On nonresonance impulsive functional differential inclusions with nonconvex valued right-hand side

M. Benchohra, A. Boucherif, and A. Ouahabi

Abstract

In this paper we investigate the existence of solutions for first and second order nonresonance impulsive functional differential inclusions. We shall rely on the Schaefer’s fixed point theorem combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values.

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1. Introduction

This paper is concerned with the existence of solutions for some classes of boundary value problems for first and second order impulsive functional differential inclusions. Initially, we will consider the first order impulsive functional differential inclusion

\[ y' - \lambda y \in F(t, y(t)), \quad \text{a.e. } t \in [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \]

\[ \Delta y|\tau_k = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \]

\[ y(t) = \phi(t), \quad t \in [-r, 0], \quad y(0) = y(T), \]

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where \( \lambda \in \mathbb{R} \), \( F : [0, T] \times D \to \mathcal{P}(\mathbb{R}^n) \) is a multivalued map, \( D = \{ \psi : [-r, 0] \to \mathbb{R}^n ; \psi \) is continuous everywhere except for a finite number of points \( \tilde{t} \) at which \( \psi(\tilde{t}) \) and \( \psi(\tilde{t}^+) \) exist and \( \psi(\tilde{t}^-) = \psi(\tilde{t}) \), \( \phi \in D, \mathcal{P}(\mathbb{R}^n) \) is the family of all subsets of \( \mathbb{R}^n \), \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T \), \( I_k \subset C(\mathbb{R}^n, \mathbb{R}^n) \) (\( k = 1, 2, \ldots, m \)). \( \Delta y|_{t=\bar{t}} = y(t_k^+) - y(t_k^-) \) and \( y(t_k^-) \) represent the left and right limits of \( y(t) \) at \( t = t_k \), respectively.

For any continuous function \( y \) defined on \( [-r, T] \setminus \{ t_1, \ldots, t_m \} \) and \( t \in [0, T] \), we denote \( y_t \) the element of \( D \) defined by \( y_t(\theta) = y(t + \theta), \theta \in [-r, 0] \). Here \( y_t(\cdot) \) represents the history of the state from time \( t - r \), up to the present time \( t \).

Later, we study the second order impulsive functional differential inclusion of the form

\[
\begin{align*}
y''(t) - \lambda y(t) & \in F(t, y_t), \quad a.e. \ t \in [0, T], \quad t \neq \bar{t}_k, \ k = 1, \ldots, m, \\
\Delta y|_{t=\bar{t}_k} & = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \\
\Delta y'|_{t=\bar{t}_k} & = \tilde{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \\
y(0) - y(\bar{t}_0) & = \mu_0, \quad y'(0) - y'(\bar{t}_0) = \mu_1,
\end{align*}
\]

where \( \lambda, F, I_k, \) and \( \phi \) are as in problem (1)–(3), \( \tilde{I}_k : \mathbb{R}^n \to \mathbb{R}^n \) and \( \mu_0, \mu_1 \in \mathbb{R}^n \).

Note that when \( \mu_0 = \mu_1 = 0 \) we have periodic boundary conditions. Differential equations with impulses are a basic tool to study evolution processes that are subjected to abrupt changes in their state. Such equations arise naturally from a wide variety of applications, such as space-craft control, inspection processes in operations research, drug administration, and threshold theory in biology. See the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [13], and Samoilenko and Perestyuk [16] and the survey paper by Rogovchenko [15] and the references cited therein. Some interesting results for impulsive functional differential equations can be found in the recent paper by Franco et al. [8].

Recently, by using the fixed point argument, existence as well as uniqueness results for nonresonance first order impulsive differential equations were given by Nieto in [14]. These results were extended to the functional case with more general boundary conditions by Benchohra and Eloe in [3]. With the aid of a fixed point theorem due to Martelli for condensing multivalued maps an existence theorem for nonresonance impulsive functional differential inclusions was obtained by Benchohra et al. in [4]. However in [4] the right-hand side \( F(t, y_t) \) was assumed to be convex. Here we shall drop this restriction and consider the problems (1)–(3) and (4)–(7) with nonconvex valued right-hand side. We shall rely on the Schaefer’s fixed point theorem combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values.

