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From p-adic to real Grassmannians via the quantum ☆

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Abstract

Let $\mathbf{F}$ be a local field. The action of $\mathrm{GL}_n(\mathbf{F})$ on the Grassmann variety $\mathrm{Gr}(m, n, \mathbf{F})$ induces a continuous representation of the maximal compact subgroup of $\mathrm{GL}_n(\mathbf{F})$ on the space of L^2-functions on $\mathrm{Gr}(m, n, \mathbf{F})$. The irreducible constituents of this representation are parameterized by the same underlying set both for Archimedean and non-Archimedean fields [G. Hill, On the nilpotent representations of $\mathrm{GL}_n(\mathcal{O})$, Manuscripta Math. 82 (1994) 293–311; A.T. James A.G. Constantine, Generalized Jacobi polynomials as spherical functions of the Grassmann manifold, Proc. London Math. Soc. 29(3) (1974) 174–192]. This paper connects the Archimedean and non-Archimedean theories using the quantum Grassmannian [M.S. Dijkhuizen, J.V. Stokman, Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians, Publ. Res. Inst. Math. Sci. 35 (1999) 451–500; J.V. Stokman, Multivariable big and little q-Jacobi polynomials, SIAM J. Math. Anal. 28 (1997) 452–480]. In particular, idempotents in the Hecke algebra associated to this representation are the image of the quantum zonal spherical functions after taking appropriate limits. Consequently, a correspondence is established between some irreducible representations with Archimedean and non-Archimedean origin.

Keywords: Representations of real and p-adic groups; Quantum Grassmannians; Multivariable orthogonal polynomials; Shifted Macdonald polynomials

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1. Introduction

This paper is concerned with relationships between the Archimedean and non-Archimedean places of a number field. Since the early works of Weil, Artin, Iwasawa, Tate [30] and the far reaching conjectures of Langlands, deep relations have been discovered between the arithmetic of a number field and the representation theory of algebraic groups over the local fields. It is within the framework of representation theory that the relations between the local fields, the *places* of the number field, will be discussed here.

Local fields occur naturally as the completions of global fields. A *global field* is either a *number field*, that is a finite extension of the rational numbers, or a *function field*, that is a field of rational functions of a curve defined over a finite field. A local field can be either *Archimedean* ($\mathbb{R}$ or $\mathbb{C}$) or *non-Archimedean* (Laurent series over a finite field or a finite extension of $\mathbb{Q}_p$). In the function field case all the completions are non-Archimedean and thus carry the same nature. In contrast, in the number field case both Archimedean and non-Archimedean completions occur, thus having a completely different nature. For example, the former is connected and the latter is totally disconnected.

Let $\mathbf{F}$ be a local field. For a non-Archimedean field, let $\mathcal{O}$ be the ring of integers and $\wp$ be the maximal ideal. Let $K^{\mathbf{F}}$ be the maximal compact subgroup of $\mathrm{GL}_n(\mathbf{F})$, for some $n \in \mathbb{N}$ which will be fixed throughout this paper. We have

$$K^{\mathbf{F}} = \begin{cases} O(n) = \text{the orthogonal group}, & \mathbf{F} = \mathbb{R}; \\ U(n) = \text{the unitary group}, & \mathbf{F} = \mathbb{C}; \\ \mathrm{GL}_n(\mathcal{O}) \simeq \varprojlim \mathrm{GL}_n(\mathcal{O}/\wp^k), & \mathbf{F} \text{ non-Archimedean}. \end{cases}$$

In particular, for Archimedean fields $K^{\mathbf{F}}$ is a Lie group while for non-Archimedean fields it is totally disconnected. In order to be able to compare between them we appeal to representation theory. In this paper we focus on a special representation of $K^{\mathbf{F}}$, the *Grassmann representation*, which arises from its natural action on $X_m^{\mathbf{F}} = \mathrm{Gr}(m, n, \mathbf{F})$, the variety of m-dimensional subspaces of a fixed n-dimensional space over $\mathbf{F}$. The natural representation space is $L^2(X_m^{\mathbf{F}})$ or its dense subspace of *smooth* functions $\mathcal{S}(X_m^{\mathbf{F}})$, with the action

$$[g \cdot f](x) = f(g^{-1}x), \quad f \in L^2(X_m^{\mathbf{F}}), g \in K^{\mathbf{F}}.$$

As far as the decomposition to irreducibles is concerned, there is no difference between the two spaces. By smooth functions we mean infinitely differentiable for Archimedean places and locally constant for non-Archimedean ones. To define the L^2 structure, the transitive action of $K^{\mathbf{F}}$ on $X_m^{\mathbf{F}}$ is used, and the measure on $X_m^{\mathbf{F}}$ is taken to be the projection of the normalized Haar measure from the group. Then, for all local fields, for the Archimedean ones [12] and for the non-Archimedean ones [11], the following decomposition holds.

Theorem 1 (*James-Constantine, Hill*). *For any local field* $\mathbf{F}$ *and* $m \leqslant [\frac{n}{2}]$, *the Grassmann representation is a multiplicity free direct sum of irreducible representations of* $K^{\mathbf{F}}$ *indexed by* Λ_m, *the set of partitions with at most m parts.*

Let $\{\mathcal{U}_\lambda^{\mathbf{F}}\}_{\lambda \in \Lambda_m}$ be the irreducible representations which occur in $L^2(X_m^{\mathbf{F}})$. In view of the independence of the labeling set on the field, it is natural to ask the following question.

Question 2. *Fix* $\lambda \in \Lambda_m$. *Are* $\{\mathcal{U}_\lambda^{\mathbf{F}}\}_{\mathbf{F}}$ *related when* $\mathbf{F}$ *runs over all local fields?*

Our goal is to address this question. For this purpose, the *Hecke algebra* of intertwining operators $\mathcal{H}_m^{\mathbf{F}} = \mathcal{S}(X_m^{\mathbf{F}} \times_{K^{\mathbf{F}}} X_m^{\mathbf{F}})$ will be used. This is the convolution algebra of smooth functions on $X_m^{\mathbf{F}} \times X_m^{\mathbf{F}}$ which are invariant under the diagonal action of $K^{\mathbf{F}}$. An element of $\mathcal{H}_m^{\mathbf{F}}$ defines an intertwining operator by realizing it as an integration kernel. The measure on $\Omega_m^{\mathbf{F}} := X_m^{\mathbf{F}} \times_{K^{\mathbf{F}}} X_m^{\mathbf{F}}$, denoted by $\mathtt{dh}^{\mathbf{F}}$, is the projection of the Haar measure from the group and described explicitly in §2.1 and §2.2. As this algebra is commutative for all local fields, the first part of Theorem 1 follows. The minimal idempotents of the algebra have been computed in [12] for Archimedean fields and in [5] for non-Archimedean fields:

- *Archimedean fields* [12]. The minimal idempotents in the Hecke algebra are naturally associated to polynomial representations of GL_m. In particular, they are parameterized by Λ_m. They are eigenfunctions of the Laplacian on the Grassmann manifold with distinct eigenvalues.
- *Non-Archimedean fields* [5]. The minimal idempotents in the Hecke algebra are naturally associated with finite quotients of $\mathcal{O}^m$, the free module of rank m. In particular, they are parameterized by Λ_m. The idempotents are computed in terms of combinatorial invariants of the lattice of submodules of $\mathcal{O}^m$.

Interestingly, geometry plays an important role in both cases; geometrically defined operators which commute with the group action are sufficient to separate representations. In the Archimedean case, it is the Laplacian on the Grassmann manifold, whereas in the non-Archimedean case, a family of discrete averaging operators plays the same role. The identical parametrization of irreducibles is reflected by the same parametrization of idempotents in the Hecke algebras for the different local fields. To show the link between the irreducibles labeled by the same partition for the different fields, the quantum Grassmannian will be used in the following scheme.

Each of the Hecke algebras $\mathcal{H}_m^{\mathbf{F}}$ is characterized by a triplet (space, measure, idempotents)

$$\left(\Omega_m^{\mathbf{F}},\ \mathtt{dh}^{\mathbf{F}}, \{\mathtt{e}_\lambda^{\mathbf{F}}\}_{\lambda \in \Lambda_m}\right).$$

These will be 'interpolated' by similar objects which arise in the quantum Grassmannian $\mathbf{U}_q(n)/\mathbf{U}_q(m) \times \mathbf{U}_q(n-m)$ (cf. [7] for a detailed discussion). The objects which will be used are

$$\left(\Omega_m^q,\ \mathtt{dS}_m^q, \{\mathtt{E}_\lambda^q\}_{\lambda\in\Lambda_m}\right).$$

The precise definition of these q-objects is given later on. Roughly, Ω_m^q is the q-exponentiation of a shift of Λ_m; $\mathtt{dS}_m^q(\mathbf{x}; a, b, t)$ is the q-Selberg measure [9,13,14,2] defined on Ω_m^q; and $\{\mathtt{E}_\lambda^q(\mathbf{x}; a, b, t)\}_{\lambda\in\Lambda_m}$ are the zonal spherical functions which occur in the quantum Grassmannian. The zonal spherical functions, also called multivariable little q-Jacobi polynomials [7,28], are orthogonal with respect to the q-Selberg measure.

