Perturbation of the Diffusion and Upper Semicontinuity of Attractors

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Abstract—We obtain uniform bounds for attractors of parabolic problems with nonlinear boundary conditions with respect to perturbations in the diffusion coefficients. These bounds are the main tool to obtain continuity properties of the attractors relatively to perturbations of the diffusion coefficient. We study two examples where the diffusion is perturbed: one related to homogenization theory and the other taken from composite materials where the diffusion goes to infinity in some subdomain. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND SETTING OF THE PROBLEM

Let \( \Omega \subset \mathbb{R}^N \) be an open bounded domain with boundary \( \Gamma \). Assume \( \Gamma = \Gamma_0 \cup \Gamma_1 \) is a regular partition of the boundary, that is, \( \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \). Note that either \( \Gamma_0 \) or \( \Gamma_1 \) could be empty. Let \( a \)
be a $C^1(\Omega)$ function such that $a(x) \geq m > 0$ for all $x \in \Omega$. Consider the parabolic problem

$$
\begin{align*}
    u_t - \text{div}(a(x)\nabla u) &= f(x,u), & \text{in } \Omega, \\
    u &= 0, & \text{on } \Gamma_0, \\
    a(x)\frac{\partial u}{\partial n} &= g(x,u), & \text{on } \Gamma_1, \\
    u(0) &= u_0,
\end{align*}
$$

where $f(x,\cdot), g(x,\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are locally Lipschitz functions uniformly in $x \in \bar{\Omega}$ and $x \in \Gamma_1$.

We consider (1.1) in the spaces $X = L^q(\Omega)$ or $W^{1,q}_{\text{loc}}(\Omega)$, $1 < q < \infty$, where $W^{1,q}_{\text{loc}}(\Omega) = \{u \in W^{1,q}(\Omega) : u = 0 \text{ in } \Gamma_0\}$. We refer the reader to [1,2] for details.

Concerning the nonlinear terms $f$ and $g$, it was shown in [2] that associated to any of these spaces $X$, there exist suitable growth restrictions on the nonlinearities $f$ and $g$, that we will call $(G)_X$, such that problem (1.1) is locally well posed in $X$. These growth restrictions are expressed as follows.

**Restrictions $(G)_X$.** Let $f(x,\cdot), g(x,\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be locally Lipschitz functions, uniformly on $x \in \Omega$ and $x \in \Gamma_1$, respectively. Assume the following.

1. If $X = L^q(\Omega)$, assume that $f$ and $g$ satisfy a relation of the form

   $$
   |f(x,u) - f(x,v)| \leq c|u - v|(|u|^\rho_f - 1 + |v|^\rho_f - 1 + 1),
   $$

   with exponents $\rho_f$ and $\rho_g$, respectively, such that, with $N \geq 2$ (respectively, $N = 1$)

   $$
   \rho_f \leq 1 + \frac{2q}{N} \quad \text{and} \quad \rho_g \leq 1 + \frac{q}{N},
   $$

   (respectively, $\rho_g < 1 + q$).

2. If $X = W^{1,q}_{\text{loc}}(\Omega)$, assume either

   (i) $q > N$,

   (ii) $q = N$ and $f, g$ satisfy that for every $\eta > 0$, there exists $c_\eta > 0$ such that

   $$
   |f(x,u) - f(x,v)| \leq c_\eta \left(\varepsilon_\eta |u|^{(N/N-1)} + \varepsilon_\eta |v|^{(N/N-1)}\right) |u - v|,
   $$

   or

   (iii) $1 < q < N$ and $f, g$ satisfy (1.2), with exponents $\rho_f$ and $\rho_g$, respectively, such that

   $$
   \rho_f \leq 1 + \frac{2q}{N - q} \quad \text{and} \quad \rho_g \leq 1 + \frac{q}{N - q}.
   $$

The results from [2] can be summarized as follows,

**Theorem 1.1.** If $X = L^q(\Omega)$ or $W^{1,q}_{\Gamma_0}(\Omega)$, $1 < q < \infty$ and $f, g$ satisfy the Growth Restriction $(G)_X$, then for any $u_0 \in X$ there exists locally a unique (in certain sense) mild solution $u(\cdot, u_0) \in C([0,\tau), X)$, of problem (1.1). This solution depends continuously on the initial data $u_0 \in X$ and it is a classical solution for $t > 0$. Also, the following regularizing effects takes place: $u(t, u_0) \in W^{1,r}_{\Gamma_0}(\Omega)$ for any $r \geq q$ and $t \in (0, \tau)$.

