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The Asymptotic Zero Distribution of Orthogonal Polynomials with Varying Recurrence Coefficients

A. B. J. Kuijlaars*

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong

and

W. Van Assche†

Celestijnenlaan 200B, 3001 Leuven, Belgium
E-mail: walter@wis.kuleuven.ac.be

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We study the zeros of orthogonal polynomials $p_{n,N}$, $n=0, 1, \ldots$, that are generated by recurrence coefficients $a_{n,N}$ and $b_{n,N}$ depending on a parameter N. Assuming that the recurrence coefficients converge whenever n, N tend to infinity in such a way that the ratio n/N converges, we show that the polynomials $p_{n,N}$ have an asymptotic zero distribution as n/N tends to $t>0$ and we present an explicit formula for the limiting measure. This formula contains the asymptotic zero distributions for various special classes of orthogonal polynomials that were found earlier by different methods, such as Jacobi polynomials with varying parameters, discrete Chebyshev polynomials, Krawtchouk polynomials, and Tricomi–Carlitz polynomials. We also give new results on zero distributions of Charlier polynomials, Stieltjes–Wigert polynomials, and Lommel polynomials.

* Current address: Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001 Leuven, Belgium. E-mail: arno@wis.kuleuven.ac.be.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The well-known Favard theorem in the theory of orthogonal polynomials says that any sequence of polynomials p_n, $n \geqslant 0$, satisfying a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n \geqslant 0, \tag{1.1}$$

with $a_n > 0$, $b_n \in \mathbf{R}$, and $p_0 \equiv 1$, $p_{-1} \equiv 0$, is a sequence of orthonormal polynomials with respect to a probability measure ρ on $\mathbf{R}$, i.e.,

$$\int p_m(x)\, p_n(x)\, d\rho(x) = \delta_{m,n}, \tag{1.2}$$

see [2]. As a consequence, orthogonal polynomials on the real line can be studied either from the orthogonality relation (1.2) or from the recurrence relation (1.1).

The zeros of the orthogonal polynomials p_n are real and simple, and they are contained in the convex hull of the support of the orthogonality measure. An object of frequent study is the asymptotic distribution of the zeros for a sequence of orthogonal polynomials. Associated with p_n is its normalized zero counting measure

$$\nu(p_n) := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j,n}}, \tag{1.3}$$

where $x_{j,n}$, $j = 1, ..., n$ are the zeros of p_n and $\delta_{x_{j,n}}$ denotes the Dirac point mass at the zero $x_{j,n}$. The measure $\nu(p_n)$ is a probability measure on the real line. We say that the probability measure μ is the asymptotic zero distribution of the sequence $\{p_n\}$, if

$$\lim_{n \to \infty} \int f\, d\nu(p_n) = \int f\, d\mu$$

for every continuous function f on $\mathbf{R}$ that vanishes at ∞.

Two basic results on the asymptotic zero distribution of orthogonal polynomials are

PROPOSITION 1.1 (see, e.g., [23, 37]). *If the sequences of recurrence coefficients $\{a_n\}$ and $\{b_n\}$ have limits $a > 0$ and $b \in \mathbf{R}$, respectively, then the polynomials p_n generated by* (1.1) *have the asymptotic zero distribution $\omega_{[\alpha,\beta]}$ with density*

$$\frac{d\omega_{[\alpha,\beta]}(t)}{dt} = \begin{cases} \dfrac{1}{\pi\sqrt{(\beta-t)(t-\alpha)}}, & \text{if} \quad t \in (\alpha, \beta), \\ 0, & \text{elsewhere}, \end{cases} \tag{1.4}$$

where

$$\alpha = b - 2a, \qquad \beta = b + 2a. \tag{1.5}$$

The measure $\omega_{[\alpha,\beta]}$ is known as the arcsine measure on the interval $[\alpha, \beta]$. It is also known as the equilibrium measure from logarithmic potential theory of $[\alpha, \beta]$, see, e.g., [30].

PROPOSITION 1.2 ([7], see also [32]). *If the measure ρ is supported on the interval $[\alpha, \beta]$ and has a density with respect to Lebesgue measure which is positive almost everywhere and if the polynomials p_n, $\deg p_n = n$, satisfy the orthogonality relation* (1.2), *then they have the asymptotic zero distribution $\omega_{[\alpha,\beta]}$ given by* (1.4).

In fact, Proposition 1.2 is a special case of Proposition 1.1, which follows from a result of Rakhmanov [26, 27], see also [19].

Extensions of Proposition 1.2 have been given for orthogonal polynomials with respect to varying weights. The prototype result here is the following:

PROPOSITION 1.3. *Let w be a continuous weight on the interval $[\alpha, \beta]$, and let $p_{n,N}$, $n = 0, 1, \ldots$, be the sequence of orthogonal polynomials with respect to the measure $w^{2N}(x)\,dx$, so that we have a two-dimensional table of orthogonal polynomials. Then any ray sequence in this table has an asymptotic zero distribution which is characterized by a minimal energy problem with external field.*

To be precise, if $\{n_j\}$, $\{N_j\}$ are two sequences of natural numbers such that $N_j \to \infty$ and $n_j/N_j \to t > 0$ as $j \to \infty$, then the polynomials p_{n_j, N_j} have the asymptotic zero distribution $\mu_{w,t}$, where $\mu_{w,t}$ is the probability measure on $[\alpha, \beta]$ that minimizes the weighted energy

$$\iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y) - \frac{2}{t} \int \log w(x)\, d\mu(x)$$

among all probability measures μ on $[\alpha, \beta]$.

See [14, 20, 21]. See also the recent monograph of Saff and Totik [30] for a comprehensive account of many results in this direction.

In view of the two different approaches to orthogonal polynomials, one expects a companion result to Proposition 1.3 that would involve zero

asymptotics for orthogonal polynomials with *varying recurrence coefficients*. Somewhat surprisingly, a general result in this direction has not appeared in the literature before and it is the goal of this paper to fill this gap.

To state our theorem we use the notation

$$\lim_{n/N \to t} X_{n,N} = X$$

to denote the property that in the doubly indexed sequence $X_{n,N}$ we have

$$\lim_{j \to \infty} X_{n_j, N_j} = X$$

whenever n_j and N_j are two sequences of natural numbers such that $N_j \to \infty$ and $n_j/N_j \to t$ as $j \to \infty$. For example, the convergence in Proposition 1.3 may be expressed by

$$\lim_{n/N \to t} \nu(p_{n,N}) = \mu_{w,t}.$$

We will use this notation throughout the rest of the paper.

Our main result is the following.

THEOREM 1.4. *Let for each $N \in \mathbf{N}$, two sequences $\{a_{n,N}\}_{n=1}^{\infty}$, $a_{n,N} > 0$ and $\{b_{n,N}\}_{n=0}^{\infty}$, $b_{n,N} \in \mathbf{R}$, of recurrence coefficients be given, together with orthogonal polynomials $p_{n,N}$ generated by the recurrence*

$$\begin{aligned} x p_{n,N}(x) = a_{n+1,N} p_{n+1,N}(x) + b_{n,N} p_{n,N}(x) \\ + a_{n,N} p_{n-1,N}(x), \qquad n \geqslant 0, \end{aligned} \tag{1.6}$$

and the initial conditions $p_{0,N} \equiv 1$ and $p_{-1,N} \equiv 0$. Suppose that there exist two continuous functions a: $(0, \infty) \to [0, \infty)$, b: $(0, \infty) \to \mathbf{R}$, such that

$$\lim_{n/N \to t} a_{n,N} = a(t), \qquad \lim_{n/N \to t} b_{n,N} = b(t) \tag{1.7}$$

whenever $t > 0$. Define the functions

$$\alpha(t) := b(t) - 2a(t), \qquad \beta(t) := b(t) + 2a(t), \qquad t > 0. \tag{1.8}$$

Then we have for every $t > 0$,

$$\lim_{n/N \to t} \nu(p_{n,N}) = \frac{1}{t} \int_0^t \omega_{[\alpha(s), \beta(s)]}\, ds. \tag{1.9}$$

Here $\omega_{[\alpha,\beta]}$ *is the measure given by* (1.4) *if* $\alpha<\beta$. *If* $\alpha=\beta$, *then* $\omega_{[\alpha,\beta]}$ *is the Dirac point mass at* α.

Remark 1.5. The measure on the right-hand side of (1.9) is the average of the equilibrium measures of the varying intervals $[\alpha(s), \beta(s)]$ for $0<s<t$. Its support is given by

$$[\inf_{0<s<t} \alpha(s), \sup_{0<s<t} \beta(s)]. \tag{1.10}$$

In particular, the support is always an interval. The support is unbounded if α or β are unbounded near 0.

Remark 1.6. Theorem 1.4 has an obvious extension to polynomials that are orthogonal with respect to a discrete measure supported on a finite set of points, so that the three-term recurrence relation terminates. In some of the examples in Section 4, such as the Krawtchouk polynomials, the recurrence (1.6) terminates at $n=N$. Then the limit relation (1.9) only holds for $t\leqslant 1$, provided, of course, that (1.7) holds for $t\leqslant 1$.

Remark 1.7. Under some additional assumptions, Theorem 1.4 was obtained earlier by Deift and McLaughlin [3] as part of their study of the continuum limit of the Toda lattice. They assumed that α and β are smooth functions having the following monotonicity properties:

(a) α is decreasing,

(b) β has at most one critical point, which, if it exists, is a maximum.