By taking appropriate limits, the q-objects interpolate between the objects related to the local fields. In the Archimedean limit $q \to 1$, the space Ω_m^q becomes dense in the Archimedean space, and the atomic q-Selberg measure approximates the continuous Selberg measure. In the non-Archimedean limit $q \to 0$, the space itself remains discrete, and the q-measure specializes to give the non-Archimedean measure. Thus, for any local field $\mathbf{F}$, the distribution $f \mapsto \int_{\Omega_m^{\mathbf{F}}} f\,\mathtt{dh}^{\mathbf{F}}$ is the limit of the distribution $f \mapsto \int_{\Omega_m^q} f\,\mathtt{dS}_m^q$ (§3, Theorem 8). Under the same limits the quantum zonal spherical functions $\{\mathtt{E}_\lambda^q\}$ are mapped to $\{\mathtt{e}_\lambda^{\mathbf{F}}\}$ (§3, Theorem 10).

1.1. Related works

Similar interpolations between p-adic and real zonal spherical functions using q-special functions have been established in several instances. For PGL_2, the zonal spherical functions which occur in the principal series of the groups $\mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{PGL}_2(\mathbb{Q}_p)$, have been shown to be limits of q-ultraspherical polynomials (see [4] for the p-adic limit and [16] for the real limit). The p-adic limit of the higher rank case appeared in the work of Macdonald [21,20], whereas the real limit was proved by Koornwinder.[1] For compact groups, such interpolation has appeared in the work of Haran [10] for the case of the maximal compact subgroup of GL_n and its action on the projective line. This has also been further generalized by Porat [24] to invariants of the GL_n-action on the projective space with respect to upper triangular matrices.

1.2. Organization of the paper and notations

The paper is organized as follows. In Section 2 we describe the Grassmann representation in its various appearances, the Archimedean in §2.1, the non-Archimedean in §2.2 and the quantum in §2.3. This section contains a description of all the ingredients required for carrying out the above plan, with the necessary adjustments and complements. In Section 3, the ingredients are glued together to establish the interpolation. Section 4 contains an example, the one-dimensional case, and Section 5 is devoted to possible extensions of this work.

Notations: Whenever possible, the notations of [21] have been followed; partitions are written in a non-increasing order and are identified with the corresponding Young

[1] Lecture at the INI program on Symmetric Functions and Macdonald Polynomials, April 2001.

diagrams. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let λ' denote the *transposed* diagram, $|\lambda| = \sum_i \lambda_i$ its *weight* and $n(\lambda) = \sum(i-1)\lambda_i$. The *rank* of the partition is the number of its nonzero parts, and its *height* is the largest part. We shall also use the notation $\lambda = (1^{\mu_1} 2^{\mu_2} \ldots)$ where $\mu_i = |\{j | \lambda_j = i\}|$.

Two partial orderings on partitions are used; The partial order defined by the *inclusion of Young diagrams* $\leqslant$, and the *dominance order* $\preceq$.[2] The set of partitions which consist of at most m parts will be denoted by Λ_m. For any ring A, we set $G(A) = \mathrm{GL}_n(A)$. In addition to q, three other parameters (a, b, t) are used. Depending on the context, they are sometimes rewritten using exponents (α, β, γ). For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m$, the vector $(q^{\lambda_1}, \ldots, q^{\lambda_m})$ is denoted by q^{λ}; the set of all such elements is denoted by q^{Λ_m}; and $\rho = (m-1, m-2, \ldots, 0)$. $\mathbb{R}$ and $\mathbb{C}$ stand for the real and complex fields, and $\mathbb{K}$ for a non-Archimedean local field with residue field of cardinality $p^r = |\mathcal{O}/\wp|$. Multivariable indeterminants such as $(x_1, \ldots, x_m)$ are abbreviated by $\mathbf{x}$, and $\mathcal{A}_m = \mathbb{C}[x_1, \ldots, x_m]^{\Sigma_m}$ is the algebra of symmetric polynomials with m variables. Integration with respect to any measure, discrete or continuous, is denoted by the integral sign.

2. The Grassmann representation

2.1. Archimedean theory

This section is concerned with the Archimedean fields $\mathbb{R}$ and $\mathbb{C}$. All the objects involved are well known (see [12,31]), but are described here for completeness. The corresponding maximal compact subgroup K is the orthogonal group $O(n)$ in the real case, and the unitary group $U(n)$ in the complex case.

2.1.1. Space and measure

Points in the space $\Omega_m^{\mathbb{R}} = X_m^{\mathbb{R}} \times_{O(n)} X_m^{\mathbb{R}}$ [resp. $\Omega_m^{\mathbb{C}} = X_m^{\mathbb{C}} \times_{U(n)} X_m^{\mathbb{C}}$] represent the relative position of two m-dimensional subspaces in the real [resp. complex] Grassmann manifold modulo the action of $O(n)$ [resp. $U(n)$]. They are given in terms of m critical angles $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_m \leqslant \pi/2$ which are conveniently rewritten [12, §5] using $u_i = \sin^2(\theta_i)$ to give[3]

$$\Omega_m^{\mathbb{R}} \simeq \Omega_m^{\mathbb{C}} \simeq \Omega_m := \{\mathbf{u} = (u_1, \ldots, u_m) | 0 \leqslant u_1 \leqslant \cdots \leqslant u_m \leqslant 1\}. \tag{2.1}$$

The projection of the normalized Haar measure from K to the orbit space Ω_m is given by special values of the parameters in the Selberg measure [26,1] which is given by

$$\mathtt{dS}_m(\mathbf{u}; \alpha, \beta, \gamma) = s_m^{\alpha,\beta,\gamma} \prod_{i=1}^{m} u_i^{\alpha/2-1}(1-u_i)^{\beta/2-1} \prod_{i<j} |u_i - u_j|^{\gamma} d\mathbf{u}, \tag{2.2}$$

[2] $\lambda \preceq \mu \Longleftrightarrow |\lambda| = |\mu|$ and $\sum_{i=1}^{j} \lambda_i \leqslant \sum_{i=1}^{j} \mu_i \quad \forall j \in \mathbb{N}$.

[3] We choose the co-ordinates $\sin^2(\theta_i)$ rather than $\cos^2(\theta_i)$, see §2.2.1.

where

$$s_m^{\alpha,\beta,\gamma} = \prod_{j=1}^{m} \frac{\Gamma(\alpha/2+\beta/2+(m+j-2)\gamma/2)\Gamma(\gamma/2)}{\Gamma(\alpha/2+(j-1)\gamma/2)\Gamma(\beta/2+(j-1)\gamma/2)\Gamma(j\gamma/2)}. \tag{2.3}$$

That this is a probability measure on Ω_m is due to Selberg [26]. We are interested in the following specializations:

$$\mathtt{dh}^{\mathbb{R}}(\mathbf{u}) = \mathtt{dS}_m(\mathbf{u}; n-2m+1, 1, 1),$$
$$\mathtt{dh}^{\mathbb{C}}(\mathbf{u}) = \mathtt{dS}_m(\mathbf{u}; 2(n-2m+1), 2, 2).$$

2.1.2. Idempotents

Define an inner product on the algebra of symmetric polynomials $\mathcal{A}_m$ by

$$\langle f, g\rangle_{\alpha,\beta,\gamma} = \int_{\Omega_m} f(\mathbf{u})\overline{g(\mathbf{u})}\ \mathtt{dS}_m(\mathbf{u}; \alpha, \beta, \gamma), \quad f, g \in \mathcal{A}_m. \tag{2.4}$$

Let $\{\mathtt{M}_\lambda\}_{\lambda\in\Lambda_m}$ be the monomial basis of $\mathcal{A}_m$

$$\mathtt{M}_\lambda(\mathbf{x}) = \sum_{\eta} x_1^{\eta_1}\cdots x_m^{\eta_m}, \tag{2.5}$$

where the summation is over all distinct permutations η of λ. The *generalized Jacobi polynomials* $\{\mathtt{E}_\lambda(\mathbf{x}; \alpha, \beta, \gamma)\}_{\lambda\in\Lambda_m}$ are defined by the following conditions [12,31]

(1) $\mathtt{E}_\lambda = d_\lambda \mathtt{M}_\lambda +$ lower terms, $d_\lambda \neq 0$,
(2) $\langle \mathtt{E}_\lambda, \mathtt{M}_\mu\rangle_{\alpha,\beta,\gamma} = 0\ \forall \mu \prec \lambda$,
(3) Normalization: $\|\mathtt{E}_\lambda\|^2 = \mathtt{E}_\lambda(\mathbf{0}; \alpha, \beta, \gamma)$.

Our normalization, which is different from the one in [12,31], is chosen so that the idempotents in the Hecke algebras are given by the generalized Jacobi polynomials for the same special values as above:

$$\mathtt{e}_\lambda^{\mathbb{R}}(\mathbf{u}) = \mathtt{E}_\lambda(\mathbf{u}; n-2m+1, 1, 1),$$
$$\mathtt{e}_\lambda^{\mathbb{C}}(\mathbf{u}) = \mathtt{E}_\lambda(\mathbf{u}; 2(n-2m+1), 2, 2).$$

The generalized Jacobi polynomials are also eigenfunctions with distinct eigenvalues of a second-order differential operator which specializes to the Laplace–Beltrami operator on the real/complex Grassmann manifolds after the parameters have been specialized.