This paper will be divided into four sections. In Section 2 we will recall briefly some basic definitions and preliminary facts which will be used later. In Sections 3 and 4 we shall establish an existence theorem for each of the problems (1)–(3) and (4)–(7). We consider the case when \( \lambda \neq 0 \). Note that when the impulses are absent (i.e., for \( I_k, \tilde{I}_k \equiv 0, k = 1, \ldots, m \)), then the above problems are nonresonance problems since the linear part in Eqs. (1) and (4) is invertible. The results of the present paper extend to the multivalued case some one obtained by Nieto in [14] and to the nonconvex valued case those obtained by Benchohra et al. in [4].
2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. Let \((a, b)\) be an open interval. \(AC^i((a, b), \mathbb{R}^n)\) \((i = 0, 1)\) is the space of \(i\)-times differentiable functions \(y: (a, b) \rightarrow \mathbb{R}^n\), whose \(i\)th derivative, \(y^{(i)}\), is absolutely continuous.

Let \(A\) be a subset of \([0, T] \times D\). \(A\) is \(L \otimes B\) measurable if \(A\) belongs to the \(\sigma\)-algebra generated by all sets of the form \(J \times D\) where \(J\) is Lebesgue measurable in \([0, T]\) and \(D\) is Borel measurable in \(D\). A subset \(A\) of \(L^1([0, T], \mathbb{R}^n)\) is decomposable if for all \(u, v \in A\) and \(J \subset [0, T]\) measurable the function \(u\chi_J + v\chi_{[0,T] - J} \in A\), where \(\chi_J\) stands for the characteristic function of the set \(J\).

Let \(E\) be a Banach space, \(X\) a nonempty closed subset of \(E\) and \(G: X \rightarrow P(E)\) a multivalued operator with nonempty closed values. \(G\) is lower semi-continuous (l.s.c.) if the set \(\{x \in X: G(x) \cap B \neq \emptyset\}\) is open for any open set \(B\) in \(E\). \(G\) has a fixed point if there is \(x \in X\) such that \(x \in G(x)\).

### Definition 2.1.
Let \(Y\) be a separable metric space and let \(N: Y \rightarrow P(L^1([0, T], \mathbb{R}^n))\) be a multivalued operator. We say \(N\) has property (BC) if

1. \(N\) is lower semi-continuous (l.s.c.);
2. \(N\) has nonempty closed and decomposable values.

In order to define the solution of the problems (1)–(3) and (4)–(7) we shall consider the following space

\[
PC = \{ y: [0, T] \rightarrow \mathbb{R}^n: y_k \in C(J_k, \mathbb{R}^n), \ k = 0, \ldots, m, \\
\text{and there exist } y(t^-_k) \text{ and } y(t^+_k), \ k = 1, \ldots, m, \\
\text{with } y(t^-_k) = y(t_k) \}
\]

which is a Banach space with the norm

\[
\|y\|_{PC} = \max \{ \|y_k\|_{L^1}, \ k = 0, \ldots, m \},
\]

where \(y_k\) is the restriction of \(y\) to \(J_k = (t_k, t_{k+1}], k = 0, \ldots, m\).

Set \(\Omega := D \cup PC\). Then \(\Omega\) is a Banach space with norm

\[
\|y\|_{\Omega} = \sup \{|y(t)|: t \in [-r, T]\}.
\]

Let \(F: [0, T] \times D \rightarrow P(\mathbb{R}^n)\) be a multivalued map with nonempty compact values. Assign to \(F\) the multivalued operator

\[
\mathcal{F}: \Omega \rightarrow P(L^1([0, T], \mathbb{R}^n))
\]

by letting

\[
\mathcal{F}(y) = \{ w \in L^1([0, T], \mathbb{R}^n): w(t) \in F(t, y_t) \text{ for a.e. } t \in [0, T] \}.
\]

The operator \(\mathcal{F}\) is called the Niemytzki operator associated to \(F\).
Definition 2.2. Let $F : [0, T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. We say $F$ is of lower semi-continuous type (l.s.c. type) if its associated Nemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.3 [5]. Let $Y$ be separable metric space and let $N : Y \to \mathcal{P}(L^1([0, T], \mathbb{R}^n))$ be a multivalued operator which has property (BC). Then $N$ has a continuous selection, i.e., there exists a continuous function (single valued) $g : Y \to L^1([0, T], \mathbb{R}^n)$ such that $g(y) \in N(y)$ for every $y \in Y$.