Without any substantial change in their arguments (which consist of a detailed WKB analysis for difference equations), the condition (a) can be replaced by

(a′) α has at most one critical point, which, if it exists, is a minimum.

It follows from (a′) and (b) that for each x, the set

$$\{s>0 : \alpha(s)\leqslant x\leqslant \beta(s)\}$$

is an interval. If we denote the end points of this interval by $t_-(x)$ and $t_+(x)$, then the support of the measure (1.9) is given by

$$\{x\in \mathbf{R} : t_-(x)\leqslant t\} \tag{1.11}$$

and the measure itself is

$$\left(\frac{1}{\pi t}\int_{t_-(x)}^{\min(t,\, t_+(x))} \frac{1}{\sqrt{(\beta(s)-x)(x-\alpha(s))}}\, ds\right) dx. \tag{1.12}$$

It should be noted that in all the examples we consider in Section 4 (and in many more that we looked at, but did not include), the functions α and β satisfy the conditions (a′) and (b). So these seem to be very natural conditions. In this connection we want to pose the following problem.

Problem 8. Give an example to Theorem 1.4 arising from a system of orthogonal polynomials, known in the literature, such that the conditions (a′) and/or (b) are not satisfied.

Any such example would be very interesting, especially if there is specific information available about the associated orthogonality measures.

Remark 1.9. In a paper on eigenvalue distributions of random Hermitian matrices [25, Theorem 2.2], Pastur presents a result on the eigenvalues of large matrices from the so-called orthogonal polynomial ensembles, which is similar to our Theorem 1.4. Pastur restricts himself to the symmetric case (i.e., $b_{n,N}=0$ for all n and N). In his (sketchy) proof he compares the eigenvalue distributions to the zero distributions of orthogonal polynomials given by varying recurrence coefficients. Extending his arguments, which are based on estimates for Stieltjes transforms and operator theory for Jacobi matrices, one may be able to obtain an alternative proof of Theorem 1.4. We believe, however, that the proof in our paper is simpler.

Note that the factor $1/\pi$ is missing in formula (2.32) of [25].

Our second result deals with the behavior of the extremal zeros of the orthogonal polynomials $p_{n,N}$. We need to assume a little more.

THEOREM 1.10. *Assume that the conditions of Theorem* 1.4 *are satisfied. Assume, in addition, that the functions $a(t)$ and $b(t)$ are continuous at $t=0$, and that* (1.7) *holds for $t=0$ as well. Let $x_1(n,N)$ denote the smallest zero of $p_{n,N}$ and $x_n(n,N)$ its largest zero. Then for every $t>0$,*

$$\lim_{n/N\to t} x_1(n,N) = \min_{0\leqslant s\leqslant t} \alpha(s), \tag{1.13}$$

$$\lim_{n/N\to t} x_n(n,N) = \max_{0\leqslant s\leqslant t} \beta(s). \tag{1.14}$$

Remark 1.11. That some additional assumption is needed in Theorem 1.10 can be seen from the following example, see also Subsection 4.1. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences with limits 1/2 and 0, respectively, then the polynomials p_n generated by (1.1) belong to the Nevai–Blumenthal class $M(1,0)$. Putting $a_{n,N}=a_n$ and $b_{n,N}=b_n$ for every N, we see that the assumptions of Theorem 1.4 are satisfied with $a(t)=1/2$ and $b(t)=0$, so that $\alpha(t)=-1$ and $\beta(t)=1$. It is easy to find $\{a_n\}$ and

$\{b_n\}$ from the class $M(1, 0)$ such that the largest zero of p_n does not converge to 1. Indeed, it is enough to take $b_0 > 1$, since b_0 is the (only) zero of p_1 and the largest zero of p_n, $n \geqslant 2$, is greater than b_0.

The proofs of Theorems 1.4 and 1.10 can be found in Section 3. As a preliminary result, which may be of independent interest, we first discuss a general theorem on ratio asymptotics of orthogonal polynomials, see Theorem 2.1 in Section 2, which generalizes an earlier result of Van Assche and Koornwinder [38]. In Section 4 we present a large number of examples, in order to show that Theorem 1.4 presents a unifying approach to various results that were treated separately in the literature. First we show that Proposition 1.1 is a special case of Theorem 1.4. We go on to discuss Jacobi polynomials with varying parameters which leads to the asymptotic zero distributions first given in [31]. Then we consider discrete Chebyshev polynomials and Krawtchouk polynomials, which satisfy a discrete orthogonality. These polynomials were studied recently in [29] and [5, 6], respectively. In Subsection 4.5 we look at regularly varying recurrence coefficients, which appear when dealing with exponential weights (Freud weights) on the real line and on unbounded discrete sets. In Section 4.6 we show how Theorem 1.4 can be applied to q-polynomials with varying parameter q. The case of the Stieltjes–Wigert polynomials is worked out in detail. Our final examples deal with Tricomi–Carlitz polynomials, whose zero distributions were studied before in [12, 13] and Lommel polynomials.

2. RATIO ASYMPTOTICS

The proof of Theorem 1.4 is based on the following result on ratio asymptotics of orthogonal polynomials with varying recurrence coefficients which may be of interest in its own right.

THEOREM 2.1. *Suppose we have for each $N \in \mathbf{N}$, the sequences $\{a_{n,N}\}_{n=1}^{\infty}$ and $\{b_{n,N}\}_{n=0}^{\infty}$ of recurrence coefficients, with $a_{n,N} > 0$ and $b_{n,N} \in \mathbf{R}$. Let $p_{n,N}$ be the orthogonal polynomials generated by the recurrence* (1.6). *Let $t > 0$ and assume that*

$$\lim_{n/N \to t} a_{n,N} = A, \qquad \lim_{n/N \to t} b_{n,N} = B, \tag{2.1}$$

for certain numbers $A \geqslant 0$ and $B \in \mathbf{R}$. Assume, in addition, that for some $t^* > t$, there exist $m \leqslant B - 2A$, $M \geqslant B + 2A$ such that all zeros of $p_{n,N}$ belong to $[m, M]$ whenever $n \leqslant t^* N$. Then

$$\lim_{n/N \to t} \frac{a_{n+1,N} p_{n+1,N}(z)}{p_{n,N}(z)} = \frac{z-B}{2} + \sqrt{\left(\frac{z-B}{2}\right)^2 - A^2} \tag{2.2}$$

uniformly on compact subsets of $\mathbf{C}\backslash[m, M]$. *In* (2.2) *that branch of the square root is taken which is positive for* $z > B + 2A$.

First we need a simple lemma which we singled out since it will be used in the next section as well.

LEMMA 2.2. *Let* $\{p_n\}$ *be a sequence of orthogonal polynomials satisfying the three-term recurrence* (1.1). *Suppose the zeros of* p_{n+1} *are in* $[m, M]$. *Then*

(a) *For all* $z \in \mathbf{C}\backslash[m, M]$,

$$\left|\frac{p_n(z)}{a_{n+1}\, p_{n+1}(z)}\right| \leqslant \frac{1}{dist(z, [m, M])}. \tag{2.3}$$

(b) *For all* $z \in \mathbf{C}$ *such that* $|z| > \max(|m|, |M|)$,

$$\left|\frac{p_n(z)}{a_{n+1}\, p_{n+1}(z)}\right| \geqslant \frac{1}{2\,|z|}. \tag{2.4}$$

Proof. The proof is based on the partial fraction decomposition

$$\frac{p_n(z)}{a_{n+1}\, p_{n+1}(z)} = \sum_{j=1}^{n+1} \frac{d_j}{z - x_j},$$

where $x_1, ..., x_{n+1}$ are the zeros of p_{n+1} and $d_1, ..., d_{n+1}$ are positive numbers that add up to 1. This follows from the fact that the zeros of p_n and p_{n+1} are interlacing combined with the fact that the leading coefficients of p_n and $a_{n+1}\, p_{n+1}$ are equal. Since all the zeros of p_{n+1} are in $[m, M]$, we have $|z - x_j| \geqslant dist(z, [m, M])$ for all j and (2.3) follows.

If $|z| > \max(|m|, |M|)$, then $|x_j/z| < 1$ for all j and therefore

$$\operatorname{Re}\left(\frac{1}{1 - x_j/z}\right) > \frac{1}{2}, \qquad j = 1, ..., n+1.$$

Hence

$$\left|\frac{p_n(z)}{a_{n+1}\, p_{n+1}(z)}\right| = \frac{1}{|z|}\left|\sum_{j=1}^{n+1} \frac{d_j}{1 - x_j/z}\right| \geqslant \frac{1}{|z|} \operatorname{Re}\left(\sum_{j=1}^{n+1} \frac{d_j}{1 - x_j/z}\right)$$
$$> \frac{1}{2\,|z|} \sum_{j=1}^{n+1} d_j = \frac{1}{2\,|z|}.$$

This proves (2.4). ∎

Proof of Theorem 2.1. We first note that the family of functions

$$\left\{\frac{p_{n,N}(z)}{a_{n+1,N}p_{n+1,N}(z)} : n, N \in \mathbf{N}, n \leqslant t^*N\right\} \tag{2.5}$$

is uniformly bounded on compact subsets of $\bar{\mathbf{C}}\backslash[m, M]$, since each of its members satisfies the estimate (2.3). Hence (2.5) is a normal family on $\bar{\mathbf{C}}\backslash[m, M]$.