2.2. Non-Archimedean theory

Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field $\mathbb{K}$. Let $\wp = (\pi)$ be the maximal ideal and p^r the cardinality of the residue field $\mathcal{O}/\wp$. By the principal divisors theorem, any finite $\mathcal{O}$-module is of the form $\oplus_{i=1}^{j} \mathcal{O}/\wp^{\lambda_i}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_j)$, which will be referred to as the *type* of the module. As an example, Λ_m above parameterizes all types of finite $\mathcal{O}$-modules with rank bounded by m. Note that in the partial order defined by the inclusion of Young diagrams, $\lambda \leqslant \mu$ if and only if a module of type λ can be embedded in a module of type μ. In such case we shall use the notation

$$\binom{\mu}{\lambda} = \#\text{ of submodules of type } \lambda \text{ contained in a module of type } \mu. \tag{2.6}$$

Elements in Λ_m with height bounded by k are denoted by $\Lambda_m^k = \{\lambda \in \Lambda_m \mid 0 \leqslant \lambda \leqslant k^m\} = \{$isomorphism types of submodules of $(\mathcal{O}/\wp^k)^m\}$.

The non-Archimedean theory is completely determined by finite quotients. More precisely, let $\mathcal{O}_k = \mathcal{O}/\wp^k$ and let I_k stand for $\mathrm{Ker}\{G(\mathcal{O}) \to G(\mathcal{O}_k)\}$. Each irreducible representation of the (pro-finite) group $G(\mathcal{O})$ factors through the groups $G(\mathcal{O}_k)$, except for a finite set of $k \in \mathbb{N}$ whose cardinality is the *level* of the representation. In particular, the Grassmann representation can be filtered as follows:

$$(0) \subset L^2(X_m^{\mathbb{K}})^{I_1} \subset \cdots \subset L^2(X_m^{\mathbb{K}})^{I_k} \subset \cdots \subset L^2(X_m^{\mathbb{K}}) \tag{$\star$}$$

and each of its irreducible constituents is contained in some finite term. The kth term in this filtration is in fact a representation of $G(\mathcal{O}_k)$, and the direct limit of this sequence is precisely the smooth part of the Grassmann representation. The finite space $I_k \backslash X_m^{\mathbb{K}}$ can be canonically identified with $X_{k^m} = \mathrm{Gr}(m, n, \mathcal{O}_k)$, the Grassmannian of free submodules of $(\mathcal{O}_k)^n$ of rank m. Thus, we may identify the representation space $L^2(X_m^{\mathbb{K}})^{I_k}$ with $\mathcal{F}_{k^m} = \mathcal{F}(X_{k^m})$, the space of $\mathbb{C}$-valued functions on X_{k^m}.

Let $\mathcal{H}_{k^m} = \mathrm{End}_{G(\mathcal{O}_k)}(\mathcal{F}_{k^m})$ be the Hecke algebra associated with the representation $\mathcal{F}_{k^m}$. It is isomorphic to the convolution algebra $\mathcal{F}(X_{k^m} \times_{G(\mathcal{O}_k)} X_{k^m})$ by interpreting elements of the latter as $G(\mathcal{O}_k)$-invariant summation kernels (see [5, §2.2] for details). The $G(\mathcal{O}_k)$-orbit of an element $(y, z) \in X_{k^m} \times X_{k^m}$ is determined by the type of the intersection $y \cap z$, giving rise to the identification $\Lambda_m^k \simeq X_{k^m} \times_{G(\mathcal{O}_k)} X_{k^m}$. The following diagram summarizes the objects involved and the maps between them.

$$\begin{array}{ccccc}
G(\mathcal{O}) & X_m^{\mathbb{K}} \times X_m^{\mathbb{K}} \quad \rightarrow & X_m^{\mathbb{K}} \times_{G(\mathcal{O})} X_m^{\mathbb{K}} & \mathtt{dh}^{\mathbb{K}} & \mathcal{H}_m^{\mathbb{K}} \\
\downarrow & \downarrow & \downarrow & \downarrow & \uparrow \\
G(\mathcal{O}_k) & X_{k^m} \times X_{k^m} \quad \rightarrow & \Lambda_m^k & \mathtt{dh}_k & \mathcal{H}_{k^m} \\
\downarrow & \downarrow & \downarrow & \downarrow & \uparrow \\
G(\mathcal{O}_{k-1}) & X_{(k-1)^m} \times X_{(k-1)^m} \rightarrow & \Lambda_m^{k-1} & \mathtt{dh}_{k-1} & \mathcal{H}_{(k-1)^m}
\end{array}$$

Here $\mathtt{dh}^{\mathbb{K}}$ and $\mathtt{dh}_k$ stand for the projection of the Haar measure from $G(\mathcal{O})$ to $X_m^{\mathbb{K}} \times_{G(\mathcal{O})} X_m^{\mathbb{K}}$ and Λ_m^k, respectively. The map from Λ_m^k to Λ_m^{k-1} is easily described using the transposed Young diagrams, and is given by $\lambda' = (\lambda'_1, \dots, \lambda'_k) \mapsto \bar{\lambda}' = (\lambda'_1, \dots, \lambda'_{k-1})$. We have

$$G(\mathcal{O}) \simeq \lim_{\leftarrow} G(\mathcal{O}_k), \quad \mathcal{H}_m^{\mathbb{K}} \simeq \lim_{\rightarrow} \mathcal{H}_{k^m},$$

$$X_m^{\mathbb{K}} \times_{G(\mathcal{O})} X_m^{\mathbb{K}} \simeq \hat{\Lambda}_m := \lim_{\leftarrow} \Lambda_m^k, \quad \mathtt{dh}^{\mathbb{K}} = \lim_{\rightarrow} \mathtt{dh}_k.$$

2.2.1. Space and measure

Points in the space $\Omega_m^{\mathbb{K}} = X_m^{\mathbb{K}} \times_{G(\mathcal{O})} X_m^{\mathbb{K}}$ correspond to the relative position of two m-dimensional spaces modulo the diagonal $G(\mathcal{O})$-action. By the above discussion it may be identified with $\hat{\Lambda}_m = \lim_{\leftarrow} \Lambda_m^k$, namely with 'partitions' $(\lambda_i)_{i=1}^m$ where the value ∞ is allowed [5, §2.3.1]. By analogy with the Archimedean space (2.1) it is convenient to rewrite it as

$$\Omega_m^{\mathbb{K}} = \{p^{-\lambda} = (p^{-\lambda_1}, \dots, p^{-\lambda_m}) | \lambda \in \hat{\Lambda}_m\} \subset \Omega_m^{\mathbb{R},\mathbb{C}}. \tag{2.7}$$

Note that this embedding is topological, and has the advantage that the origin $\mathbf{0} = (0, \dots, 0)$ is the common representative of the orbit of the trivial relative position for all local fields, namely, $\mathbf{0} = [(x, x)] \in X_m^{\mathbf{F}} \times_{K_{\mathbf{F}}} X_m^{\mathbf{F}}$. This is also the reason for the choice of the co-ordinates $u_i = \sin^2(\theta_i)$ rather than $u_i = \cos^2(\theta_i)$ for the Archimedean spaces.

The following proposition computes the measures $\mathtt{dh}_k$ and $\mathtt{dh}^{\mathbb{K}}$. Note that the measure $\mathtt{dh}^{\mathbb{K}}$ vanishes outside the set $\dot{\Omega}_m^{\mathbb{K}} = \{p^{-\lambda} | \lambda \in \Lambda_m\} \subset \Omega_m^{\mathbb{K}}$.

Notation: Let $[i]_q = 1 - q^i$ for $i \in \mathbb{N}$, $[i]_q! = [i]_q[i-1]_q \cdots [1]_q$ and $\left[{i \atop i'}\right]_q = \frac{[i]_q!}{[i']_q![i-i']_q!}$. The index q is omitted whenever $q = p^{-r} = |\mathcal{O}/\wp|^{-1}$.

Proposition 3.

(1)
$$\mathtt{dh}_1(\lambda) = \frac{\begin{bmatrix} m \\ \lambda'_1 \end{bmatrix}\begin{bmatrix} n-m \\ m-\lambda'_1 \end{bmatrix}}{\begin{bmatrix} n \\ m \end{bmatrix}} p^{-r\lambda'_1(n-2m+\lambda'_1)} \quad (k=1)$$

$$\frac{\mathtt{dh}_k(\lambda)}{\mathtt{dh}_{k-1}(\bar{\lambda})} = \begin{bmatrix} \lambda'_{k-1} \\ \lambda'_k \end{bmatrix} \frac{[n-2m+\lambda'_{k-1}]!}{[n-2m+\lambda'_k]!} p^{-r\lambda'_k(n-2m+\lambda'_k)} \quad (k>1)$$

(2)
$$\mathtt{dh}^{\mathbb{K}}(p^{-\lambda}) = \frac{\begin{bmatrix} m \\ m-\lambda'_1, \lambda'_1-\lambda'_2, \dots \end{bmatrix} \frac{[n-m]!}{[m-\lambda'_1]![n-2m]!}}{\begin{bmatrix} n \\ m \end{bmatrix}} p^{-r\sum(\lambda'_i)^2 - r(n-2m)\sum \lambda'_i}.$$

Proof. The proof of part (2) follows directly from part (1) using

$$\mathtt{dh}^{\mathbb{K}}(p^{-\lambda}) = \prod_{k \geqslant 1} \frac{\mathtt{dh}_k(\lambda)}{\mathtt{dh}_{k-1}(\bar{\lambda})}, \tag{2.8}$$

where $\mathtt{dh}_0 = 1$.