Let us introduce the following hypotheses which are assumed hereafter:

(H1) $F : [0, T] \times D \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact multivalued map such that:
   (a) $(t, u) \mapsto F(t, u)$ is $L \otimes B$ measurable;
   (b) $u \mapsto F(t, u)$ is lower semi-continuous for a.e. $t \in [0, T]$.

(H2) There exists a function $h \in L^1([0, T], \mathbb{R}^+)$ such that
     \[ \|F(t, u)\| := \sup\{ |v| : v \in F(t, u) \} \leq h(t) \] for a.e. $t \in [0, T]$ and for $u \in D$.

The following lemma is crucial in the proof of our main theorem:

Lemma 2.4 [9]. Let $F : [0, T] \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume (H1) and (H2) hold. Then $F$ is of l.s.c. type.

3. First order nonresonance impulsive FDIs

In this section we give an existence result for the periodic BVP (1)–(3).

Definition 3.1. A function $y \in \Omega \cap \bigcup_{k=0}^m AC((t_k, t_{k+1}), \mathbb{R}^n)$ is said to be a solution of (1)–(3) if $y$ satisfies the differential inclusion $y'(t) - \lambda y(t) \in F(t, y(t))$ a.e. on $[0, T] = [t_1, \ldots , t_m]$, the conditions $\Delta y|_{t=k} = I_k(y(t_k))$, $k = 1, \ldots , m$, $y(t) = \phi(t)$, for $t \in [-r, 0]$, and $y(0) = y(T)$.

Theorem 3.2. Suppose that hypotheses (H1), (H2) are satisfied and

(H3) there exist positive constants $c_k$ such that
     \[ |I_k(x)| \leq c_k \] for $x \in \mathbb{R}^n$, $k = 1, \ldots , m$;

then the impulsive periodic boundary value problem (1)–(3) has at least one solution.

Proof. (H1) and (H2) imply by Lemma 2.4 that $F$ is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function $f : \Omega \to L^1([0, T], \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. 

Consider the following problem:

\[ y'(t) - \lambda y(t) = f(y_t), \quad t \in [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \]  
\[ \Delta y|_{t=t_k} = I_k\{y(t^-_k)\}, \quad k = 1, \ldots, m, \]  
\[ y(t) = \phi(t), \quad t \in [-r, 0], \quad y(0) = y(T). \]  

(8) (9) (10)

Transform the problem into a fixed point problem. Consider the operator \( G : \Omega \to \Omega \) defined by:

\[
G(y)(t) = \begin{cases} 
\phi(t), & \text{if } t \in [-r, 0], \\
\int_0^T H(t, s)f(y_s)\, ds + \sum_{k=1}^m H(t, t_k)I_k\{y(t^-_k)\}, & \text{if } t \in [0, T],
\end{cases}
\]

where

\[
H(t, s) = (e^{-\lambda T} - 1)^{-1} \begin{cases} 
e^{-\lambda (T+s-t)}, & 0 \leq s \leq t \leq T, \\
e^{-\lambda (s-t)}, & 0 \leq t < s \leq T.
\end{cases}
\]

Remark 3.3. We can easily show (see [4]) that the fixed points of \( G \) are solutions to (8)–(10) and hence a solution to the problem (1)–(3).

We shall show that \( G \) satisfies the assumptions of the Schaefer’s fixed point theorem (see [17, p. 29]). The proof will be given in several steps.

Step 1. \( G \) is continuous.