Define

$$\phi(z) := \left(\frac{z-B}{2} + \sqrt{\left(\frac{z-B}{2}\right)^2 - A^2}\right)^{-1}. \tag{2.6}$$

We are going to prove inductively that if we have sequences n_j, $N_j \to \infty$ with $n_j/N_j \to t$ such that the functions

$$f_j(z) := \frac{p_{n_j,N_j}(z)}{a_{n_j+1,N_j}p_{n_j+1,N_j}(z)} \tag{2.7}$$

converge uniformly on compact subsets of $\bar{\mathbf{C}}\backslash[m, M]$, then

$$f(z) := \lim_{j\to\infty} f_j(z) = \phi(z) + \mathcal{O}(z^{-k}) \qquad (z \to \infty). \tag{2.8}$$

This is clear if $k = 1$, since $f_j(z) = \mathcal{O}(1/z)$, for every j, because of (2.3) and $\phi(z) = 1/z + \mathcal{O}(z^{-2})$, which follows immediately from the definition (2.6).

Now suppose that (2.8) holds for some $k \geqslant 1$. Let $n_j, N_j \to \infty$ with $n_j/N_j \to t$ such that f_j defined by (2.7) converges to f uniformly on compact subsets of $\bar{\mathbf{C}}\backslash[m, M]$. Since $t < t^*$ we may assume without loss of generality that $n_j < t^*N_j$ for every j. Put

$$g_j(z) := \frac{p_{n_j-1,N_j}(z)}{a_{n_j,N_j}p_{n_j,N_j}(z)}, \qquad z \in \mathbf{C}\backslash[m, M].$$

Then g_j belongs to the family (2.5), which is normal, and therefore has a subsequence that converges uniformly on compact subsets of $\bar{\mathbf{C}}\backslash[m, M]$. Passing to such a subsequence, we may assume that the functions g_j converge on $\bar{\mathbf{C}}\backslash[m, M]$ with limit g, say. From the induction hypothesis we obtain that

$$g(z) = \phi(z) + \mathcal{O}(z^{-k}) \qquad (z \to \infty). \tag{2.9}$$

Rewriting the recurrence relation (1.6), we have

$$\frac{a_{n_j+1,N_j}p_{n_j+1,N_j}(z)}{p_{n_j,N_j}(z)} = z - b_{n_j,N_j} - a^2_{n_j,N_j}\frac{p_{n_j-1,N_j}(z)}{a_{n_j,N_j}p_{n_j,N_j}(z)}.$$

Taking the limit $j \to \infty$ and using (2.1), we find

$$\frac{1}{f(z)} = z - B - A^2 g(z), \qquad z \in \overline{\mathbf{C}} \backslash [m, M].$$

By (2.9) this leads to

$$\frac{1}{f(z)} = z - B - A^2 \phi(z) + \mathcal{O}(z^{-k}) = \frac{1}{\phi(z)} + \mathcal{O}(z^{-k}), \tag{2.10}$$

where we used that ϕ satisfies $z - B - A^2 \phi(z) = 1/\phi(z)$, which is easy to obtain from (2.6). Since $\phi(z) = 1/z + \mathcal{O}(z^{-2})$, we obtain from (2.10)

$$f(z) = \left(\frac{1}{\phi(z)} + \mathcal{O}(z^{-k}) \right)^{-1} = \frac{\phi(z)}{1 + \mathcal{O}(z^{-k-1})} = \phi(z) + \mathcal{O}(z^{-k-2}).$$

So we see that (2.8) holds with k replaced by $k+2$.

Therefore, by mathematical induction, we find that (2.8) holds for all k. The uniqueness of the Laurent expansion around infinity, then implies that $f(z) = \phi(z)$, whenever f is the limit of a sequence of functions as in (2.7).

Now, to complete the proof, it is enough to observe that for $n_j, N_j \to \infty$ with $n_j/N_j \to t$, we have that the functions f_j defined by (2.7) belong to the normal family (2.5) if j is sufficiently large. By normality, the sequence $\{f_j\}$ has a subsequence that converges uniformly on compact subsets of $\overline{\mathbf{C}} \backslash [m, M]$, and by what has been proved above, the limit of any such subsequence is equal to ϕ. Therefore, by a standard compactness argument, the full sequence $\{f_j\}$ converges to ϕ and this gives (2.2). ∎

Remark 2.3. Theorem 2.1 is an extension of a result of Van Assche and Koornwinder [38], who obtained the ratio asymptotics (2.2) under additional hypotheses in the recurrence coefficients.

3. PROOFS

In this section we give the proof of Theorems 1.4 and 1.10.

Proof of Theorem 1.4. Assume that the conditions of the theorem are satisfied and let $t > 0$. First we prove (1.9) under the additional assumption that there is a number $t^* > t$ such that

$$M := \sup \{ |b_{n,N}| : n \leqslant t^* N \} + 2 \sup \{ a_{n,N} : n \leqslant t^* N \} < \infty. \tag{3.1}$$

This finiteness assumption deals with the behavior of the recurrence coefficients for small n/N, as it follows from the convergence (1.7) that the

recurrence coefficients are uniformly bounded if n/N is restricted to a compact subset of $(0, \infty)$.

From the above assumption it follows that the zeros of the orthogonal polynomials $p_{n,N}$, $n \leqslant t^*N$, are all in the interval $[-M, M]$, see e.g. [37]. From (1.7) and (3.1) it is further clear that $M \geqslant b(s) + 2a(s)$ and $-M \leqslant b(s) - 2a(s)$ for every $s \in [0, t^*]$.

We use $P_{n,N}$ to denote the monic orthogonal polynomials. That is, $P_{n,N} = p_{n,N}/\gamma_{n,N}$, where $\gamma_{n,N}$ is the leading coefficient of $p_{n,N}$. Since $a_{k+1,N} p_{k+1,N}$ and $p_{k,N}(z)$ have the same leading coefficient, we have for every k,

$$\frac{P_{k+1,N}(z)}{P_{k,N}(z)} = \frac{a_{k+1,N} p_{k+1,N}(z)}{p_{k,N}(z)},$$

and it follows that

$$P_{n,N}(z) = \prod_{k=0}^{n-1} \frac{a_{k+1,N} p_{k+1,N}(z)}{p_{k,N}(z)}.$$

Then we obtain

$$\begin{aligned} \frac{1}{n} \log |P_{n,N}(z)| &= \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{a_{k+1,N} p_{k+1,N}(z)}{p_{k,N}(z)} \right| \\ &= \int_0^1 \log \left| \frac{a_{[sn]+1,N} p_{[sn]+1,N}(z)}{p_{[sn],N}(z)} \right| ds. \end{aligned} \tag{3.2}$$

Here $[sn]$ denotes the integer part of sn. As $n/N \to t$, we have $[sn]/N \to st$ for every $s > 0$. Thus it follows from (1.7) and Theorem 2.1 that

$$\lim_{n/N \to t} \frac{a_{[sn]+1,N} p_{[sn]+1,N}(z)}{p_{[sn],N}(z)} = \frac{z - b(st)}{2} + \sqrt{\left(\frac{z - b(st)}{2}\right)^2 - a(st)^2} \tag{3.3}$$

for every $z \in \mathbf{C} \backslash [-M, M]$. Furthermore, by Lemma 2.2, we have the estimates

$$dist(z, [-M, M]) \leqslant \left| \frac{a_{k+1,N} p_{k+1,N}(z)}{p_{k,N}(z)} \right| \leqslant 2\,|z| \tag{3.4}$$

for $|z| > M$. Then we get by (3.2)–(3.4) and Lebesgue's dominated convergence theorem, for $|z| > M$,

$$\begin{aligned} \lim_{n/N \to t} \frac{1}{n} \log |P_{n,N}(z)| &= \int_0^1 \log \left| \frac{z - b(st)}{2} + \sqrt{\left(\frac{z - b(st)}{2}\right)^2 - a(st)^2} \right| ds \\ &= \frac{1}{t} \int_0^t \log \left| \frac{z - b(s)}{2} + \sqrt{\left(\frac{z - b(s)}{2}\right)^2 - a(s)^2} \right| ds. \end{aligned}$$

Using the fact that the logarithmic potential of the measure $\omega_{[\alpha(s),\,\beta(s)]}$ is equal to

$$U^{\omega_{[\alpha(s),\,\beta(s)]}}(z) := \int \log \frac{1}{|z-y|}\, d\omega_{[\alpha(s),\,\beta(s)]}(y)$$
$$= -\log \left| \frac{z-b(s)}{2} + \sqrt{\left(\frac{z-b(s)}{2}\right)^2 - a(s)^2} \right|$$

for $z \in \mathbf{C} \backslash [\alpha(s), \beta(s)]$, see [30], we get that for $|z| > M$,

$$\lim_{n/N \to t} \frac{1}{n} \log |P_{n,N}(z)| = -\frac{1}{t} \int_0^t U^{\omega_{[\alpha(s),\,\beta(s)]}}(z)\, ds = -U^{\sigma}(z), \tag{3.5}$$

where σ is the measure

$$\sigma = \frac{1}{t} \int_0^t \omega_{[\alpha(s),\,\beta(s)]}\, ds.$$

As the zeros of $P_{n,N}$ are in $[-M, M]$ for $n \leqslant t^*N$, it follows that (3.5) holds for all $z \in \mathbf{C} \backslash [-M, M]$. This gives by a standard argument, see [30], that the polynomials $P_{n,N}$ have σ as limiting zero distribution as $n/N \to t$. So we have proved (1.9) under the additional assumption (3.1).