To prove (1), start with $k = 1$. The measure $\mathtt{dh}_1$ appeared in connection with the q-Johnson association scheme [6], but is included here for completeness. Fix spaces $z_1 \subset y_1$ of dimensions $\lambda'_1 \leqslant m$ in $(\mathcal{O}/\wp)^n$. Then

$$n_1 = |\{y|\dim\, y = m,\, y \cap y_1 = z_1\}| = \begin{bmatrix} n-m \\ m-\lambda'_1 \end{bmatrix}_{p^r} p^{r(m-\lambda'_1)^2},$$

$$n_2 = \#\text{ of choices for } z_1 = \binom{1^m}{1^{\lambda'_1}} = \begin{bmatrix} m \\ \lambda'_1 \end{bmatrix}_{p^r},$$

$$n_3 = \#\text{ of } m\text{-dimensional subspaces in } (\mathcal{O}/\wp)^n = \begin{bmatrix} n \\ m \end{bmatrix}_{p^r}$$

and $\mathtt{dh}_1(\lambda) = n_1 n_2/n_3$, which together with the relation $\left[\begin{smallmatrix} i \\ i' \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i \\ i' \end{smallmatrix}\right]_{1/q} q^{i'(i-i')}$, gives the desired formula.

For $k > 1$, fix two $\mathcal{O}_k$-modules $f \subset F$ of types $\phi = k^m \leqslant k^n = \Phi$. Let $\bar{F} = F/\wp^{k-1}F$ where $z \mapsto \bar{z}$ is the quotient map. For any module y, let $\mathtt{t}(y)$ denote its isomorphism type. Then for any $\lambda \leqslant k^m$:

$$\begin{aligned} \frac{\mathtt{dh}_k(\lambda)}{\mathtt{dh}_{k-1}(\bar{\lambda})} &= \frac{\binom{\Phi}{\phi}^{-1}}{\binom{\bar{\Phi}}{\bar{\phi}}^{-1}} \frac{\left|\{z \mid \mathtt{t}(z \cap f) = \lambda,\ \mathtt{t}(z) = \phi\}\right|}{\left|\{\bar{z} \mid \mathtt{t}(\bar{z} \cap \bar{f}) = \bar{\lambda},\ \mathtt{t}(\bar{z}) = \bar{\phi}\}\right|} \quad \text{(Haar} \to \text{counting measure)} \\ &= \frac{\binom{\bar{\Phi}}{\bar{\phi}}}{\binom{\Phi}{\phi}} \cdot \frac{\binom{\phi}{\lambda}}{\binom{\bar{\phi}}{\bar{\lambda}}} \cdot \frac{\left|\{z \mid z \cap f = y_0,\ \mathtt{t}(z) = \phi\}\right|}{\left|\{\bar{z} \mid \bar{z} \cap \bar{f} = \bar{y}_0,\ \mathtt{t}(\bar{z}) = \bar{\phi}\}\right|} \quad (y_0 \text{ fixed of type } \lambda) \\ &= \left(p^{-rm(n-m)}\right)\left(\begin{bmatrix} \lambda'_{k-1} \\ \lambda'_k \end{bmatrix} p^{r\lambda'_k(m-\lambda'_k)}\right)\left(\frac{[n-2m+\lambda'_{k-1}]!}{[n-2m+\lambda'_k]!} p^{r(m-\lambda'_k)(n-m)}\right). \end{aligned}$$

The computation of the first two terms is straightforward (alternatively, use the explicit formulas in [5, §4.1]), and for the third term we argue as follows. Let $y_0 \subset f$ be a fixed submodule of type λ. Let $\bar{z}$ be a fixed submodule of $\bar{F}$ of type $\bar{\phi}$ such that $\bar{f} \cap \bar{z} = \bar{y}_0$. Then $\frac{|\{z|z\cap f=y_0,\ \mathtt{t}(z)=\phi\}|}{|\{\bar{z}|\bar{z}\cap\bar{f}=\bar{y}_0,\ \mathtt{t}(\bar{z})=\bar{\phi}\}|}$ counts the number of submodules $z \subset F$ of type

ϕ which fit into the following diagram:

$$\begin{array}{ccc} z & \longmapsto & \bar{z} \\ \cup & & \cup \\ y_0 = z \cap f & \longmapsto & \bar{y}_0 \\ \cap & & \cap \\ f & \longmapsto & \bar{f} \end{array}$$

That is, we need to count liftings of $\bar{z}$ which intersect f precisely in y_0. First, observe that we may assume $\lambda'_k = 0$. This amounts to moding out a $k^{\lambda'_k}$-type summand of y_0. A second observation is that counting different liftings of $\bar{z}$ is equivalent to deforming a fixed basis of a chosen lifting. Namely, let z be a lifting of $\bar{z}$ and let $\mathcal{B}_z$ be a basis for z. Complete this basis to a basis $\mathcal{B}_F$ of F. If z' is another lifting of $\bar{z}$, then it has a basis $\mathcal{B}_{z'}$ which is a deformation of $\mathcal{B}_z$ with elements from $\wp^{k-1}\mathcal{B}_F$. There are in fact many such bases, however, if we deform only with elements from $\wp^{k-1}(\mathcal{B}_F \setminus \mathcal{B}_z)$ we get that $z' = z'' \Longleftrightarrow \mathcal{B}_{z'} = \mathcal{B}_{z''}$. Putting the last two observations together, we now fix z which fits into the diagram above together with a basis $\mathcal{B}_z$, and count proper deformations which also fit the diagram. Let $\mathcal{B}_z = \coprod_{i=0}^{k} \mathcal{B}_z^i$ and $\mathcal{B}_f = \coprod_{i=0}^{k} \mathcal{B}_f^i$ be bases of z and f respectively such that $\coprod_i \pi^{k-i}\mathcal{B}_z^i = \coprod_i \pi^{k-i}\mathcal{B}_f^i$ is a basis for y_0. The assumption $\lambda'_k = 0$ implies that $\mathcal{B}_f^k = \mathcal{B}_z^k = \emptyset$. Elements of $\mathcal{B}_z \setminus \mathcal{B}_z^{k-1}$ can be deformed arbitrarily, and there are $|\wp^{k-1}(\mathcal{B}_F \setminus \mathcal{B}_z)| \cdot |\mathcal{B}_z \setminus \mathcal{B}_z^{k-1}| = p^{r(n-m)(m-\lambda'_{k-1})}$ such deformations. However, when deforming the jth basis element of $\mathcal{B}_z^{k-1}$, elements from $\wp^{k-1}(\mathcal{B}_f \setminus \mathcal{B}_f^{k-1})$ together with the span of the previously chosen $j-1$ elements must be avoided in order not to enlarge the intersection y_0. Thus, the number of possible deformations of this element is $(p^{r(n-m)} - p^{r(m-\lambda'_{k-1}+j-1)})$. Multiplying all contributions gives the desired result for the third term

$$(p^{r(n-m)} - p^{r(m-\lambda'_{k-1})}) \cdots (p^{r(n-m)} - p^{r(m-1)}) \cdot p^{r(n-m)(m-\lambda'_{k-1})}$$

and completes the proof of the proposition. $\square$

2.2.2. Idempotents

We have the following inner product on $\mathcal{H}_m^{\mathbb{K}}$:

$$\langle f, g \rangle_{\mathbb{K}} = \int_{\Omega_m^{\mathbb{K}}} f\bar{g}\,\mathrm{dh}^{\mathbb{K}}, \qquad \forall f, g \in \mathcal{H}_m^{\mathbb{K}}. \tag{2.9}$$

The idempotents in the algebra $\mathcal{H}_m^{\mathbb{K}}$, considered as functions on $\Omega_m^{\mathbb{K}}$, are orthogonal with respect to the measure $\mathrm{dh}^{\mathbb{K}}$. Since $\dot{\Omega}_m^{\mathbb{K}} = p^{-\Lambda_m}$ is an open dense subset of $\Omega_m^{\mathbb{K}} = p^{-\hat{\Lambda}_m}$ (see [5, §2.3]), and carries the full measure of the space by Proposition 3, it suffices to know the restrictions of functions to $\dot{\Omega}_m^{\mathbb{K}}$. The explicit computation of the minimal idempotents in $\mathcal{H}_m^{\mathbb{K}}$ has been carried out in [5, §4.2]. The algebra $\mathcal{H}_m^{\mathbb{K}}$ is equipped with the following natural bases:

- $\{\mathrm{g}_\lambda^{\mathbb{K}}\}_{\lambda\in\Lambda_m}$—*geometric basis* (delta functions supported on points in $\dot{\Omega}_m^{\mathbb{K}}$).
- $\{\mathrm{c}_\lambda^{\mathbb{K}}\}_{\lambda\in\Lambda_m}$—*cellular basis*.
- $\{\mathrm{e}_\lambda^{\mathbb{K}}\}_{\lambda\in\Lambda_m}$—*algebraic basis* (minimal idempotents).