2. **GLOBAL EXISTENCE AND ATTRACTORS**

In order to obtain that all solutions of (1.1) are globally defined, we will assume some sign conditions on the nonlinear terms. These sign conditions are independent of the space $X$ and can be expressed in the following form.

(S) Assume there exist $B_0, C_0 \in \mathbb{R}$ and $B_1, C_1 \geq 0$ such that the following holds:

$$
\begin{align*}
    u f(x,u) &\leq -C_0u^2 + C_1|u|, \\
    u g(x,u) &\leq -B_0u^2 + B_1|u|,
\end{align*}
$$

for $u \in \mathbb{R}$, $x \in \Omega$ and $x \in \Gamma_1$, respectively.

Then we have the following result on global existence.
THEOREM 2.1. Let \( X = L^q(\Omega) \) or \( W_{\Gamma_0}^{1,q}(\Omega) \), \( 1 < q < \infty \). Assume that Growth Condition \((G)_X\) and the sign condition \((S)\) hold. Then, for any \( u_0 \in X \), the solution \( u(t, u_0) \) of (1.1) starting at \( u_0 \) exists for all \( t \geq 0 \).

This result allows us to define in \( X \) the semigroup \( \{S(t), t \geq 0\} \) associated to (1.1), by \( S(t)u_0 = u(t, u_0), t \geq 0 \).

To prove the existence of a global attractor for problem (1.1) we will impose, besides \((S)\), some dissipation condition on the flow given by (1.1). This dissipative condition is expressed as follows.

\[ \text{(D) Assume \((S)\) holds and that with } C_0 \text{ and } B_0 \text{ from } (S), \text{ the first eigenvalue, } \lambda_1, \text{ of the following problem is positive} \]

\[ \begin{align*}
-\text{div}(a(x)\nabla u) + C_0 u &= \lambda u, & \text{in } \Omega \\
u &= 0, & \text{on } \Gamma_0, \\
a(x) \frac{\partial u}{\partial n} + B_0 u &= 0, & \text{on } \Gamma_1.
\end{align*} \] (2.2)

With all these, it is possible to show the following result.

THEOREM 2.2. Let \( X = L^q(\Omega) \) or \( W_{\Gamma_0}^{1,q}(\Omega) \), \( 1 < q < \infty \), and assume that Growth Condition \((G)_X\) and dissipative condition \((D)\) hold. Then we have the following.

- The semigroup \( \{S(t), t \geq 0\} \) associated to (1.1) has a global attractor, \( A_X \), in \( X \). Moreover, \( A_X \) is a compact subset of \( W_{\Gamma_0}^{1,r}(\Omega) \) for any \( r \geq q \).
- For any other space \( Y = L^r(\Omega) \) or \( W_{\Gamma_0}^{1,r}(\Omega) \), \( 1 < r < \infty \), with \( Y \hookrightarrow X \), we have the existence of the attractor \( A_Y \). Moreover, \( A_X = A_Y \) and it attracts bounded sets of \( X \) in the topology of \( Y \).

3. UNIFORM BOUNDS ON THE ATTRACTORS

Now we obtain bounds on the attractors which are uniform with respect to variations on the diffusion coefficient \( a \). In this direction, a very important first result is a pointwise bound on the attractor \( A_X \) given by the following result.

