph K3 is an MOP. (b) For an MOP H1 embedded in the plane with vertex lines on the exterior face F1, let H2 be obtained by joining a new vertex to the two vertices of an edge on F1. Then H2 is an MOP. (c) Any MOP can be constructed by finite steps of applications of above (a) and (b).
Clearly, Propositions 3.4–3.7 are consequences of the following theorems.

Theorem 3.1. For a connected graph G with at least 2 vertices, it is an MOP iff the following hold: (a) for any vertex v of G, its neighbors induce a path in G, (b) G is 2-degenerate.

Corollary 3.2. Any vertex v of an MOP G and its neighbors form a maximal Fan in G with v as the central vertex.
A maximal Fan partition (MFP) of an MOP $G$ is a set $\{F_i\}_{i=1}^{k}$ of maximal Fans in $G$ such that their union is $G$, and the central vertex of each $F_i$ is not in any other $F_j$.

Notice $G$ has a following MFP. First, choose any maximal Fan in $G$ as $F_1$. If $G$ has a vertex $v_2$ not in $F_1$, then let the maximal Fan in $G$ formed by $v_2$ and its neighbors be $F_2$ (see Corollary 3.2). Note the central vertex of $F_1$ is not in $F_2$. Otherwise, there is a common edge that is a spoke edge in $F_1$ and a path edge in $F_2$. This contradicts to Lemma 3.3. Repeating such a procedure finite times, we obtain an MFP $\{F_i\}_{i=1}^{k}$. So MFP is well-defined for any MOP. Generally, MFP of $G$ may not be unique.

**Lemma 3.3.** For any two maximal Fans $F_1$ and $F_2$ of an MOP $G$ with the central vertex of $F_2$ not in $F_1$, they cannot have a common edge which is a spoke edge in $F_1$ and a path edge in $F_2$.

**Proof.** Assume $F_1$ and $F_2$ have a common edge $v_1v_3$ which is a spoke edge in $F_1$ and a path edge in $F_2$. Here $v_1$ is the central vertex of $F_1$. Note the central vertex $v_4$ of $F_2$ is not in $F_1$. Let $F_1 = K_1 \lor P_n := \{v_1\} \lor (w_1w_2\cdots w_n)$. Then both $v_4, w_1, w_2, \ldots, w_n$ and $v_4, w_n, w_{n-1}, \ldots, w_1$ cannot be a path in $G$; otherwise, either $\{v_1\} \lor (v_4w_1w_2\cdots w_n)$ or $\{v_1\} \lor (v_4w_nw_{n-1}\cdots w_1)$ will be a Fan, which contradicts to maximality of $F_1$. Thus, $v_3$ is an interior vertex of path $P_n$. Let the neighbors of $v_3$ in $P_n$ be $v_2$ and $v_5$. The induced subgraph on $\{v_1, \ldots, v_5\}$ has a $K_{2,3}$ minor (see Fig. 9). A contradiction. \(\square\)

**Proposition 3.4.** Given an MFP $\{F_i\}_{i=1}^{k}$ of an MOP $G$. Then any two $F_i$ and $F_j$ ($i \neq j$) have at most one common pathedge; any edge of $G$ can be a common pathedge of at most two $F_i$.

**Proof.** Assume there are two $F_i$ and $F_j$ ($i \neq j$) having 2 common path edges, then these path edges have at least 3 vertices, say $v_2, v_3, v_4$. (In Fig. 7, the 2 common path edges have 3 vertices due to adjacency. But they need not to be adjacent and may have 4 vertices.) Let central vertices of $F_i$ and $F_j$ be $v_1$ and $v_5$, respectively. Then induced subgraph on $\{v_1, \ldots, v_5\}$ has a $K_{2,3}$ minor, which is impossible!
Let $G$ be an MOP with an MFP $\{F_i\}_{i=1}^k$ ($k \geq 2$). If $F_1$ and $F_2$ have 2 common vertices, then the 2 vertices form a common path edge of $F_1$ and $F_2$. 

**Proposition 3.5.** Let $G$ be an MOP with an MFP $\{F_i\}_{i=1}^k$ ($k \geq 2$). If $F_1$ and $F_2$ have 2 common vertices, then the 2 vertices form a common path edge of $F_1$ and $F_2$. 