The cellular basis is an intermediate basis which plays an important role in the non-Archimedean theory and also for the interpolation. It is lower triangular with respect to the geometric basis, defined explicitly by

$$\mathrm{c}_\lambda^{\mathbb{K}} = \sum_{\mu\geqslant\lambda} \binom{\mu}{\lambda} \mathrm{g}_\mu^{\mathbb{K}}. \tag{2.10}$$

On the other hand, it is upper triangular with respect to the algebraic basis; The subspaces

$$\mathcal{J}_\lambda^{\mathbb{K}} = \mathrm{Span}_{\mathbb{C}}\{\mathrm{c}_\mu^{\mathbb{K}} | \mu\leqslant\lambda\}, \quad \mathcal{J}_{\lambda^-}^{\mathbb{K}} = \mathrm{Span}_{\mathbb{C}}\{\mathrm{c}_\mu^{\mathbb{K}} | \mu<\lambda\}$$

are in fact ideals, and $\{\mathcal{J}_\lambda^{\mathbb{K}}/\mathcal{J}_{\lambda^-}^{\mathbb{K}}\}_{\lambda\in\Lambda_m}$ exhaust the irreducible $\mathcal{H}_m^{\mathbb{K}}$-modules (hence the term cellular basis). As $\mathrm{e}_\lambda^{\mathbb{K}}$ is by definition the idempotent which corresponds to the representation $\mathcal{J}_\lambda^{\mathbb{K}}/\mathcal{J}_{\lambda^-}^{\mathbb{K}}$ we have

$$\langle \mathrm{e}_\lambda^{\mathbb{K}}, \mathrm{c}_\mu^{\mathbb{K}}\rangle_{\mathbb{K}} = 0, \quad \forall\mu\leqslant\lambda. \tag{2.11}$$

2.3. Quantum Grassmannians and some symmetric functions

In this section, we describe the q-objects which interpolate between the objects related to the local fields. For more details, the reader is referred to [14,2] for the measure theoretic considerations (the q-Selberg measure), to [28,29] for the spherical functions analysis (multivariable little q-Jacobi polynomials), to [23,22,18,19] for the generalized binomial coefficients and the shifted Macdonald polynomials and to [7] for the description of quantum Grassmannians. Throughout this section the parameters q, t, a and b are used, where the first two are the standard Macdonald parameters. In some parts restrictions are set on their values.

2.3.1. The q-Selberg measure

The q-Selberg measure is a multivariable generalization of the q-beta measure [3]. Let $q, t, a, b \in (0, 1)$ and let $\rho = (m-1, m-2, \ldots, 0)$. Let

$$\Omega_m^q = \{q^\lambda t^\rho = (q^{\lambda_1} t^{m-1}, q^{\lambda_2} t^{m-2}, \ldots, q^{\lambda_m}) | \lambda \in \Lambda_m\} \subset \Omega_m^{\mathbb{R},\mathbb{C}} \tag{2.12}$$

and denote

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x), \tag{2.13}$$

$$(x_1,\dots,x_l)_\infty = \prod_{i=1}^{l}(x_i)_\infty. \tag{2.14}$$

The q-Selberg measure is given by

$$\begin{aligned} &\mathrm{dS}_m^q(\mathbf{x};a,b,t) \\ &\quad = \prod_{j=1}^{m} \frac{(at^{m-j}, bt^{j-1}, t^j, qx_j)_\infty}{(abt^{m+j-2}, t, q, bx_j)_\infty} a^{\lambda_j} t^{2j-2} \prod_{j<i} \frac{(qx_j/tx_i)_\infty}{(tx_j/x_i)_\infty}\left(1-\frac{x_j}{x_i}\right) \end{aligned} \tag{2.15}$$

for $\mathbf{x} = q^\lambda t^\rho \in \Omega_m^q$. Askey conjectured [3, §2] that $\mathrm{dS}_m^q(\mathbf{x};a,b,t)$ is a probability measure supported on Ω_m^q for $t = q^\gamma, \gamma \in \mathbb{N}$. This was proved independently by Habsieger [9] and Kadell [13], and was further generalized by Aomoto [2, Proposition 2] for any $\gamma \in \mathbb{R}_{>0}$. Our notation follows [2] with the dictionary

$$\begin{aligned} &a \leftrightarrow q^{\alpha-(m-1)(2\gamma-1)}, \quad m \leftrightarrow n, \\ &b \leftrightarrow q^{\beta+1}, \quad x_j \leftrightarrow q^{-1}t_{n-j+1}, \\ &t \leftrightarrow q^\gamma, \quad t^\rho \leftrightarrow q^{-1}\xi_F \end{aligned}$$

and Proposition 2 in [2] translates into

$$\int_{\Omega_m^q} \mathrm{dS}_m^q = \sum_{q^\lambda t^\rho \in \Omega_m^q} \mathrm{dS}_m^q(q^\lambda t^\rho; a,b,t) = 1. \tag{2.16}$$

Note that the order of the variables is reversed with respect to [2,28,14] since partitions there are written in non-decreasing order while here they are written in non-increasing order. Also, we avoid the use of the q-Jackson integral, which is illuminating when one takes the Archimedean limit but is less adapted for taking the non-Archimedean limit.

2.3.2. Multivariable little q-Jacobi polynomials

Define an inner product on the algebra of symmetric polynomials $\mathcal{A}_m$ by

$$\langle f,g\rangle_{q,a,b,t} = \int_{\Omega_m^q} f(\omega)\overline{g(\omega)}\mathrm{dS}_m^q(\omega;a,b,t), \quad f,g \in \mathcal{A}_m.$$

Definition 4 (*Stokman, [28]*). The multivariable little q-Jacobi polynomials $\{\mathtt{E}^q_\lambda(\mathbf{x}; a, b, t)\}_{\lambda\in\Lambda_m}$ are the unique polynomials defined by

(1) $\mathtt{E}^q_\lambda = d_\lambda \mathtt{M}_\lambda +$ lower terms, $d_\lambda \neq 0$,
(2) $\langle \mathtt{E}_\lambda, \mathtt{M}_\mu\rangle_{q,a,b,t} = 0$, $\forall \mu \prec \lambda$,
(3) Normalization:[4] $\|\mathtt{E}^q_\lambda\|^2 = \mathtt{E}^q_\lambda(\mathbf{0}; a, b, t)$.

The multivariable little q-Jacobi polynomials have interpretation as zonal spherical functions in the representation of $\mathbf{U}_q(n)$ which arises from its action on $\mathbf{U}_q(n)/\mathbf{U}_q(m) \times \mathbf{U}_q(n-m)$. As we focus only on the zonal spherical functions, the reader is referred to [7] for a detailed discussion on the quantum Grassmannian.

2.3.3. The basis $\{\mathcal{C}^q_\lambda\}$

In the absence of an explicit formula for the multivariable little q-Jacobi polynomials, a key role in the interpolation between the idempotents in the Hecke algebras is played by a q-version of the non-Archimedean cellular basis (2.10). In short, it consists of a symmetrized and normalized version of the shifted Macdonald polynomials. We review their definition and some of their properties. The only parameters to be used here are (q, t). Let

$$v_\lambda = v_\lambda(q,t) = \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}),$$

$$v'_\lambda = v'_\lambda(q,t) = \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}).$$

The shifted Macdonald polynomials, also known as interpolation Macdonald polynomials, were defined in [23,15,25]. They were further studied in [22], in which an integral representation is given and a binomial formula. They are defined as follows [22, §1].

Definition 5. The shifted Macdonald polynomials $\{P^\star_\lambda(\mathbf{x}; q, t)\}_{\lambda\in\Lambda_m}$ are polynomials in m variables defined by the following conditions:

(1) $P^\star_\lambda$ has degree $|\lambda|$,
(2) $P^\star_\lambda$ is symmetric in the variables $x_i t^{-i}$,
(3) $P^\star_\lambda(q^\mu; q, t) = 0$ unless $\lambda \leqslant \mu$,
(4) $P^\star_\lambda(q^\lambda; q, t) = (-1)^{|\lambda|} t^{-2n(\lambda)} q^{n(\lambda')} v'_\lambda$.

The values of these polynomials on points q^μ with $\lambda \leqslant \mu$ are connected to two-parameter generalized binomial coefficients, defined in [18, §4; 22, §1]

[4] Note the different normalization comparing to [28].

Definition 6. The generalized binomial coefficients $\binom{\mu}{\lambda}_{q,t}$ are defined by the identity

$$(v'_\lambda)^{-1} P_\lambda(\mathbf{x}; q, t) \prod_{i=1}^{m} (x_i; q)_\infty^{-1} = \sum_\mu \binom{\mu}{\lambda}_{q,t} t^{n(\mu)-n(\lambda)} (v'_\mu)^{-1} P_\mu(\mathbf{x}; q, t)$$

where the P_λ's are the Macdonald polynomials.

The connection between the generalized binomial coefficients and the shifted Macdonald polynomials is given by

$$\binom{\mu}{\lambda}_{q,t} = \frac{P^\star_\lambda(q^\mu)}{P^\star_\lambda(q^\lambda)}, \quad [19, §7; 22, §1] \tag{2.17}$$

The q-analogue of the non-Archimedean cellular basis which was described in §2.2.2, is the following symmetrized and normalized version of the shifted Macdonald polynomials

Definition 7. The basis $\{\mathrm{C}^q_\lambda(\mathbf{x}; t)\}_{\lambda\in\Lambda_m}$ of $\mathcal{A}_m$ is defined by

$$\mathrm{C}^q_\lambda(x_1, \ldots, x_m; t) = \frac{P^\star_\lambda(x_1 t^{1-m}, x_2 t^{2-m}, \ldots, x_m; q, t)}{P^\star_\lambda(q^\lambda; q, t)}\ , \quad \lambda \in \Lambda_m. \tag{2.18}$$

3. Interpolation

We are now in a position to state our results concerning the interpolation. Most of them are multidimensional generalizations of Haran's work [10] regarding interpolation between projective spaces (Grassmannians of lines) over local fields. By interpolation, we mean that the q-objects described in §2.3 have limits which are the local objects described in §2.1 and §2.2. The functions or measures of which we take limits are of m variables and might carry one, two or three parameters, in addition to q.