PROPOSITION 3.1. Let \( X = L^q(\Omega) \) or \( W_{\Gamma_0}^{1,q}(\Omega) \), \( 1 < q < \infty \), and assume that Growth Restriction \((G)_X\) and dissipative condition \((D)\) hold. Denote by \( \phi \) the solution of

\[ \begin{align*}
-\text{div}(a(x)\nabla \phi) + C_0 \phi &= C_1, & \text{in } \Omega, \\
\phi &= 0, & \text{on } \Gamma_0, \\
a(x) \frac{\partial \phi}{\partial n} + B_0 \phi &= B_1, & \text{on } \Gamma_1.
\end{align*} \] (3.1)

Then, \( 0 \leq \phi \leq L^\infty(\Omega), \lim_{t \to \infty} |u(t, x, u_0)| \leq \phi(x) \), uniformly in \( x \in \tilde{\Omega} \) and for \( u_0 \) in bounded subsets of \( X \). In particular, we have that for any \( v \in A_X \) we have \( |v(x)| \leq \phi(x) \).

In fact, we will prove the following.

THEOREM 3.2. Let \( X = L^q(\Omega) \) or \( W_{\Gamma_0}^{1,q}(\Omega) \), \( 1 < q < \infty \), and assume that Growth Restriction \((G)_X\) and dissipative condition \((D)\) hold; then, we have the following.

\( (i) \) \( \sup_{v \in A_X} \{ \|v\|_{L^\infty(\Omega)} \} \leq K_0 \), where \( K_0 \) depends on the domain \( \Omega \), the first eigenvalue of (2.2), \( \lambda_1, m = \inf\{a(x) : x \in \Omega\} > 0, C_0, C_1, B_0, \) and \( B_1 \). Furthermore, if one of the following two conditions hold:

- \( B_0 > 0 \),
- \( B_0 = B_1 = 0 \) and \( C_0 > 0 \),

then \( K_0 \) depends on \( |\Omega| \) rather than on \( \Omega \).
(ii) We have
\[ \sup_{v \in \mathcal{A}_X} \|v\|_{H^1(\Omega)} \leq K, \]
for some constant \( K \) that depends on \( \Omega, m, \lambda_1, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, f \) and \( g \).

(iii) Let \( u(t, u_0) \) be the solution to (1.1) starting at \( u_0 \in \mathcal{A}_X \). Then
\[ \sup_{u_0 \in \mathcal{A}_X} \sup_{t \in \mathbb{R}} \{\|u(t, u_0)\|_{H^1(\Omega)} + \|u(t, u_0)\|_{L^\infty(\Omega)}\} \leq K, \]
for some constant \( K \) as in (ii).

(iv) For some \( \nu \in (0, 1) \), \( \sup_{u \in \mathcal{A}_X} \{\|u\|_{C^\nu(\Omega)}\} \leq K_\nu \), where \( K_\nu \) depends on the same as \( K_0 \) plus on \( \nu \), on \( M = \sup\{a(x); x \in \Omega\} \), on \( \sup\{\|f'(\psi)\|_{L^\infty(\Omega)} : \psi \in \mathcal{A}_X\} \), and on \( \sup\{\|g'(\psi)\|_{L^\infty(\Gamma_1)} : \psi \in \mathcal{A}_X\} \).

The proposition above is proved from Condition (D) and comparison arguments. With this and some uniform estimates obtained through elliptic regularity results in the spirit of [3], we get point (i) of the theorem. This allows us to truncate the nonlinearities and work with globally Lipschitz nonlinearities which let us pose the problem in \( H^1_{f_0}(\Omega) \). With this, we obtain point (ii) and the \( H^1 \) estimate of \( u_t \). Again by comparison, we prove the \( L^\infty \) estimate of \( u_t \). Finally, viewing the equation, for fixed \( t \), as an elliptic problem and with some results on Hölder regularity like the ones in [3] we get point (iv).

4. PERTURBATION OF THE DIFFUSION

The uniform estimates on the attractors obtained in the previous section are the first step in studying the continuity properties of the attractors for a family of problems depending on a parameter. We will focus on upper semicontinuity of the family of attractors. For this concept, we provide the following well-known definition, see [4,5].