**Proof.** For $F_1$ and $F_2$, let $v_1$ and $v_3$ be their central vertices respectively, and $v_2$ and $v_4$ the 2 common vertices. If $v_2$ and $v_4$ cannot form an edge in $F_1$ or $F_2$, say in $F_1$, then there must be a path $(v_2v_5w_1\cdots w_tv_4)$ in $F_1$ and induced subgraph of $G$ on $\{v_1, v_2, v_3, v_4, v_5, w_1, \ldots, w_7\}$ have a $K_{2,3}$ minor with vertex set $\{v_1, \ldots, v_5\}$ (see Fig. 12). This is impossible. So $v_2$ and $v_4$ form an edge in both $F_1$ and $F_2$. Clearly, this common edge is a path edge of $F_1$ and $F_2$. 

**Proposition 3.6.** Let $G$ be an MOP with an MFP $\{F_i\}_{i=1}^k$ ($k \geq 3$). If $F_1$ and $F_2$ have a common vertex, then any $F_j$ ($3 \leq j \leq k$) has at most 3 common vertices with $F_1 \cup F_2$. 

**Proof.** We only prove

$$|V(F_2) \cap (V(F_1) \cup V(F_2))| \leq 3.$$  

(3.1)

For convenience, let $|V(F_2) \cap V(F_1)| \leq |V(F_2) \cap V(F_1)|$. If $|V(F_2) \cap V(F_2)| \geq 3$, then by Proposition 3.5, $F_1$ and $F_2$ have not less than 2 common path edges. This contradicts to Proposition 3.4. So

$$|V(F_2) \cap V(F_1)| \leq |V(F_2) \cap V(F_2)| \leq 2.$$  

(3.2)

Assume (3.1) is not true. By Propositions 3.4 and 3.5, $F_1$ and $F_2$ have at most 2 common vertices (one is $v_1$). Then by (3.2), $|V(F_2) \cap V(F_1)| = |V(F_2) \cap V(F_2)| = 2$, and any common vertex of $F_1$ and $F_2$ is not in $F_3$. Let $V(F_2) \cap V(F_1) = \{v_2, v_3\}$, $V(F_2) \cap V(F_2) = \{v_5, v_7\}$, and $(v_1u_1\cdots u_7v_2)$ be the $v_1$-$v_2$ path in $F_1$ not containing the central vertex $v$ of $F_1$. Let $(\cdots v_2v_3w_{11}\cdots w_5v_7\cdots)$ be the path of $F_2$, $v_6$ and $v_9$ the central vertices of $F_2$ and $F_3$, respectively. See Figs. 10 and 11. Then induced subgraph of $G$ on $\{v_1, u_1, \ldots, u_7, v_2, u_1, \ldots, w_5, v_4, v_6, v_7\}$ have a $K_{2,3}$ minor with vertex set $\{v_2, v_4, v_5, v_6, v_7\}$ (Figs. 10 and 11). A contradiction. Thus (3.1) holds. 

Let $G$ be an MOP with an MFP $\{F_i\}_{i=1}^k$ ($k \geq 2$). Then by Proposition 3.5, any two $F_i$ and $F_j$ ($i \neq j$) having 2 common vertices have a common path edge. By Proposition 3.6, if $F_1$ and $F_2$ have a common vertex, then any other $F_j$ (if exists) has at most a
common path edge with $F_1$ or $F_2$. Thus we have

**Proposition 3.7.** If $G$ is an MOP with an MFP $\{F_i\}_{i=1}^k$, then $rc(G) \leq \sum_{i=1}^k rc(F_i)$.

4. The computation of the upper bound of MOPs

For a graph $G$, a vertex is called simplicial if its neighborhood induces a clique in $G$. A simplicial elimination ordering (perfect elimination ordering) is an ordering $v_n, \ldots, v_2, v_1$ for the deletion of vertices so that each vertex $v_i$ is simplicial in the remaining graph induced by $\{v_1, \ldots, v_i\}$, whose reverse is a simplicial construction ordering of $G$. The simplicial construction ordering of a chordal graph $G$ can be found by Maximum Cardinality Search (MCS) in time $O(n(G) + e(G))$, where $n(G)$ and $e(G)$ are vertex number and edge number of $G$, respectively.
The MCS algorithm is a simple linear time algorithm that processes first the vertex $x$ for which $f(x) = 1$, where $f : V(G) \rightarrow \{1, \ldots, n(G)\}$ is a function, and continues generating an elimination ordering in reverse. In addition, it maintains, for each vertex $v$, an integer weight $l(v)$ that is the cardinality of the already processed neighbors of $v$; produces a simplicial construction ordering when a chordal graph is the input (see Step 2 of our MCS-$R$ algorithm).