3.1. *Definition of the limits*

Two kinds of limits are considered; For $f^q = f^q(\mathbf{x}; a, b, t) \in \mathbb{C}[\mathbf{x}]$ define the *Archimedean limit* by

$$\left[\lim_{\mathbf{Arch}} f^q\right](\mathbf{u}; \alpha, \beta, \gamma) = \lim_{q\to 1} f^q(\mathbf{u}; q^{\alpha/2}, q^{\beta/2}, q^{\gamma/2}), \quad \mathbf{u} \in \Omega_m^{\mathbb{R},\mathbb{C}} \tag{3.1}$$

and, the *Non-Archimedean limit* by

$$\left[\lim_{\mathbf{NonArch}} f^q\right](p^{-\lambda};\alpha,\beta,\gamma)=\lim_{q\to 0} f^q(q^{\lambda}p^{-\gamma\rho};p^{-\alpha},p^{-\beta},p^{-\gamma}),\quad p^{-\lambda}\in\Omega_m^{\mathbb{K}}. \tag{3.2}$$

In both limits, the parameter q disappears and the parameters a, b, t are replaced by α, β, γ. In practice, the non-Archimedean limit amounts to first substituting $(a, b, t) = (p^{-\alpha}, p^{-\beta}, p^{-\gamma})$, and then substituting $q = 0$. To get interpretation of these functions in the Hecke algebras $\mathcal{H}_m^{\mathbf{F}}$, set

$$(\alpha,\beta,\gamma)=r(n-2m+1,1,1), \tag{3.3}$$

where for non-Archimedean places $r = [\mathcal{O}/\wp : \mathbb{F}_p]$, the degree of the residue field over its prime field, and since this degree is the same as $[\mathbb{K} : \mathbb{Q}_p]$ for non-ramified extensions, we set $r = 1$ for a real place and $r = 2$ for a complex place.

3.2. Interlude for setting the strategy

We shall follow the following plan. First, we observe that the Archimedean and non-Archimedean weak limits of the distribution

$$f\mapsto\int_{\Omega_m^q} f\,\mathrm{d}\mathrm{S}_m^q,$$

are the distributions

$$f\mapsto\int_{\Omega_m^{\mathbf{F}}} f\,\mathrm{d}\mathrm{h}^{\mathbf{F}},$$

when substituting the appropriate parameters (3.3). Second, we observe that the flag which is used to define the zonal spherical functions in the quantum Grassmannian converges to the flag which is used to define the idempotents $\{\mathrm{e}_\lambda^{\mathbf{F}}\}$. We then conclude that the zonal spherical functions in the quantum Grassmannian converge to the idempotents.

3.3. Limits of the measure

We now prove (the non-Archimedean part of)

Theorem 8. *For any local field* $\mathbf{F}$ *the measure on the space* $X_m^{\mathbf{F}}\times_{K_{\mathbf{F}}}X_m^{\mathbf{F}}$ *is a limit of the q-Selberg measure.*

Proof. For the Archimedean limit see [28]; As $q\to 1$ the space Ω_m^q approximates the space Ω_m, and the distribution $f\mapsto\int_{\Omega_m^q} f\,\mathrm{d}\mathrm{S}_m^q$ weakly converges to the distribution

$f \mapsto \int_{\Omega_m} f \, \mathtt{dS}_m$. In fact, the possible existence of the Archimedean limit was the main motivation for introducing the q-Selberg measure.

For the non-Archimedean limit, we show that the function $\mathtt{dh}^{\mathbb{K}}$ is a limit of the function $\mathtt{dS}_m^q$. Substituting a typical element $\omega_\lambda = q^\lambda t^\rho$ in the q-Selberg measure gives

$$\mathtt{dS}_m^q(q^\lambda t^\rho; a, b, t) = f_1 \cdot f_2 \cdot f_3 \ , \quad \lambda \in \Lambda_m,$$

where

$$f_1 = \prod_{j=1}^{m} \frac{(at^{m-j}, bt^{j-1}, t^j)_\infty}{(abt^{m+j-2}, t, q)_\infty} \quad \text{(normalization constant)},$$

$$f_2 = \prod_{j=1}^{m} \frac{(q^{\lambda_j+1}t^{m-j})_\infty}{(bq^{\lambda_j}t^{m-j})_\infty} a^{\lambda_j} t^{2\lambda_j(j-1)} \quad \text{(local factors)},$$

$$f_3 = \prod_{j<i} \frac{(q^{\lambda_j-\lambda_i+1}t^{i-j-1})_\infty}{(q^{\lambda_j-\lambda_i}t^{i-j+1})_\infty} (1 - q^{\lambda_j-\lambda_i}t^{i-j}) \quad \text{(mixed factors).}$$

Taking the non-Archimedean limit of these expressions gives

$$\left[\lim_{\text{NonArch}} f_1\right](p^{-\lambda}; \alpha, \beta, \gamma) = \prod_{j=1}^{m} \frac{(1 - p^{-\alpha-(m-j)\gamma})(1 - p^{-\beta-(j-1)\gamma})(1 - p^{-j\gamma})}{(1 - p^{-\alpha-\beta-(m+j-2)\gamma})(1 - p^{-\gamma})},$$

$$\left[\lim_{\text{NonArch}} f_2\right](p^{-\lambda}; \alpha, \beta, \gamma) = \prod_{j=1}^{m} p^{-\lambda_j[\alpha+2(j-1)\gamma]} \prod_{\{j:\lambda_j=0\}} (1 - p^{-\beta-(m-j)\gamma})^{-1},$$

$$\left[\lim_{\text{NonArch}} f_3\right](p^{-\lambda}; \alpha, \beta, \gamma) = \prod_{\substack{j<i \\ \lambda_i=\lambda_j}} \frac{1 - p^{-(i-j)\gamma}}{1 - p^{-(i-j+1)\gamma}},$$

so that the product of these terms is $[\lim_{\text{NonArch}} \mathtt{dS}_m^q](p^{-\lambda}; \alpha, \beta, \gamma)$. To get the non-Archimedean measure for the Grassmannian we specialize $(\alpha, \beta, \gamma) = r(n-2m+1, 1, 1)$ and get

$$\left[\lim_{\text{NonArch}} f_1\right](p^{-\lambda}; r(n-2m+1), r, r) = \frac{[m]![n-m]!}{[1]^m \left[{n \atop m}\right] [n-2m]!},$$

$$\left[\lim_{\text{NonArch}} f_2\right](p^{-\lambda}; r(n-2m+1), r, r) = \frac{1}{[m-\lambda_1']!} p^{-r\sum_{j=1}^{m} \lambda_j(n-2m+2j-1)},$$

$$\left[\lim_{\text{NonArch}} f_3\right](p^{-\lambda}; r(n-2m+1), r, r) = \frac{[1]^m}{\prod_{i=0}^{k} [\lambda_i' - \lambda_{i+1}']!},$$

which will agree with the second part of Proposition 3 once we show that the exponents of p are the same, that is

$$-\sum_{j=1}^{m} \lambda_j (n-2m+2j-1) = -\sum (\lambda_i')^2 - (n-2m) \sum \lambda_i'.$$

However, since $\sum \lambda_j' = \sum \lambda_i$, the last equality reduces to

$$\sum_{j=1}^{m} \lambda_j (2j-1) = \sum (\lambda_i')^2$$

and this equality follows from the fact that both sides evaluate the cardinality of $\mathrm{End}_{\mathcal{O}}(\oplus \mathcal{O}/\wp^{\lambda_i})$. Thus, we conclude that

$$\mathtt{dh}^{\mathbb{K}}(p^{-\lambda}) = \left[\lim_{\textbf{NonArch}} \mathtt{ds}_m^q\right] \left(p^{-\lambda}; r(n-2m+1), r, r\right). \qquad \square$$

3.4. Limits of functions

In [17, §2], Koornwinder has given an alternative proof for Haran's non-Archimedean limit of little q-Jacobi polynomials which involves the one variable shifted Macdonald polynomials. This section consists of a generalization of this proof to the multidimensional case.

Proposition 9.
$$\left[\lim_{\textbf{NonArch}} \mathtt{C}_\lambda^q\right] (p^{-\mu}; p^{-r}) = \mathtt{c}_\lambda^{\mathbb{K}}(p^{-\mu}).$$

Proof. Using 2.17, Definition 7 and the definition of the non-Archimedean limit, we observe that

$$\left[\lim_{\textbf{NonArch}} \mathtt{C}_\lambda^q\right] (p^{-\mu}; p^{-r}) = \mathtt{C}_\lambda^q (q^{\mu} p^{-r\rho}; p^{-r})\big|_{q=0} = \binom{\mu}{\lambda}_{0,p^{-r}}. \tag{3.4}$$

Hence, recalling 2.10, we should show that $\binom{\mu}{\lambda}_{0,p^{-r}} = \binom{\mu}{\lambda} = \mathtt{c}_\lambda^{\mathbb{K}}(p^{-\mu})$. Indeed

$$\binom{\mu}{\lambda}_{q,t} = t^{-n(\mu)+n(\lambda)}\ J_{\mu/\lambda}(1,t,t^2,\ldots;q,t) \quad [18, §15] \tag{3.5}$$

where $J_{\mu/\lambda}$ are symmetric functions defined in terms of the integral Macdonald polynomials $J_\nu = v_\nu P_\nu$ by

$$J_{\mu/\lambda} = \frac{v_\mu'}{v_\lambda'} \sum_\nu (v_\nu')^{-1} f_{\nu,\lambda}^{\mu} J_\nu \quad [18, §15; 21, \mathrm{VI}(7.5)] \tag{3.6}$$

with $f^{\mu}_{\nu,\lambda} = f^{\mu}_{\nu,\lambda}(q,t)$ defined by

$$P_{\lambda}(\mathbf{x}; q,t) P_{\nu}(\mathbf{x}; q,t) = \sum_{\mu} f^{\mu}_{\nu,\lambda}(q,t) P_{\mu}(\mathbf{x}; q,t) \quad \text{[21, VI (7.1')]} \tag{3.7}$$