Let \( \{y_n\} \) be a sequence such that \( y_n \to y \) in \( \Omega \). Then

\[
\|G(y_n) - G(y)\|_\Omega \leq \int_0^T |H(t, s)||f(y_n) - f(y_s)|\, ds \\
+ \sum_{k=1}^m |H(t, t_k)||I_k(y_n(t_k)) - I_k(y(t_k))| \\
\leq \int_0^T |H(t, s)||f(y_n) - f(y_s)|\, ds \\
+ \sum_{k=1}^m |H(t, t_k)||I_k(y_n(t_k)) - I_k(y(t_k))|.
\]

Since the functions \( H, f \) and \( I_k, k = 1, \ldots, m, \) are continuous, then

\[
\|G(y_n) - G(y)\|_\Omega \leq \frac{1}{|1 - e^{-\lambda T}|} \|f(y_n) - f(y)\|_{L^1} \\
+ \frac{1}{|1 - e^{-\lambda T}|} \sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))| \to 0 \quad \text{as } n \to \infty.
\]

Step 2. \( G \) maps bounded sets into bounded sets in \( \Omega \).
Indeed, it is enough to show that for any \( q > 0 \) there exists a positive constant \( \ell \) such that for each \( y \in B_q = \{ y \in \Omega : \| y \|_\Omega \leq q \} \) one has \( \| G(y) \|_\Omega \leq \ell \). Let \( y \in B_q \). Then for \( t \in [0, T] \) we have

\[
G(y)(t) = \int_0^T H(t, s)f(y_s)\,ds + \sum_{k=1}^m H(t, t_k)I_k(y(t_k)).
\]

By (H2) we have for each \( t \in [0, T] \)

\[
|G(y)(t)| \leq \int_0^T \| H(t, s) \|f(y_s)\|ds + \sum_{k=1}^m \| H(t, t_k) \|I_k(y(t_k))\|
\]

\[
\leq \int_0^T \| H(t, s) \|h(s)\|ds + \sum_{k=1}^m \| H(t, t_k) \|c_k.
\]

Then for each \( y \in B_q \) we have

\[
\| G(y) \|_\Omega \leq \frac{1}{|1 - e^{-\lambda T}|} \int_0^T h(s)\,ds + \frac{1}{|1 - e^{-\lambda T}|} \sum_{k=1}^m c_k = \ell.
\]

**Step 3.** \( G \) maps bounded set into equicontinuous sets of \( \Omega \).

Indeed, it is enough to show that \( G(y)'(t) \) is bounded on \([0, T]\). Let \( B_q \) be a bounded set of \( \Omega \) as in Step 2 and \( y \in B_q \). Then for each \( t \in [0, T] \) we have

\[
G(y)(t) = \int_0^T H(t, s)f(y_s)\,ds + \sum_{k=1}^m H(t, t_k)I_k(y(t_k)).
\]

Thus for each \( t \in [0, T] \setminus \{ t_1, \ldots, t_m \} \)

\[
|G(y)'(t)| = \left| \int_0^T \frac{\partial}{\partial t} H(t, s)f(y_s)\,ds + \sum_{k=1}^m \frac{\partial}{\partial t} H(t, t_k)I_k(y(t_k)) \right|
\]

\[
= \left| \int_0^T \lambda H(t, s)f(y_s)\,ds + \sum_{k=1}^m \lambda H(t, t_k)I_k(y(t_k)) \right|
\]

\[
\leq \lambda H_0 \int_0^T h(s)\,ds + \lambda H_0 \sum_{k=1}^m c_k := d,
\]

where

\[
H_0 = \sup\{ H(t, s) : (t, s) \in [0, T] \times [0, t] \}.
\]

As a consequence of Steps 1–3 together with the Arzela–Ascoli theorem we can conclude that \( G : \Omega \to \Omega \) is completely continuous.
Step 4. Now it remains to show that the set
\[ E(G) := \{ y \in \Omega : y = \sigma G(y), \text{ for some } 0 < \sigma < 1 \} \]
is bounded.