To prove the theorem in the general case, we write for $\delta > 0$,

$$a_{n,N}^{(\delta)} := a_{n+[\delta N],N}, \qquad b_{n,N}^{(\delta)} := b_{n+[\delta N],N}, \tag{3.6}$$

where $[\delta N]$ is the integer part of δN. We denote the orthogonal polynomials generated by these recurrence coefficients by $p_{n,N}^{(\delta)}$. These polynomials are known as the $[\delta N]$th associated polynomials.

From (1.7) and (3.6) we see that

$$\lim_{n/N \to t} a_{n,N}^{(\delta)} = a(t+\delta), \qquad \lim_{n/N \to t} b_{n,N}^{(\delta)} = b(t+\delta)$$

and, in addition, we get that the recurrence coefficients (3.6) are uniformly bounded for $n \leqslant t^*N$. Thus, by what has been proved before, for $t > \delta$,

$$\lim_{n/N \to t} \nu(p_{n-[\delta N],N}^{(\delta)}) = \frac{1}{t-\delta} \int_0^{t-\delta} \omega_{[\alpha(s+\delta),\,\beta(s+\delta)]}\, ds$$
$$= \frac{1}{t-\delta} \int_\delta^t \omega_{[\alpha(s),\,\beta(s)]}\, ds. \tag{3.7}$$

Next, it is well known that the zeros of $p_{n,N}$ separate the zeros of its $[\delta N]$th associated polynomial of degree $n-[\delta N]$, that is, the zeros of $p_{n-[\delta N],N}^{(\delta)}$. This means that there are $n-[\delta N]$ zeros of $p_{n,N}$ that interlace

with the zeros of $p_{n-[\delta N]}^{(\delta)}$. In view of (3.7), it follows that these zeros are asymptotically distributed as the right hand side of (3.7) as $n/N \to t$. Hence they give a contribution

$$\frac{1-\delta/t}{t-\delta} \int_{\delta}^{t} \omega_{[\alpha(s),\, \beta(s)]}\, ds = \frac{1}{t} \int_{\delta}^{t} \omega_{[\alpha(s),\, \beta(s)]}\, ds \tag{3.8}$$

to the asymptotic zero distribution of $p_{n,N}$ as $n/N \to t$.

What remains are $[\delta N] \approx (\delta/t)\, n$ zeros of $p_{n,N}$. Their contribution to the asymptotic zero distribution of $p_{n,N}$ is negligible as $\delta \to 0$. Thus we can let $\delta \to 0$ in (3.8) to obtain (1.9). This completes the proof of Theorem 1.4. ∎

Proof of Theorem 1.10. From Theorem 1.4 it follows that the zeros of the orthogonal polynomials $p_{n,N}$ are dense in the support of the measure

$$\frac{1}{t} \int_{0}^{t} \omega_{[\alpha(s),\, \beta(s)]}\, ds$$

whenever $N \to \infty$ and $n/N \to t$. Thus in view of (1.10)

$$\limsup_{n/N \to t} x_1(n, N) \leqslant \min_{0 \leqslant s \leqslant t} \alpha(s),$$

$$\liminf_{n/N \to t} x_n(n, N) \geqslant \max_{0 \leqslant s \leqslant t} \beta(s).$$

To prove the converse, we use the inequality

$$x_1(n, N) \geqslant \min_{0 \leqslant k \leqslant n-1} (b_{k,N} - a_{k,N} - a_{k+1,N}), \tag{3.9}$$

and a similar inequality for $x_n(n, N)$, see [37, p. 437]. From the assumptions of the theorem, it follows easily that the right-hand side of (3.9) tends to $\min_{0 \leqslant s \leqslant t} \alpha(s)$ as $n/N \to t$. This immediately gives

$$\liminf_{n/N \to t} x_1(n, N) \geqslant \min_{0 \leqslant s \leqslant t} \alpha(s).$$

Similarly

$$\limsup_{n/N \to t} x_n(n, N) \leqslant \max_{0 \leqslant s \leqslant t} \beta(s)$$

and we have proved (1.13) and (1.14). This completes the proof of Theorem 1.10. ∎

4. EXAMPLES

Theorem 1.4 can be used to obtain asymptotic zero distributions of many classes of orthogonal polynomials, that are known in the literature. We present here a selection of the results that can be obtained. For definitions and special properties of the orthogonal polynomials appearing below, we refer the reader to [2, 33] and especially to the useful report [16].

4.1. *The Nevai–Blumenthal Class $M(a, b)$*

If the recurrence coefficients in (1.1) satisfy

$$\lim_{n\to\infty} a_n = a/2, \qquad \lim_{n\to\infty} b_n = b \in \mathbf{R},$$

then the orthogonal polynomials are said to belong to the class $M(a, b)$ introduced by Nevai [23]. If we define for each N,

$$a_{n,N} := a_n, \qquad b_{n,N} := b_n,$$

then the condition (1.7) in Theorem 1.4 is satisfied with constant functions $a(t) = a/2$ and $b(t) = b$. Hence, according to Theorem 1.4, the asymptotic zero distribution is equal to $\omega_{[b-a,\, b+a]}$. Thus Proposition 1.1 is a special case of Theorem 1.4.

4.2. *Jacobi Polynomials*

Orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$, are in the class $M(1, 0)$ since they have recurrence coefficients

$$a_n = 2\sqrt{\frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2\,(2n+\alpha+\beta+1)}}, \tag{4.1}$$

and

$$b_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}. \tag{4.2}$$

Thus the Jacobi polynomials have the arcsine measure $\omega_{[-1,1]}$ as asymptotic zero distribution.

More interesting limiting behavior can be obtained if the parameters $\alpha = \alpha_N$ and $\beta = \beta_N$ depend on N. Jacobi polynomials with varying parameters were considered before in a number of papers [1, 4, 8, 11, 15,

22]. The corresponding recurrence coefficients are denoted here by $a_{n,N}$ and $b_{n,N}$. Suppose the following limits exist

$$\lim_{N\to\infty}\frac{\alpha_N}{N}=A\geqslant 0,\qquad \lim_{N\to\infty}\frac{\beta_N}{N}=B\geqslant 0.$$

Then it is easy to get from (4.1), (4.2) that for $t>0$,

$$\lim_{n/N\to t} a_{n,N}=\frac{2\sqrt{t(t+A+B)(t+A)(t+B)}}{(2t+A+B)^2}, \tag{4.3}$$

and

$$\lim_{n/N\to t} b_{n,N}=\frac{B^2-A^2}{(2t+A+B)^2}. \tag{4.4}$$

So the functions $\alpha(t)$ and $\beta(t)$ from (1.8) are here

$$\alpha(t)=\frac{B^2-A^2-4\sqrt{t(t+A+B)(t+A)(t+B)}}{(2t+A+B)^2} \tag{4.5}$$

$$\beta(t)=\frac{B^2-A^2+4\sqrt{t(t+A+B)(t+A)(t+B)}}{(2t+A+B)^2}. \tag{4.6}$$

See Fig. 1 for the curves α and β with $A=1$ and $B=2$.

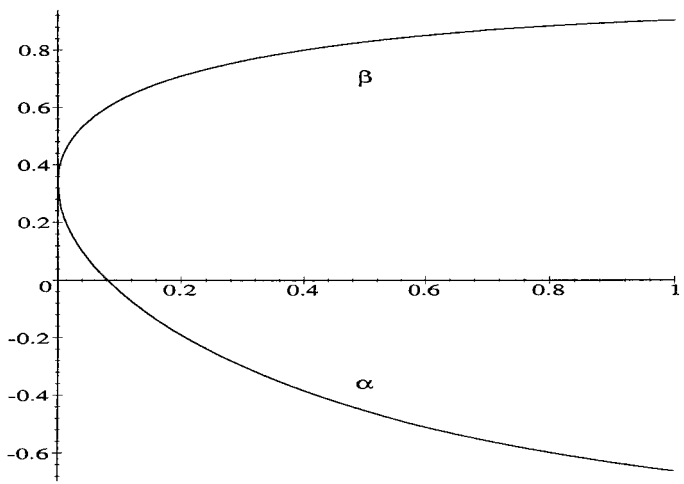

FIG. 1. The curves α and β for $A=1$ and $B=2$.

From Theorem 1.4 we deduce that the asymptotic zero distribution of the Jacobi polynomials $p_n^{(\alpha_N, \beta_N)}$, $n/N \to t$, is given by

$$\frac{1}{t} \int_0^t \omega_{[\alpha(s), \beta(s)]}\, ds \tag{4.7}$$

with α and β as in (4.5) and (4.6). The function $\alpha(s)$ is decreasing and $\beta(s)$ is increasing. Therefore, by Remark 1.5, the support of the measure (4.7) is $[\alpha(t), \beta(t)]$. This is a smaller interval than $[-1, 1]$ unless A and B are both 0. For each $x \in [\alpha(t), \beta(t)]$, there is a unique $t_- = t_-(x) \in [0, t]$ such that either $\alpha(t_-(x)) = x$, or $\beta(t_-(x)) = x$. Thus the density of the measure (4.7) is

$$\frac{1}{\pi t} \int_{t_-(x)}^t \frac{1}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}\, ds, \qquad x \in [\alpha(t), \beta(t)],$$

see also (1.12). It is somewhat remarkable that this integral can be evaluated explicitly in terms of elementary functions, leading to the density

$$\frac{2t + A + B}{2\pi t} \frac{\sqrt{(\beta(t) - x)(x - \alpha(t))}}{1 - x^2}, \qquad x \in [\alpha(t), \beta(t)].$$

In Fig. 2 we have plotted this density for $A = 1$, $B = 2$ and various values of t.