We now substitute (3.6) into (3.5) and specialize to the case $q = 0$. As $v'_{\lambda}(0,t) \equiv 1$ we get

$$\binom{\mu}{\lambda}_{0,t} = t^{-n(\mu)+n(\lambda)} \sum_{\nu} f^{\mu}_{\nu,\lambda}(0,t) J_{\nu}(1,t,t^2,\ldots;0,t), \tag{3.8}$$

however,

$$J_{\nu}(1,t,t^2,\ldots;q,t) = t^{n(\nu)} \quad \text{[21, p. 366(9)]}$$
$$f^{\mu}_{\nu,\lambda}(0,t) = t^{n(\mu)-n(\lambda)-n(\nu)} g^{\mu}_{\nu,\lambda}(t^{-1}) \quad \text{[21, p. 217(3.6) and p. 343(7.2)(ii)]}$$

where $g^{\mu}_{\nu,\lambda}$ are the Hall polynomials [21, Chapter II]. Thus

$$\binom{\mu}{\lambda}_{0,t} = \sum_{\nu} g^{\mu}_{\nu,\lambda}(t^{-1}) \tag{3.9}$$

Since $g^{\mu}_{\nu,\lambda}(p^r)$ is by definition the number of $\mathcal{O}$-submodules of type λ and co-type ν in an $\mathcal{O}$-module of type μ, summing over all the co-types ν, gives the total number of submodules of type λ, and we have

$$\left[\lim_{\mathbf{NonArch}} \mathtt{C}^{q}_{\lambda}\right](p^{-\mu}; p^{-r}) = \binom{\mu}{\lambda}_{0,p^{-r}} = \binom{\mu}{\lambda} = \mathtt{c}^{\mathbb{K}}_{\lambda}(p^{-\mu}). \qquad \square$$

With this in hand we can prove (the non-Archimedean part of)

Theorem 10. *For any local field* **F** *the idempotents in the Hecke algebra associated with the Grassmann representation are limits of multivariable little q-Jacobi polynomials.*

Proof. The partial orderings $\leqslant$ and $\preceq$ can be completed simultaneously to a total ordering, e.g. the lexicographical ordering. Let $\mathcal{M} = \text{Flag}\{\mathtt{M}_{\lambda} | \lambda \in \Lambda_m\}$ be the flag defined by the monomial basis of $\mathcal{A}_m$ with respect to such total ordering. The multivariable little q-Jacobi are obtained by applying the Gram-Schmidt procedure to the flag $\mathcal{M}$, with respect to the inner product $\langle \cdot, \cdot \rangle_{q,a,b,t}$ (Definition 4).

The Archimedean limit, $\mathtt{E}_{\lambda} = \lim_{\mathbf{Arch}} \mathtt{E}^{q}_{\lambda}$, follows as this inner product deforms continuously to the inner product $\langle \cdot, \cdot \rangle_{\alpha,\beta,\gamma}$, which is used to define the generalized multivariable polynomials (§2.1), see [29] for details.

As for the non-Archimedean limit, we observe that the flag $\mathcal{M}$ is also defined by the basis $\{\mathtt{C}_\lambda^q\}_{\lambda\in\Lambda_m}$; Indeed,

$$\mathtt{C}_\lambda^q = P_\lambda^\star(q^\lambda)^{-1}P_\lambda + \text{lower terms w.r.t. } \leqslant \quad \text{(by the binomial formula [22, (1.12)])}$$

$$P_\lambda = \mathtt{M}_\lambda + \text{lower terms w.r.t. } \preceq \quad \text{(by definition [21, VI(4.7)])}$$

Thus, using the total ordering which refines both partial orderings, $\{\mathtt{C}_\lambda^q\}$ and $\{\mathtt{M}_\lambda\}$ define the same flag $\mathcal{M}$. The idempotents basis in the non-Archimedean Hecke algebra $\mathcal{H}_m^{\mathbb{K}}$ are obtained by applying the Gram–Schmidt procedure to the cellular basis $\{\mathtt{c}_\lambda^{\mathbb{K}}\}$. As the q-Selberg measure deforms continuously to the non-Archimedean measure $\mathtt{dh}^{\mathbb{K}}$ (by Theorem 8), and the basis $\{\mathtt{C}_\lambda^q\}$ converges to the cellular basis (by Proposition 9), the multivariable little q-Jacobi polynomials converge to the idempotents up to constants. Our normalization in Definition 4 is designed to eliminate these constants, as for idempotents one has $\|\mathtt{e}_\lambda^{\mathbb{K}}\|^2 = \mathtt{e}_\lambda^{\mathbb{K}}(\mathbf{0})$. $\square$

4. Example

This section is devoted to the one-dimensional case which was treated in [10] (see also [17]), as it admits a completely explicit description. For $m = 1$, the representation of $K^{\mathbf{F}}$ arises from its action on the projective space $X_1^{\mathbf{F}} = \mathbb{P}_{\mathbf{F}}^{n-1}$. The representation $L^2(\mathbb{P}_{\mathbf{F}}^{n-1})$ decomposes into irreducible representations $\{\mathcal{U}_\lambda^{\mathbf{F}}\}_{\lambda\in\Lambda_1}$ where $\Lambda_1 = \mathbb{N}_0$. The space $\mathbb{P}_{\mathbf{F}}^{n-1} \times_{K^{\mathbf{F}}} \mathbb{P}_{\mathbf{F}}^{n-1}$, which describes the $K^{\mathbf{F}}$-relative position of two lines, is given by $[0, 1]$ (normalized angles) for an Archimedean place and by $\{p^{-\lambda}\}_{\lambda\in\mathbb{N}_0\cup\{\infty\}} \subseteq [0, 1]$ for a non-Archimedean place. The triplets (space, measure, idempotents) are given as follows.

Archimedean. For $u \in [0, 1]$ and $\lambda \in \mathbb{N}_0$ let

$$\mathtt{dS}(u; \alpha, \beta) = \frac{\Gamma(\frac{\alpha}{2}+\frac{\beta}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})} u^{\frac{\alpha}{2}-1}(1-u)^{\frac{\beta}{2}-1}du$$

$$\mathtt{E}_\lambda(u; \alpha, \beta) = \frac{(\frac{\alpha}{2})_\lambda(\frac{\alpha}{2}+\frac{\beta}{2})_\lambda}{(\frac{\beta}{2})_\lambda\ \lambda!} \frac{2\lambda-1+\frac{\alpha}{2}+\frac{\beta}{2}}{\lambda-1+\frac{\alpha}{2}+\frac{\beta}{2}}\, {}_2F_1\left[{-\lambda, \lambda+\frac{\alpha}{2}+\frac{\beta}{2}-1 \atop \alpha}; u\right]$$

where $(y)_j = y(y+1)\cdots(y+j-1)$ is the shifted factorial and ${}_2F_1$ is the hypergeometric function.[5] $\mathtt{dS}(u; \alpha, \beta)$ is the normalized beta measure on the unit interval, and $\{\mathtt{E}_\lambda(u; \alpha, \beta)\}_{\lambda\in\mathbb{N}_0}$ are the normalized Jacobi polynomials. For the special values $(\alpha, \beta) = (n-1, 1)$ and $(\alpha, \beta) = 2(n-1, 1)$ the triplet $([0, 1], \mathtt{dS}(u; \alpha, \beta), \{\mathtt{E}_\lambda(u; \alpha, \beta)\}_{\lambda\in\mathbb{N}_0})$ specializes to the real and complex triplets $([0, 1], \mathtt{dh}^{\mathbb{R}}, \{\mathtt{e}_\lambda^{\mathbb{R}}\}_{\lambda\in\mathbb{N}_0})$ and $([0, 1], \mathtt{dh}^{\mathbb{C}}, \{\mathtt{e}_\lambda^{\mathbb{C}}\}_{\lambda\in\mathbb{N}_0})$. The dimensions of the irreducible representations

[5] ${}_2F_1\left[{\alpha_1, \alpha_2 \atop \alpha_3}; u\right] = \sum_{j=0}^\infty \frac{(\alpha_1)_j(\alpha_2)_j}{(\alpha_3)_j\, j!}u^j$.

are given by

$$\dim \mathcal{U}_\lambda^{\mathbb{R}} = \mathtt{E}_\lambda(0; n-1, 1) = \frac{2\lambda + \frac{n}{2} - 1}{\lambda + \frac{n}{2} - 1} \frac{(\frac{n-1}{2})_\lambda (\frac{n}{2})_\lambda}{(\frac{1}{2})_\lambda \lambda!} ,$$

$$\dim \mathcal{U}_\lambda^{\mathbb{C}} = \mathtt{E}_\lambda(0; 2n-2, 2) = \frac{2\lambda + n - 1}{n-1} \binom{n+\lambda-2}{\lambda}^2 .$$