DEFINITION 4.1. A family \( \{\mathcal{A}_\epsilon, 0 \leq \epsilon \leq \epsilon_0\} \), \( \epsilon_0 > 0 \), of subsets of a given metric space \( X \) is called upper semicontinuous at \( \epsilon = 0 \) if given \( \eta > 0 \), there is an \( \epsilon_\eta > 0 \) such that \( A_\epsilon \subset N_\eta(A_0) \) for all \( 0 \leq \epsilon \leq \epsilon_\eta \), where \( N_\eta(A_0) \) is an \( \eta \)-neighborhood of \( A_0 \) in \( X \).

We will deal with two different perturbation problems. In both of them, the diffusion coefficient is perturbed. We will denote by \( a_\epsilon \) the perturbed diffusion and by (1.1), the original problem (1.1) with the perturbed diffusion.

PROBLEM 1. We will consider the case in which the perturbed diffusion is uniformly bounded in \( L^\infty \). Actually, assume the following hypothesis on the diffusions \( a_\epsilon \), which is taken from the context of homogenization, see [6,7].

(H) Assume that the family of \( C^1 \)-diffusion coefficients \( a_\epsilon, 0 \leq \epsilon \leq \epsilon_0 \), satisfy that there exists \( m \) and \( M \) positive numbers such that \( m \leq a_\epsilon(x) \leq M \) for all \( x \in \Omega \) and that for every \( f \in L^2(\Omega) \) and \( g \in L^2(\Gamma_1) \), the solution of
\[
-\text{div}(a_\epsilon(x) \nabla u^\epsilon) + u^\epsilon = f, \quad \text{in } \Omega, \\
u^\epsilon = 0, \quad \text{on } \Gamma_0, \\
a_\epsilon(x) \frac{\partial u^\epsilon}{\partial n} = g, \quad \text{on } \Gamma_1
\]
converges weakly in \( H^1_{f_0}(\Omega) \) (respectively, strongly), as \( \epsilon \to 0 \), to the solution of the problem above with \( \epsilon = 0 \).

Observe that a sufficient condition for the strong convergence is that \( a_\epsilon \to a_0 \) strongly in \( L^1(\Omega) \), and that, as example, we can set \( \Omega = B(0, 1) \subset \mathbb{R}^N \) and \( a_\epsilon(x) = 1 + |x|^{1/\epsilon} \) and \( a_0 = 1 \).

From this, we have the following result.
THEOREM 4.2. Assume that the family $a_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$, satisfies convergence hypothesis (H). Let $X = L^q(\Omega)$ or $W^{1,q}_0(\Omega)$, $1 < q < \infty$, and assume that $f$ and $g$ satisfy Growth Restriction $(G)_X$ and sign condition $(S)$. If $(1.1)_0$ satisfies dissipation condition $(D)$, then problem $(1.1)_\varepsilon$ has an attractor $A^X_\varepsilon$. Moreover, the family of attractors $\{A^X_\varepsilon : 0 \leq \varepsilon \leq \varepsilon_0\}$ is upper semicontinuous at $\varepsilon = 0$ in $C(\overline{\Omega})$ and in $H^1_{1,0}(\Omega)$ with the weak topology (respectively, with the strong topology).

PROBLEM 2. In this case, we are concerned with the problem in which the diffusion coefficient becomes large in a subregion which is interior to $\Omega$. Notice that this situation can be found, for example, in composite materials, where the heat diffusion properties can change significantly from one part of the region to another; that is, heat may diffuse much faster in some subregions than in others.

In the following, we borrow the notations from [8]. Let $\Omega_0$ be an interior subdomain of $\Omega$ and let $\Gamma_0 = \partial \Omega_0$. Denote by $\Omega_1 = \Omega \setminus \overline{\Omega_0}$ and note that its boundary is given by $\Gamma_0 \cup \Gamma_0 \cup \Gamma_1$.