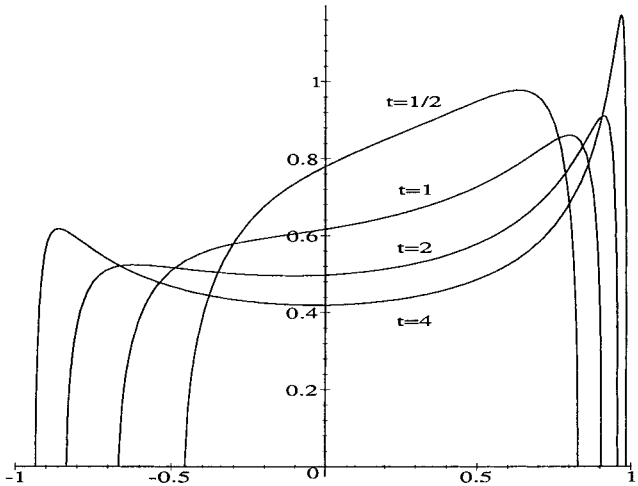

FIG. 2. Some densities for the asymptotic zero distribution.

This limiting density for the zeros of Jacobi polynomials was obtained earlier by Saff *et al.* [31] in their work on incomplete polynomials, see also Chen and Ismail [1] and Gawronski and Shawyer [11] for strong asymptotics.

If at least one of A and B is different from 0, then the limiting relations (4.3) and (4.4) are also valid for $t=0$. Then Theorem 1.10 gives us the limits of the smallest and largest zeros, as obtained earlier by Moak *et al.* [22].

Recently, Dette and Studden [4], see also [8], considered Jacobi polynomials with parameters α_N and β_N tending to infinity essentially faster than N. If $(\alpha_N+\beta_N)/N \to \infty$ and $\alpha_N/\beta_N \to c \in [0, \infty]$, then the measure (4.7) reduces to the Dirac measure at the point $(1-c)/(1+c)$. By rescaling the independent variables as well, they were able to find interesting limit distributions of the zeros of

$$P_n^{(\alpha_N, \beta_N)}(\gamma_N(x-(1-c)/(1+c)))$$

with suitably chosen $\gamma_N \to 0$. Dette and Studden used continued fraction techniques, but these results can also be derived from the three-term recurrence relations via our Theorem 1.4.

The same techniques used for Jacobi polynomials (i.e., changing the parameters and rescaling the independent variable) can be used to obtain asymptotic zero distributions of many systems of orthogonal polynomials for which the recurrence coefficients are explicitly known. In this way one may rederive, for example, the results on zero distributions of varying Hermite and Laguerre polynomials given in [1, 4, 8, 9, 10].

4.3. *Hahn Polynomials*

Recently, classical orthogonal polynomials of a discrete variable have been studied from the point of view of extremal problems in potential theory. In order to describe the asymptotic zero distribution, Rakhmanov [29] found that the minimal energy problem stated in Proposition 1.3 should be combined with an upper constraint for the extremal measure. This constraint arises in a natural way from the fact that any two zeros of an orthogonal polynomial are separated by at least one point of the support of the orthogonality measure. Rakhmanov studied discrete Chebyshev polynomials $t_{n,N}$, $n \leqslant N$. These polynomials are orthogonal on the finite set of points $\{0, 1, ..., N-1\}$,

$$\frac{1}{N}\sum_{x=0}^{N-1} t_{n,N}(x)\, t_{m,N}(x) = 0, \qquad m < n \leqslant N,$$

see, e.g., [2]. The rescaled polynomials

$$T_{n,N}(x) := t_{n,N}\left(\frac{N-1}{2}(x+1)\right)$$

are orthogonal on the points $-1+2(k-1)/(N-1)$, $k=1, ..., N$, which are in $[-1, 1]$. Rakhmanov showed that for $t \in (0, 1)$,

$$\lim_{n/N \to t} \nu(T_{n,N}) = \mu_t,$$

where μ_t minimizes the logarithmic energy

$$\iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y)$$

among all probability measures μ on $[-1, 1]$ satisfying the constraint $d\mu(x) \leqslant 1/(2t)\, dx$. In addition, he gave an explicit formula for μ_t.

We now show how to obtain the measures μ_t from our Theorem 1.4. The recurrence coefficients, in orthonormal form, for the polynomials $T_{n,N}$ are

$$a_{n,N} = \frac{n\sqrt{N^2-n^2}}{(N-1)\sqrt{(2n-1)(2n+1)}}, \qquad b_{n,N} = 0.$$

Thus, for $t \in (0, 1]$, the limits (1.7) exist and the functions α and β from (1.8) are

$$\alpha(t) = -\sqrt{1-t^2}, \qquad \beta(t) = \sqrt{1-t^2}, \qquad 0 \leqslant t \leqslant 1. \tag{4.8}$$

See Fig. 3.

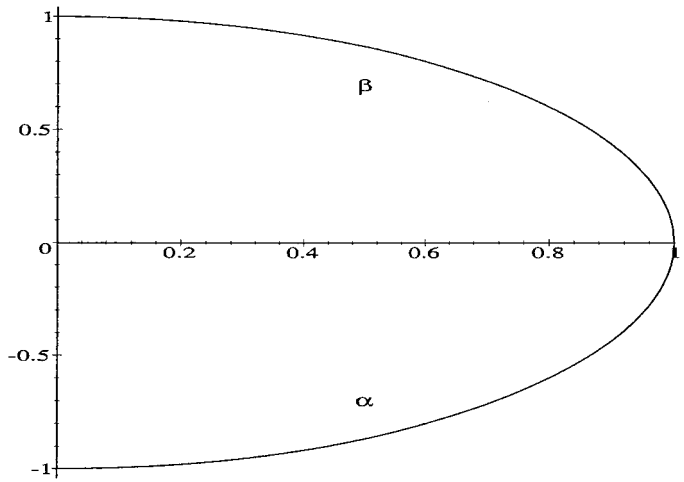

FIG. 3. The functions α and β for discrete Chebyshev polynomials.

Note that α and β are defined on $[0, 1]$ only. This clearly does not affect the result of Theorem 1.4 as long as $t \leqslant 1$, see Remark 1.6. Thus we have from (1.9)

$$\lim_{n/N \to t} \nu(T_{n,N}) = \frac{1}{t} \int_0^t \omega_{[\alpha(s), \beta(s)]}\, ds, \qquad 0 < t \leqslant 1. \tag{4.9}$$

The density of the measure on the right hand side of (4.9) is given by

$$\frac{1}{\pi t} \int_0^{\min(t, \sqrt{1-x^2})} \frac{1}{\sqrt{1-s^2-x^2}}\, ds.$$

This is easily evaluated and the result is

$$\begin{cases} \dfrac{1}{\pi t} \arcsin\left(\dfrac{t}{\sqrt{1-x^2}}\right), & \text{if} \quad |x| < \sqrt{1-t^2}, \\ \dfrac{1}{2t}, & \text{if} \quad |x| \geqslant \sqrt{1-t^2}, \end{cases} \tag{4.10}$$

as was proved by Rakhmanov [29]. See Fig. 4 for the asymptotic density corresponding to various values of t.

Observe that the density is equal to the constant $1/(2t)$ for $x \in [-1, \alpha(t)] \cup [\beta(t), 1]$. This is the region where the constraint $1/(2t)\, dx$ is effective.

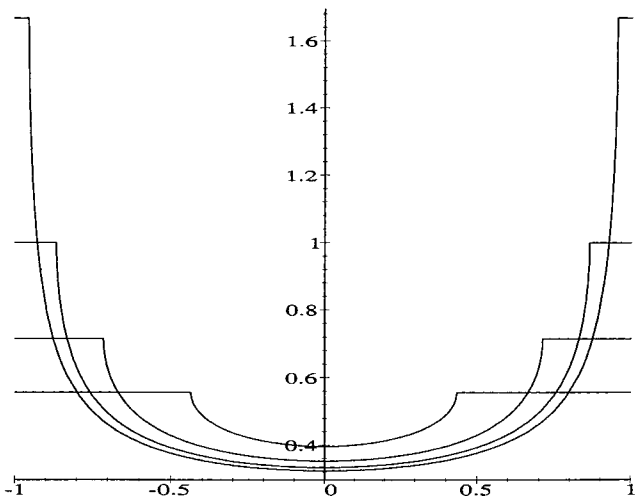

FIG. 4. Densities for the zeros of discrete Chebyshev polynomials ($t = 0.3, 0.5, 0.7, 0.9$).