Note an MOP $G$ can be embedded in the plane such that every vertex lines on the boundary of the exterior face, all exterior edges form a Hamiltonian cycle $[v_1 v_2 \cdots v_n v_1]$, and Hamiltonian degree sequence $D = (d_1, d_2, \ldots, d_n, d_1)$ of $G$ is the degree sequence of vertices $v_1, v_2, \ldots, v_n, v_1$. Recall [9] gave an Algorithm MOP which takes a Hamiltonian degree sequence and produces the unique corresponding maximal outerplanar graph in linear time.

**Theorem 4.1** [9]. An MOP $G$ is determined uniquely up to isomorphisms by its Hamiltonian degree sequence $D = (d_1, d_2, \ldots, d_n, d_1)$.

Algorithm computing the rainbow number bound and giving a rainbow coloring of maximal outerplanar graph (MCS-R):

**Input:** A maximal outerplanar graph $G$.

**Output:** A bound of $rc(G)$ and a rainbow coloring of $G$.

**Step 1:** Finding a Hamiltonian degree sequence $D = (d_1, d_2, \ldots, d_n, d_1)$ of $G$.

**Step 2:** In this step, we have a simplicial construction ordering of vertices.

begin
  for all vertices $v$ in $G$ do $l(v) = 0$; for $i = 1$ up to $n$ do
  Choose an unnumbered vertex $z$ of maximum weight; $f(z) = i$;
  for all unnumbered vertices $y \in N(z)$ do $l(y) = l(y) + 1$;
end

**Step 3:** In this step, we have all maximal Fans of $G$.

begin
  for $i = 1$ up to $n$ do $i : N(f^{-1}(i)) = \{f^{-1}(i)\}$;
  for $i = 1$ up to $n$
    if $\{f^{-1}(i)\} < d_i - 1 + 1$ do
      for $j = 1, \ldots, i - 1, i + 1, \ldots, n$ do
        if $f^{-1}(j)$ and $f^{-1}(i)$ are adjacent, $N(f^{-1}(i)) \leftarrow N(f^{-1}(i)) \cup \{f^{-1}(j)\}$, $j \leftarrow j + 1$;
        otherwise $j \leftarrow j + 1$; else $i \leftarrow i + 1$;
    end
end

**Step 4:** Computing the rainbow number and giving a rainbow coloring for $G$.

Given a down ordering of $\{N(f^{-1}(i))\}$ such that when $s < t$, $|N(f^{-1}(i_s))| \geq |N(f^{-1}(i_t))|$. Note $N(f^{-1}(i_1)) \setminus \{f^{-1}(i_s)\}$ and $N(f^{-1}(i_s))$ form respectively a path and a maximal Fan.

begin
  $n_2 \leftarrow 0$, $n_3 \leftarrow 0$, $n_7 \leftarrow 0$, and $N = N(f^{-1}(i_1))$;
  if $|N(f^{-1}(i_s))| = 3$, then $n_2 = 1$ and color it as Proposition 2.1 showing;
  if $4 \leq |N(f^{-1}(i_s))| \leq 7$, then $n_3 = 1$ and color it as Proposition 2.1 showing;
  if $|N(f^{-1}(i_s))| \geq 8$, then $n_7 = 1$ and color it as Proposition 2.1 showing;
  for $s = 2$ up to $n$ do
    if $N \cap N(f^{-1}(i_s)) \neq \emptyset$, $f^{-1}(i_s) \notin N$, $N(f^{-1}(i_s))$ intersects any $N(f^{-1}(i_s))$ from $N$ with a common path edge (Propositions 3.4 and 3.5) or at most 1 vertex, and $N(f^{-1}(i_s))$ and any two $N(f^{-1}(i_s))$ and $N(f^{-1}(i_s))$ from $N$ satisfy Proposition 3.6, then $N \leftarrow N \cup N(f^{-1}(i_s))$;
    if $|N(f^{-1}(i_s))| = 3$, then $n_2 \leftarrow n_2 + 1$ and color it with new colors by Proposition 2.1;
    if $4 \leq |N(f^{-1}(i_s))| \leq 7$, then $n_3 \leftarrow n_3 + 1$ and color it with new colors by Proposition 2.1;
    if $|N(f^{-1}(i_s))| \geq 8$, then $n_7 \leftarrow n_7 + 1$ and color it with new colors by Proposition 2.1;
end