The diffusion coefficients $a_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$, are assumed to be regular, bounded functions in $\Omega$ satisfying

$$0 < m \leq a_\varepsilon(x) \leq M,$$

for every $x \in \Omega$ and $0 \leq \varepsilon \leq \varepsilon_0$. We also assume that the diffusion becomes very large on $\Omega_0$ as $\varepsilon$ approaches zero. More precisely, we assume that, as $\varepsilon \to 0$,

$$a_\varepsilon(x) \to \begin{cases} a_0(x), & \text{uniformly on } \Omega_1, \\ \infty, & \text{uniformly on compact subsets of } \Omega_0. \end{cases} \quad (4.1)$$

With these notations, we consider the family of parabolic equations $(1.1)_\varepsilon$. From physical considerations, we intuitively guess that for small values of $\varepsilon$, the solution of problem $(1.1)_\varepsilon$ should be approximately constant on $\Omega_0$ as time increases. Therefore, it seems natural to guess that the limiting problem should be (see [8]):

$$\begin{align*}
&u_t - \text{div}(a_0(x)\nabla u) = f(u), \quad \text{in } \Omega_1, \\
&u = 0, \quad \text{on } \Gamma_0, \\
&a_0 \frac{\partial u}{\partial n_0} = g(u), \quad \text{on } \Gamma_1, \\
&u|_{\Omega_0} = u_0, \quad \text{in } \Omega_0, \\
&u(0) = u_0.
\end{align*} \quad (4.2)$$

Actually, the following result can be proved.

THEOREM 4.3. Assume that the family $a_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$, satisfies convergence hypothesis $(4.1)$. Let $X = L^q(\Omega)$ or $W^{1,q}_0(\Omega)$, $1 < q < \infty$, and assume that $f$ and $g$ satisfy Growth Restriction $(G)_X$ and sign condition $(S)$. Assume that the first eigenvalue of the following problem is $\lambda_1 > 0$:

$$\begin{align*}
&-\text{div}(a_0(x)\nabla u) + C_0 u = \lambda u, \quad \text{in } \Omega_1, \\
&u = 0, \quad \text{on } \Gamma_0, \\
&a_0 \frac{\partial u}{\partial n} + B_0 u = 0, \quad \text{on } \Gamma_1, \\
&\frac{1}{|\Omega_0|} \int_{\Gamma_0} a_0 \frac{\partial u}{\partial n} + C_0 u|_{\Omega_0} = \lambda u|_{\Omega_0}, \\
&u \in H^1_{1,0,\Omega_0}(\Omega),
\end{align*} \quad (4.3)$$

where $H^1_{1,0,\Omega_0}(\Omega) = \{u \in H^1_{1,0}(\Omega) : u \text{ is constant in } \Omega_0\}$. 

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Then problem (1.1) has an attractor $A_X^\epsilon$ in $X$ and (4.2), after a suitable truncation of the nonlinearities, has an attractor $A_0^\epsilon$ in $H_{1,0,\Omega_0}^1(\Omega)$. Moreover, the family of attractors $\{A_X^\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is upper semicontinuous at $\epsilon = 0$ in $C(\bar{\Omega})$ and in $H_{1,0}^1(\Omega)$.

Both results are proved along the following lines. From the eigenvalue condition for the limiting problem, we obtain that Hypothesis (D) holds for problem (1.1), which implies, from Theorem 2.2, the existence of the attractor $A_X$ and from Theorem 3.2, we obtain uniform $H^1$ and $L^\infty$ bounds on $A_X^\epsilon$.

For Problem 1, point (iv) of Theorem 3.2 is satisfied. This estimate gives us a compactness argument that allows us to prove the upper semicontinuity of the attractors in $C(\bar{\Omega})$.

For Problem 2, the lack of uniform Hölder estimates is overcome, in a nontrivial way, with the fact that the functions converge to a constant in $\Omega_0$.

Using these uniform estimates, we consider a family of global solutions lying on the attractors $A_X^\epsilon$ and with the aid of Ascoli-Arzela’s Theorem, we obtain a subsequence which converges to a global and bounded solution of the limiting problem. Therefore, this solution lies on $A_0^\epsilon$ and upper semicontinuity follows, like in [4].

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