The discrete Chebyshev polynomials belong to the Hahn class of orthogonal polynomials. These polynomials depend on N and on two additional parameters. The asymptotic distribution of the contracted zeros of any member of this class has the density (4.10) if these additional parameters are kept fixed. If we let them vary with N, we can obtain other asymptotic zero distributions.

4.4. *Krawtchouk Polynomials*

Other polynomials of a discrete variable that have been studied by Rakhmanov's method of constrained equilibrium problems include the Krawtchouk polynomials [5, 6] and the Meixner polynomials [17]. Since the recurrence coefficients for these polynomials are known, the asymptotic zero distributions for these systems can be obtained from our Theorem 1.4 as well. In particular, orthonormal Krawtchouk polynomials $k_n(x, p, N)$ satisfy the orthogonality

$$\sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} k_n(i)\, k_m(i) = \delta_{m,n}, \qquad m, n \leqslant N,$$

and their recurrence coefficients are

$$a_n = \sqrt{(N-n+1)\, npq}, \qquad b_n = (N-n)\, p + nq,$$

where $0 < p < 1$ and $q = 1 - p$. Of interest here are the rescaled polynomials $k_n(Nx, p, N)$, that have the recurrence coefficients $a_{n,N} = a_n/N$ and $b_{n,N} = b_n/N$. Then we immediately find

$$\lim_{n/N \to t} a_{n,N} = \sqrt{pqt(1-t)}, \qquad \lim_{n/N \to t} b_{n,N} = (1-t)\, p + tq,$$

where we have to restrict t to the interval $[0, 1]$. The functions α and β now are

$$\begin{aligned} \alpha(t) &= (1-t)\, p + tq - 2\sqrt{pqt(1-t)}, \\ \beta(t) &= (1-t)\, p + tq + 2\sqrt{pqt(1-t)}, \end{aligned} \tag{4.11}$$

which are the lower and upper part of an ellipse, as shown in Fig. 5.

Observe that α attains its minimum 0 at $t = p$ and β its maximum 1 at $t = q$. Let us assume that $p \leqslant 1/2$ (the case $p > 1/2$ is similar). Then the support of the asymptotic zero distribution is $[\alpha(t), \beta(t)]$ when $t \leqslant p$, it is equal to $[0, \beta(t)]$ when $p < t < q$, and it is $[0, 1]$ when $t \geqslant q$. The density of the asymptotic zero distribution can be computed from (1.12). Using $t_-(x)$ and $t_+(x)$ as in Remark 1.7, we get for $t \leqslant p$,

$$\frac{1}{\pi t} \int_{t_-(x)}^{t} \frac{ds}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}, \qquad x \in [\alpha(t), \beta(t)] \tag{4.12}$$

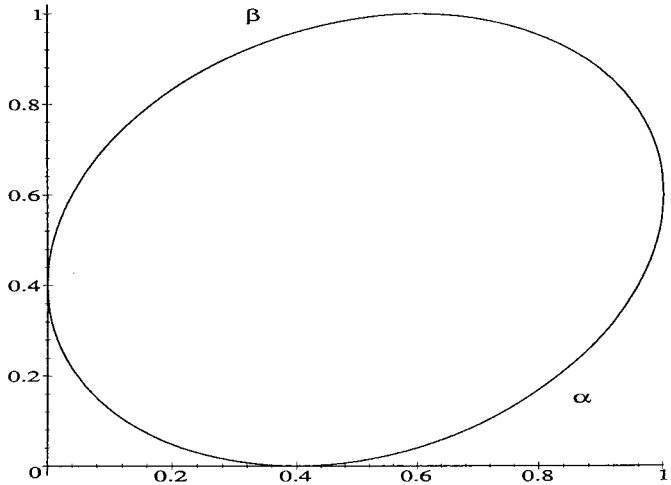

FIG. 5. The functions α and β for Krawtchouk polynomials with $p = 2/5$.

with α and β given by (4.11). For $p < t < q$, the density is

$$\frac{1}{\pi t} \int_{t_-(x)}^{t_+(x)} \frac{ds}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}, \qquad x \in [0, \alpha(t)], \tag{4.13}$$

and

$$\frac{1}{\pi t} \int_{t_-(x)}^{t} \frac{ds}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}, \qquad x \in [\alpha(t), \beta(t)]. \tag{4.14}$$

Finally, for $t \geqslant q$, the density is

$$\frac{1}{\pi t} \int_{t_-(x)}^{t_+(x)} \frac{ds}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}, \qquad x \in [0, \alpha(t)] \cup [\beta(t), 1], \tag{4.15}$$

and

$$\frac{1}{\pi t} \int_{t_-(x)}^{t} \frac{ds}{\sqrt{(\beta(s) - x)(x - \alpha(s))}}, \qquad x \in [\alpha(t), \beta(t)]. \tag{4.16}$$

All these integrals can be evaluated explicitly and they lead to the formulas given by Dragnev and Saff [5, 6], who obtained these densities from the constrained energy problem associated with Krawtchouk polynomials. The constraint here is the measure $(1/t)\, dx$ on $[0, 1]$ and there is also an external field depending on t and p.

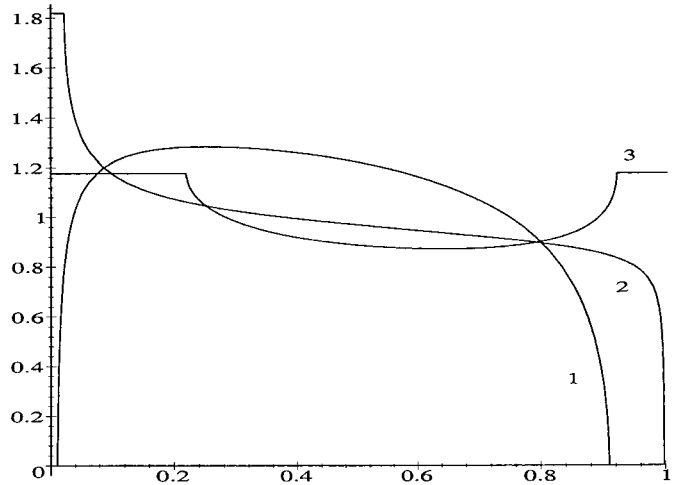

FIG. 6. Some densities for zeros of Krawtchouk polynomials with $p = 2/5$ (1 is for $t < p$, 2 for $p < t < q$, and 3 is for $t > q$).

See Fig. 6 for the asymptotic densities of the scaled zeros of Krawtchouk polynomials. Here one sees that the constraint $(1/t)\,dx$ is effective in case $p < t < q$ on the interval $[0, \alpha(t)]$ and in case $t > q$ on the two intervals $[0, \alpha(t)]$ and $[\beta(t), 1]$. Indeed, it can be verified that the integrals in (4.13) and (4.15) are equal to $1/t$. In case $t < p$, the constraint is not effective.

4.5. *Regularly Varying Recurrence Coefficients*

Quite often the recurrence coefficients of a system of orthogonal polynomials are unbounded. In such a case, it is convenient to have a method to describe the rate at which the coefficients tend to infinity. Regularly varying functions are often of help in this situation. A non-negative function $\phi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is regularly varying at infinity if for every $t > 0$ we have

$$\lim_{x \to \infty} \frac{\phi(xt)}{\phi(x)} = t^{\gamma},$$

where γ is a real number, which is the exponent (or index) of regular variation. If the recurrence coefficients are such that

$$\lim_{n \to \infty} \frac{a_n}{\phi(n)} = a > 0, \qquad \lim_{n \to \infty} \frac{b_n}{\phi(n)} = b \in \mathbf{R},$$

where ϕ is regularly varying with $\gamma > 0$, then the recurrence coefficients are unbounded and we say that they are regularly varying with index γ. Many

systems of orthogonal polynomials have regularly varying recurrence coefficients, e.g., the Hermite polynomials for which $a_n = \sqrt{n/2}$ and $b_n = 0$ (hence $\gamma = 1/2$), the Laguerre polynomials for which $a_n = \sqrt{n(n+a)}$ and $b_n = 2n + a + 1$ (so that $\gamma = 1$), the Meixner–Pollaczek polynomials with $a_n = \sqrt{(\delta^2+1)\, n(n+\eta-1)}$ and $b_n = (2n+\eta)\,\delta$ (again with $\gamma = 1$), and the Meixner polynomials with $a_n = \sqrt{cn(n+\beta-1)}/(1-c)$ and $b_n = [(1+c)n + \beta c]/(1-c)$ (also with $\gamma = 1$). Also, all orthogonal polynomials with Freud weights $d\rho(x) = \exp(-|x|^{1/\gamma})\, dx$ have regularly varying recurrence coefficients with index γ (Freud's conjecture, proved in [18]).