**Lemma 4.2.** The graph $N$ in Step 4 of the algorithm is the MOP $G$ up to isomorphisms.

**Proof.** Notice $N$ contains every vertex of $G$, and has the same Hamiltonian degree sequence up to cyclic permutations as that of $G$. By Theorem 4.1, $N$ is the MOP $G$ up to isomorphisms. □

**Lemma 4.3.** The coloring given by the algorithm is a rainbow coloring of the MOP $G$.

**Proof.** Note the $n_2 + n_3 + n_7$ resulting Fans $\{F_i\}_{i=1}^{n_2+n_3+n_7}$ in Step 4 of the algorithm form an MFP of $G$. Given any two different vertices $u$ and $v$. They are rainbow connected if they belong to some $F_i$, and connected by a path whose edges belong to some Fans from $\{F_i\}_{i=1}^{n_2+n_3+n_7}$ otherwise. Since in the algorithm, the color sets of different $F_i$s are disjointed, and vertices in a fixed $F_i$ are rainbow connected, the just mentioned path in the latter case is a rainbow path. □

**Theorem 4.4.** For the MOP $G$, $rc(G) \leq n_2 + 2n_3 + 3n_7 := \ell$, and it can be given a rainbow coloring (clearly $k$-rainbow coloring for any $k \geq \ell$) in polynomial time.

**Proof.** By Propositions 2.1 and 3.7, and Lemma 4.2, $rc(G) \leq \ell$. For the rest, see Section 6. □
Fig. 13. A maximal outerplanar graph with a simplicial construction ordering and a rainbow coloring.

Fig. 14. MOP^n.

Table 1
The outcome of Step 3 in the algorithm MCS-R of the example.

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5. An example

In row c of Table 1, the numbers 1, 2, 3, 4 are obtained by enumerating neighbors of c in the simplicial construction ordering, and the last number is just the degree of c. Note N(c) = [k, b, a, d] ∪ {c} forms a maximal Fan with c as the central vertex. Other rows are similar to row c.

In Step 4, first choose N(a) = {f, b, c, d, e, h} ∪ {a} and color it with colors c1 and c2; next let N(g) = {e, d, i, l} ∪ {g} be the second Fan and colored by colors c3 and c4; finally, choose N(k) = {b, c} ∪ {k} and N(j) = {f, b} ∪ {j} as the last two Fans and color them by c5 and c6, respectively. See Fig. 13. So rainbow connection number of the graph is not more than 2 + 2 + 1 + 1 = 6.

**Theorem 5.1.** For any $n \geq 1$, there is an MOP $M_n$ such that $rc(M_n) = diam(M_n) = 2n$.

**Proof.** In Fig. 14, we show an MOP^n which consists of n Fans. It is an MOP of diameter 2n. By Theorem 4.4, $rc(MOP^n) = 2n$. □

6. Concluding remarks

The Hamiltonian cycle $C_G$ of MOP G can be obtained by a linear time algorithm presented in [10] through the canonical representation of G. Then select any vertex of G as the initial vertex of $C_G$, we can obtain a Hamiltonian degree sequence in time $O(n^3)$, where $n = n(G)$. Note the time of Step 2 is $O(n + e)$ which is not crucial, but may reduce time in Step 3. Here $e = e(G)$. Step 3 takes time at most $O(n^3)$ and Step 4 takes time at most $O(n^2)$. Note Lemma 4.3. Then we have a polynomial time algorithm to compute an upper bound of $rc(G)$ and to give G a rainbow coloring. By Theorem 5.1, our result is the best possible for MOPs.
Acknowledgments

We thank the referees for their valuable comments which improve the quality of the paper greatly.

References