Non-Archimedean. The case $m = 1$ is greatly simplified by the fact that the terms in the filtration §2.2 ($\star$) are in bijection with the irreducibles, and each step in the filtration contains exactly one new irreducible representation. It follows that λ and k are identified and

$$\dim \mathcal{U}_\lambda^{\mathbb{K}} = |\mathbb{P}^{n-1}_{\mathcal{O}/\wp^\lambda}| - |\mathbb{P}^{n-1}_{\mathcal{O}/\wp^{\lambda-1}}| = \begin{cases} \frac{(1-p^{-r(n-1)})}{(1-p^{-r})} p^{r(n-1)}, & \lambda = 1, \\ \frac{(1-p^{-rn})(1-p^{-r(n-1)})}{(1-p^{-r})} p^{r(n-1)\lambda}, & \lambda \geqslant 2 \end{cases}$$

and is equal to 1 for $\lambda = 0$, where $|\mathbb{P}^{n-1}_{\mathcal{O}/\wp^\lambda}| = \frac{1-p^{-rn}}{1-p^{-r}} p^{r(n-1)\lambda}$ for $\lambda \geqslant 1$, and $|\mathbb{P}^{n-1}_{\mathcal{O}/\wp^0}| = 1$. The measure is easily seen to be

$$\mathtt{dh}^{\mathbb{K}}(p^{-\lambda}) = \begin{cases} \frac{|\mathbb{P}^{n-1}_{\mathcal{O}/\wp}| - 1}{|\mathbb{P}^{n-1}_{\mathcal{O}/\wp}|} = \frac{1-p^{-r(n-1)}}{1-p^{-rn}}, & \lambda = 0, \\ \frac{1}{|\mathbb{P}^{n-1}_{\mathcal{O}/\wp}|} \frac{|\mathbb{A}^n_{\wp/\wp^\lambda}| - 1}{|\mathbb{A}^n_{\wp/\wp^\lambda}|} = \frac{(1-p^{-r})(1-p^{-r(n-1)})}{(1-p^{-rn}) p^{-r(n-1)\lambda}}, & \lambda \geqslant 1, \end{cases}$$

where $\mathbb{A}^n$ stands for the affine n-space. For the idempotents, we use again the fact that the filtration admits only one new irreducible in each step, but this time on the level of the Hecke algebras. The Hecke algebra $\mathcal{H}_1^{\mathbb{K}}$ is the direct limit the algebras $\{\mathcal{H}_\lambda\}_{\lambda \in \mathbb{N}_0}$ ($\lambda = k^1$ of §2.2). Each of these algebras contains a unit element $\mathbf{1}_\lambda$, which as a function on the orbits space is given by $|\mathbb{P}^{n-1}_{\mathcal{O}/\wp^\lambda}| \mathbb{1}_{\{p^{-\mu} | \mu \geqslant \lambda\}}$. Thus on $p^{-\mathbb{N}_0}$ we have

$$\mathtt{e}_\lambda^{\mathbb{K}} = \begin{cases} \mathbf{1}_0 = \mathbb{1}_{p^{-\mathbb{N}_0}}, & \lambda = 0, \\ \mathbf{1}_\lambda - \mathbf{1}_{\lambda-1} = |\mathbb{P}^{n-1}_{\mathcal{O}/\wp^\lambda}| \mathbb{1}_{\{p^{-\mu}|\mu \geqslant \lambda\}} - |\mathbb{P}^{n-1}_{\mathcal{O}/\wp^{\lambda-1}}| \mathbb{1}_{\{p^{-\mu}|\mu \geqslant \lambda-1\}}, & \lambda \geqslant 1, \end{cases}$$

$$\mathtt{c}_\lambda^{\mathbb{K}} = \sum_{\mu \geqslant \lambda} \mathtt{g}_\mu^{\mathbb{K}} = \mathbb{1}_{\{p^{-\mu}|\mu \geqslant \lambda\}}$$

and the non-Archimedean triplet is $(p^{-\mathbb{N}_0} \cup \{0\}, \mathtt{dh}^{\mathbb{K}}, \{\mathtt{e}_\lambda^{\mathbb{K}}\}_{\lambda \in \mathbb{N}_0})$.

Quantum. For $q \in (0, 1)$ and $\lambda \in \mathbb{N}$ let

$$\mathtt{dS}^q(q^\lambda; a, b) = \frac{(a;q)_\infty}{(ab;q)_\infty}\frac{(b;q)_\lambda}{(q;q)_\lambda}a^\lambda,$$

$$\mathtt{E}^q_\lambda(x; a, b) = \frac{(1-abq^{2\lambda-1})(abq^{-1};q)_\lambda(a;q)_\lambda}{(1-abq^{-1})(q;q)_\lambda(b;q)_\lambda a^\lambda}\,{}_2\phi_1\left[\begin{matrix} q^{-\lambda}, q^{\lambda-1}ab \\ a \end{matrix}; q, qx\right],$$

$$\mathtt{C}^q_\lambda(x) = \frac{(x;q^{-1})_\lambda}{(q^\lambda;q^{-1})_\lambda} = \frac{(x-1)(x-q)\cdots(x-q^{\lambda-1})}{(q^\lambda-1)(q^\lambda-q)\cdots(q^\lambda-q^{\lambda-1})},$$

where $(y;q)_j = (1-y)(1-yq)\cdots(1-yq^{j-1})$ is the q-shifted factorial and ${}_2\phi_1$ is the basic hypergeometric function.[6] $\mathtt{dS}^q(q^\lambda; a, b)$ is the normalized q-beta measure on the set $\{q^\lambda\}_{\lambda=0}^\infty$ and the associated orthogonal base consists of the normalized little q-Jacobi polynomials [8,17], $\{\mathtt{E}^q_\lambda(x; a, b)\}_{\lambda\in\mathbb{N}_0}$. Then the q-triplet is given by $(q^{\mathbb{N}_0},\ \mathtt{dS}^q(q^\lambda; a, b), \{\mathtt{E}^q_\lambda(x; a, b)\}_{\lambda\in\mathbb{N}_0})$.

Remarks.

- The parameter t does not appear in the one-dimensional case.
- The formula

$$\mathtt{D}^q_\lambda(a, b) = \mathtt{E}^q_\lambda(\mathbf{0}; a, b) = \frac{(1-abq^{2\lambda-1})(abq^{-1};q)_\lambda(a;q)_\lambda}{(1-abq^{-1})(q;q)_\lambda(b;q)_\lambda a^\lambda}$$

  interpolates between the dimensions of the irreducible representations $\mathcal{U}^{\mathbf{F}}_\lambda$.
- The non-Archimedean limit for $\mathtt{dS}^q$, $\mathtt{D}^q_\lambda$ and $\mathtt{C}^q_\lambda$ is immediate. For more details regarding this limit see [10;17, §2]. The Archimedean limit of the q-beta measure and basic hypergeometric series is discussed in [3, §1; 8, pp. 1–28].

5. Related problems

5.1. The module of intertwining operators $\mathcal{S}(X^{\mathrm{F}}_{m_1} \times_{K^{\mathrm{F}}} X^{\mathrm{F}}_{m_2})$

For $m_1 \leqslant m_2 \leqslant [n/2]$, one can consider in a similar manner the module of intertwining operators between the representations $\mathcal{S}(X^{\mathbf{F}}_{m_2})$ and $\mathcal{S}(X^{\mathbf{F}}_{m_1})$. This results in a very similar discussion, where the only difference occurs in the parameters α, β and γ, while the geometry remains as in the equal dimension case for $m = m_1$. As an example see [17] for the case $m_1 = 1$.

[6] ${}_2\phi_1\left[\begin{matrix} a_1, a_2 \\ a_3 \end{matrix}; u\right] = \sum_{j=0}^\infty \frac{(a_1;q)_j(a_2;q)_j}{(a_3;q)_j(q;q)!}u^j.$

5.2. Dimensions of the irreducible representations

The q-dimension of the irreducible representation $\mathcal{U}_\lambda^q$ in the quantum Grassmannian, which independently on the normalization is given by

$$\mathrm{D}_\lambda^q(a,b,t) = \frac{\mathrm{E}_\lambda^q(\mathbf{0};a,b,t)^2}{\|\mathrm{E}_\lambda^q(\mathbf{x};a,b,t)\|^2} \tag{5.1}$$

interpolates between the dimensions of the irreducible representations which correspond to λ for all local fields.

5.3. Haran's process

The case $m=1$ was studied extensively by Haran in [10]. Haran also constructs discrete random processes in order to obtain the bases for the Archimedean places, the non-Archimedean places and the q-case. The bases are defined on the Martin boundary of the processes. It would be interesting to find a generalization of these processes in the case of Grassmannians.

5.4. Other algebraic groups

A natural venue for further study is to consider other maximal compact subgroups $K^{\mathbf{F}}$ of reductive algebraic groups and natural multiplicity free representations of them $V^{\mathbf{F}}$. The finite analogue of such representations can be found in [27], in which such representations of Chevalley groups over finite fields are studied. These can be considered as the *level zero* part of representations of the maximal compact subgroups. Roughly, the picture is

$$\begin{array}{ccl} V^{\mathbb{R}}, V^{\mathbb{C}} \longleftarrow q \longrightarrow & V^{\mathbb{K}} & \\ & \uparrow & \\ & V^{\mathcal{O}/\wp} & = \text{level zero part of } V^{\mathbb{K}}. \end{array}$$

For example, the particular case of $\mathrm{GL}_n(\mathcal{O}/\wp)$, which admits the $|\mathcal{O}/\wp|^{-1}$-Hahn polynomials as idempotents, is just the first term in the filtration described in §2.2.

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