In order to apply our Theorem 1.4 to this situation, we consider the scaled polynomials $p_n(\phi(N)\, x)$ which have recurrence coefficients

$$a_{n,N} = \frac{a_n}{\phi(N)}, \qquad b_{n,N} = \frac{b_n}{\phi(N)}.$$

The property of regular variation then gives the functions

$$a(t) = \lim_{n/N \to t} \frac{a_n}{\phi(n)} \frac{\phi(n)}{\phi(N)} = at^\gamma, \qquad b(t) = \lim_{n/N \to t} \frac{b_n}{\phi(n)} \frac{\phi(n)}{\phi(N)} = bt^\gamma,$$

so that the corresponding functions α and β are

$$\alpha(t) = (b-2a)\, t^\gamma, \qquad \beta(t) = (b+2a)\, t^\gamma.$$

The asymptotic zero distribution is thus given by the measure

$$\frac{1}{t} \int_0^t \omega_{[(b-2a)\, s^\gamma,\, (b+2a)\, s^\gamma]}\, ds.$$

Observe that the measure $\omega_{[\alpha,\beta]}$ has a density $\omega'_{[\alpha,\beta]}(x)$ with the property $\omega'_{[\alpha s, \beta s]}(xs) = \omega'_{[\alpha,\beta]}(x)/s$, so that the asymptotic zero distribution has the density

$$\frac{1}{t} \int_0^t \omega'_{[b-2a,\, b+2a]}(xs^{-\gamma})\, ds = \frac{1}{\gamma t} \int_0^{t^\gamma} \omega'_{[b-2a,\, b+2a]}(x/y)\; y^{1/\gamma - 1} \frac{dy}{y}.$$

This integral is a Mellin convolution of the density $\omega'_{[b-2a,\, b+2a]}$ and the density $y^{1/\gamma-1}/(t\gamma)$ on $[0, t^\gamma]$ and is known as the Nevai–Ullman density. This measure is the zero distribution of orthogonal polynomials with Freud weights (this is the case when $b = 0$), which was obtained by Rakhmanov [28] and Mhaskar and Saff [20]. Observe that for $b^2 - 4a^2 < 0$ the support of the measure is $[(b-2a)\, t^\gamma, (b+2a)\, t^\gamma]$, but that for $b^2 - 4a^2 > 0$ the support is $[0, (b+2a)\, t^\gamma]$ when $b - 2a > 0$, or $[(b-2a)\, t^\gamma, 0]$ when $b + 2a < 0$. The case $b^2 - 4a^2 < 0$ corresponds to non-symmetric Freud weights [36], and the case $b^2 - 4a^2 > 0$ corresponds to Freud weights on

a discrete set, such as Meixner polynomials [17] in the sense that the Nevai–Ullman measure is indeed the asymptotic zero distribution for these families of orthogonal polynomials. At present we do not know whether the recurrence coefficients of these orthogonal polynomials are regularly varying, but we conjecture that this is indeed the case.

Theorem 2.1 for regularly varying recurrence coefficients was first proved in [35], see also [34, Theorem 4.10, p. 117]. Theorem 1.10 for log-slowly varying coefficients, i.e., when $\phi(\log x)$ is regularly varying with index 0, can be found in [24, Lemma 5]. Observe that every regularly varying function ϕ is also log-slowly varying.

For Charlier polynomials $C_n^{(a)}$, $a>0$, the orthogonality is on the non-negative integers with respect to the Poisson distribution

$$\sum_{k=0}^{\infty} C_n^{(a)}(k)\, C_m^{(a)}(k)\, \frac{a^k}{k!} = 0. \qquad m \neq n.$$

The recurrence coefficients are $a_n = \sqrt{an}$ and $b_n = n+a$, so that a_n is regularly varying with index 1/2 and b_n is regularly varying with index 1. For the scaling we need to use the largest index and consider the polynomials $C_n^{(a)}(Nx)$. Then

$$a(t) = \lim_{n/N \to t} \frac{a_n}{N} = 0, \qquad b(t) = \lim_{n/N \to t} \frac{b_n}{N} = t,$$

so that $\alpha(t) = \beta(t) = t$. The asymptotic distribution of the zeros of $C_n^{(a)}(Nx)$ is therefore given by

$$\frac{1}{t} \int_0^t \delta_s \, ds,$$

where δ_s is the Dirac measure with mass 1 at the point s. This measure is the uniform distribution on $[0, t]$, which indicates that the distribution of the zeros is completely determined by the constraint imposed by the discrete set on which the polynomials are orthogonal. If we allow the parameter a to grow with N, then we can find other zero distributions. For the polynomials $C_n^{(aN)}(Nx)$ we have

$$a(t) = \lim_{n/N \to t} \frac{\sqrt{aNn}}{N} = \sqrt{at}, \qquad b(t) = \lim_{n/N \to t} \frac{n+aN}{N} = t+a,$$

so that the functions α and β are

$$\alpha(t) = t + a - 2\sqrt{at} = (\sqrt{a} - \sqrt{t})^2, \qquad \beta(t) = t + a + 2\sqrt{at} = (\sqrt{a} + \sqrt{t})^2.$$

Observe that the minimum of α is 0, which is attained when $t=a$, and β is monotonically increasing. The support of the asymptotic zero distribution is therefore equal to $[\alpha(t), \beta(t)]$ when $t<a$, and it is equal to $[0, \beta(t)]$ whenever $t \geqslant a$. The density can be obtained explicitly by evaluating the integral (1.12), which gives for $t<a$ the density

$$\frac{1}{2t}-\frac{1}{\pi t}\arcsin\left(\frac{x+a-t}{2\sqrt{ax}}\right), \qquad (\sqrt{a}-\sqrt{t})^2 \leqslant x \leqslant (\sqrt{a}+\sqrt{t})^2,$$

and for $t \geqslant a$ the density

$$\begin{cases} \dfrac{1}{t}, & \text{if} \quad 0 \leqslant x \leqslant (\sqrt{a}-\sqrt{t})^2, \\ \dfrac{1}{2t}-\dfrac{1}{\pi t}\arcsin\left(\dfrac{x+a-t}{2\sqrt{ax}}\right), & \text{if} \quad (\sqrt{t}-\sqrt{a})^2 < x \leqslant (\sqrt{t}+\sqrt{a})^2. \end{cases}$$

Observe that for $t>a$ we again get a uniform distribution on $[0, (\sqrt{a}-\sqrt{t})^2]$, which shows that the constraint imposed by the discrete set is effective on that subinterval.

4.6. *q-Orthogonal Polynomials*

q-Orthogonal polynomials typically satisfy a recurrence relation whose recurrence coefficients are functions of q^n. Setting $q=c^{1/N}$ we are able to use Theorem 1.4 to obtain interesting asymptotic zero distributions. Van Assche and Koornwinder [38] found the asymptotic zero distribution of the Wall polynomials with varying parameter $q=c^{1/N}$. As an example of the use of Theorem 1.4, we will discuss the Stieltjes–Wigert polynomials $S_n(x; q)$, which are orthogonal with respect to the log-normal distribution,

$$\int_0^\infty S_n(x; q)\, S_m(x; q)\, w(x; q)\, dx = 0, \qquad n \neq m,$$

$$w(x; q) = \frac{\gamma}{\sqrt{\pi}} \exp(-\gamma^2 \log^2(x)), \qquad x>0,$$

with $\gamma^2 = -1/(2 \log q)$ and $0<q<1$.

The recurrence coefficients are

$$a_n = \sqrt{1-q^n}\, q^{2n-1}, \qquad b_n = \frac{1+q-q^{n+1}}{q^{2n+1}}.$$

Putting $q=c^{1/N}$, $0<c<1$, and letting $n/N \to t>0$, we find

$$a(t) = \frac{\sqrt{1-c^t}}{c^{2t}}, \qquad b(t) = \frac{2-c^t}{c^{2t}}.$$

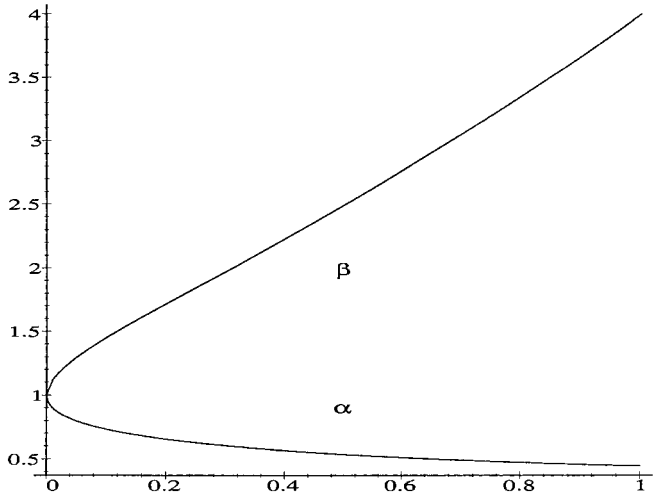

FIG. 7. The curves α and β for Stieltjes–Wigert polynomials ($c = 3/4$).

Thus we can apply Theorem 1.4 with

$$\alpha(t) = \frac{2 - c^t - 2\sqrt{1-c^t}}{c^{2t}} = \frac{(1-\sqrt{1-c^t})^2}{c^{2t}} = \frac{1}{(1+\sqrt{1-c^t})^2},$$

and

$$\beta(t) = \frac{2 - c^t + 2\sqrt{1-c^t}}{c^{2t}} = \frac{(1+\sqrt{1-c^t})^2}{c^{2t}} = \frac{1}{(1-\sqrt{1-c^t})^2}.$$

The graphs of these functions are in Fig. 7.

It follows that the asymptotic zero distribution of the Stieltjes–Wigert polynomials $S_n(x; c^{1/N})$, $n/N \to t > 0$, has a density given by

$$\frac{1}{\pi t} \int_{t\_(x)}^{t} \frac{1}{\sqrt{c^{-2s} + 2(2-c^s)\, c^{-2s} c - x^2}}\, ds, \qquad x \in [\alpha(t), \beta(t)],$$

where $t\_(x) \in [0, t]$ is such that either $\alpha(t\_(x)) = x$ or $\beta(t\_(x)) = x$. Note that α is strictly decreasing and β is strictly increasing with $\alpha(0) = \beta(0) = 1$, so that $t\_(x)$ is well-defined. After some computation one arrives at

$$\frac{1}{\pi t x \log(1/c)} \arctan\left(\frac{\sqrt{4x - (xc^t + 1)^2}}{xc^t + 1}\right), \qquad x \in [\alpha(t), \beta(t)],$$

for the density of the asymptotic zero distribution. See Fig. 8 for some densities corresponding to various values of t.

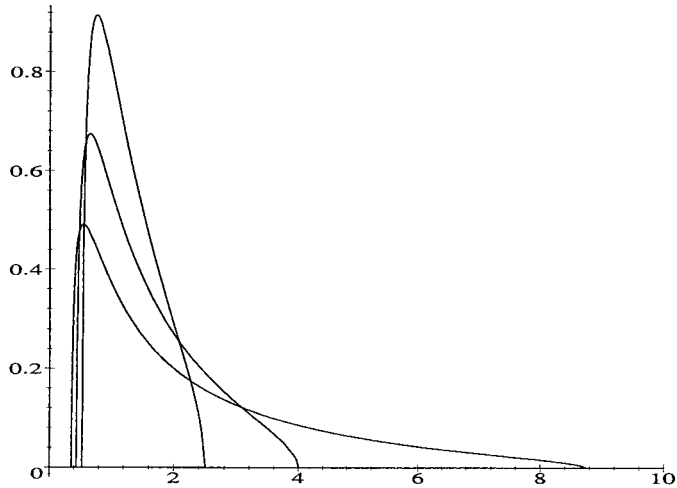

FIG. 8. Some densities for the zeros of Stieltjes–Wigert polynomials ($t = 1/2$, 1, 2).

4.7. *Tricomi–Carlitz and Lommel Polynomials*

Finally, we mention the work of Goh and Wimp [12, 13] on the zero distribution of Tricomi–Carlitz polynomials $f_n^{(\alpha)}$, see [2]. The asymptotic zero distribution of these polynomials is the Dirac point mass at zero. To obtain more interesting asymptotics, one has to rescale the independent variable by a factor $N^{-1/2}$. The polynomials $f_n^{(\alpha)}(N^{-1/2}x)$ have recurrence coefficients, in orthonormal form,

$$a_{n,N} = \sqrt{\frac{Nn}{(n+\alpha-1)(n+\alpha)}}, \qquad b_{n,N} = 0.$$

Thus the functions α and β from (1.8) are

$$\alpha(t) = -\frac{2}{\sqrt{t}}, \qquad \beta(t) = \frac{2}{\sqrt{t}}, \qquad t > 0,$$

see Fig. 9.

We now have the interesting feature that the functions α and β are unbounded near $t = 0$. Still Theorem 1.4 applies also to this case, and it follows that the Tricomi–Carlitz polynomials $f_n^{(\alpha)}(N^{-1/2}x)$ have asymptotic zero distribution

$$\frac{1}{\pi t} \int_0^{\min(t,\, 4/x^2)} \frac{1}{\sqrt{4/s - x^2}}\, ds, \qquad x \in \mathbf{R}$$

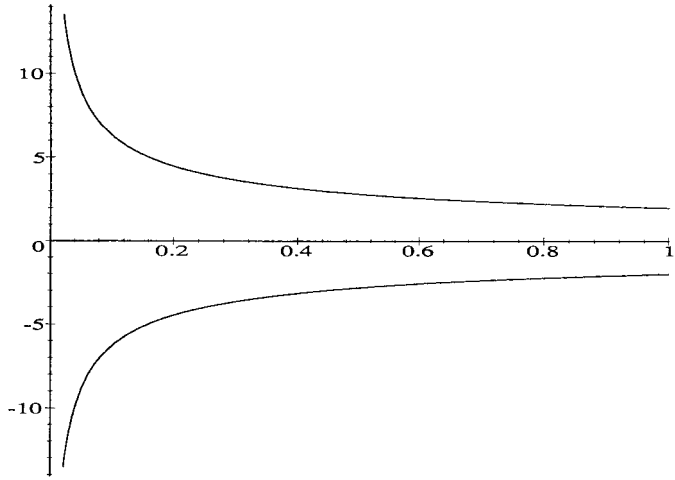

FIG. 9. The functions α and β for Tricomi–Carlitz polynomials.

as $n/N \to t$. The measure is now supported on the whole real line. The integral is easy to evaluate and the result is

$$\begin{cases} \dfrac{4}{\pi t\,|x|^3}\left(\arcsin(|x|\sqrt{t}/2) - |x|/4\sqrt{t(4-x^2t)}\right), & \text{if} \quad |x| < \dfrac{2}{\sqrt{t}}, \\ \dfrac{2}{t\,|x|^3}, & \text{if} \quad |x| \geqslant \dfrac{2}{\sqrt{t}}, \end{cases} \tag{4.17}$$

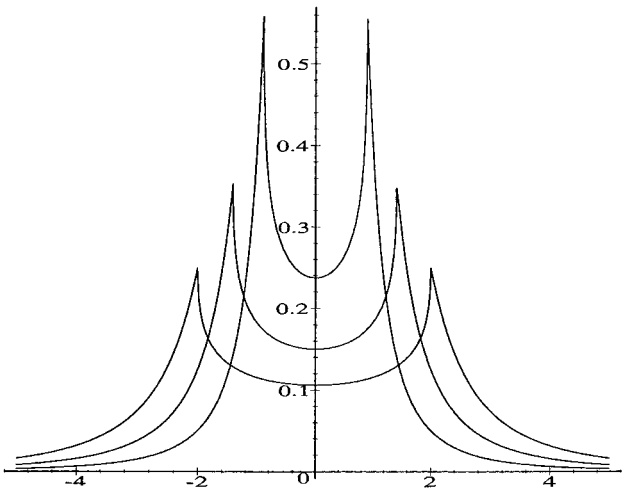

FIG. 10. Some densities for the zeros of Tricomi–Carlitz polynomials for $t = 1$, 2, and 5.

see Fig. 10. This agrees with the asymptotic distribution found by Goh and Wimp [13]. Their proof was based on the integral representation of the Tricomi–Carlitz polynomials.

Tricomi–Carlitz polynomials are orthogonal on the set $\{\pm(k+\alpha)^{-1/2}: k=0, 1, 2, ...\}$, which is discrete with one accumulation point at the origin. Between two points of this set there can be at most one zero of the orthogonal polynomial $f_n^{(\alpha)}$. Similar to the discrete Chebyshev polynomials (Subsection 4.3), the distribution of the rescaled points $\{\pm N^{1/2}/(k+\alpha)^{1/2}\}$ is a natural constraint for the density of the zeros. The part of the density (4.17) for $|x| \geqslant 2/\sqrt{t}$ comes from this constraint.

Note that the density (4.17) does not depend on the parameter α. Varying $\alpha = \alpha_N$ linearly with N, we find other non-trivial asymptotic zero distributions for Tricomi–Carlitz polynomials. These measures are again supported on bounded intervals.

A similar analysis can be made for modified Lommel polynomials $h_{n,\nu}$, see [2]. The orthonormal polynomials have recurrence coefficients

$$a_n^2 = \frac{1}{4(n+\nu)(n+\nu-1)}, \qquad b_n = 0,$$

so that $\lim_{n\to\infty} a_n = 0$, as is the case for Tricomi–Carlitz polynomials. For the rescaled polynomials $h_{n,\nu}(x/N)$ we have recurrence coefficients $a_{n,N} = Na_n$ and $b_{n,N} = 0$, hence the functions α and β are

$$\alpha(t) = -\frac{1}{2t}, \qquad \beta(t) = \frac{1}{2t}.$$

These are again unbounded near $t=0$. The distribution of the zeros of $h_{n,\nu}(x/N)$ is given by

$$\frac{1}{\pi t}\int_0^{\min(t,\, 1/2\,|x|)} \frac{1}{\sqrt{1/(4s^2)-x^2}}\, ds, \qquad x \in \mathbf{R},$$

as $n/N \to t$. This integral can be evaluated explicitly, giving

$$\begin{cases} \dfrac{1}{\pi t x^2}\left(1-\sqrt{1-4x^2t^2}\right), & \text{if} \quad |x| < \dfrac{1}{2t}, \\[2ex] \dfrac{1}{\pi t x^2}, & \text{if} \quad |x| \geqslant \dfrac{1}{2t}. \end{cases} \tag{4.18}$$

This density is again supported on the whole real line, and the part for $|x| \geqslant 1/(2t)$ reflects the constraint posed by the distribution of the points of the support of the orthogonality measure. Indeed, the modified Lommel polynomials are orthogonal on the set $\{1/j_{\nu-1,k}: k=0, \pm 1, \pm 2, ...\}$, where

$\{j_{\nu-1,k}\}$ are the zeros of the Bessel function $J_{\nu-1}$. The asymptotic distribution of the points $\{N/j_{\nu-1,k}\}$ near the origin can be determined from the behavior of the large zeros of the Bessel function, which is $j_{\nu-1,k} = k\pi + O(1)$ as $k \to \infty$, and this explains the density $1/(\pi t x^2)$.

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