Let \( y \in E(G) \). Then \( y = \sigma G(y) \) for some \( 0 < \sigma < 1 \). Thus for each \( t \in [0, T] \)
\[ y(t) = \sigma \left[ \int_0^T |H(t, s)| f(y_s) \, ds + \sum_{k=1}^m H(t, t_k) I_k(y(t_k^-)) \right]. \]
This implies by (H2) and (H3) that for each \( t \in [0, T] \) we have
\[ |y(t)| \leq \frac{1}{|1 - e^{-\lambda T}|} \int_0^T h(t) \, ds + \frac{1}{|1 - e^{-\lambda T}|} \sum_{k=1}^m c_k := l. \]
Then
\[ \|y\|_{\Omega} \leq \max\{\|\phi\|_{D, l}\}, \]
where \( l \) depends only on the functions \( h \) and \( c_k \), \( k = 1, \ldots, m \). This shows that \( E(G) \) is bounded. As a consequence of Schaefer’s theorem we deduce that \( G \) has a fixed point which is a solution to the problem (8)–(10). Then from Remark 3.3 this fixed point is a solution to the problem (1)–(3). \( \blacksquare \)

4. Second order nonresonance impulsive FDIs

In this section we give an existence result for the BVP (4)–(7).

**Definition 4.1.** A function \( y \in \Omega \cap \bigcup_{k=0}^m AC^1([t_k, t_{k+1}], \mathbb{R}^n) \) is said to be a solution of (4)–(7) if \( y \) satisfies the differential inclusion \( y''(t) - \lambda y(t) \in F(t, y_t) \) a.e. on \([0, T] - \{t_1, \ldots, t_m\}\), the conditions \( \Delta y|_{t=t_k} = I_k(y(t_k^-)), \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), k = 1, \ldots, m \), \( y(t) = \phi(t), t \in [-r, 0], y(0) - y(T) = \mu_0, y'(0) - y'(T) = \mu_1 \).

We now consider the following “linear problem”
\[ y''(t) - \lambda y(t) = g(t), \quad t \neq t_k, k = 1, \ldots, m, \quad (11) \]
subjected to Eqs. (5)–(7), and where \( g \in L^1([t_k, t_{k+1}], \mathbb{R}^n) \). For brevity, we shall refer to (5)–(7), (11) as (LP). Note that (LP) is not really a linear problem since the impulsive functions are not necessarily linear. However, if \( I_k, \bar{I}_k, k = 1, \ldots, m \), are linear, then (LP) is a linear impulsive problem.

We need the following auxiliary result:

**Lemma 4.2.** \( y \in \Omega \cap \bigcup_{k=0}^m AC^1([t_k, t_{k+1}], \mathbb{R}^n) \) is a solution of (LP), if and only if \( y \in \Omega \) is a solution of the following impulsive integral functional equation
\[ y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \int_0^T M(t, s)g(s) \, ds + M(t, 0)\mu_1 + N(t, 0)\mu_0 \\ + \sum_{k=1}^m \left[ M(t, t_k)I_k(y(t_k^-)) + N(t, t_k)\bar{I}_k(y(t_k^-)) \right], & t \in [0, T], \end{cases} \quad (12) \]
where
\[ M(t, s) = \frac{-1}{2\sqrt{\lambda}(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} + e^{\sqrt{\lambda}(t-s)}, & 0 \leq s \leq t \leq T, \\ e^{\sqrt{\lambda}(T+t-s)} + e^{\sqrt{\lambda}(t-s)}, & 0 \leq t < s \leq T, \end{cases} \]
and
\[ N(t, s) = \frac{\partial}{\partial t} M(t, s) = \frac{1}{2(e^{\sqrt{\lambda}T} - 1)} \begin{cases} e^{\sqrt{\lambda}(T+s-t)} - e^{\sqrt{\lambda}(t-s)}, & 0 \leq s \leq t \leq T, \\ e^{\sqrt{\lambda}(s-t)} - e^{\sqrt{\lambda}(T+t-s)}, & 0 \leq t < s \leq T. \end{cases} \]

**Proof.** We omit the proof since it is similar with the results in [7]. \(\square\)

**Theorem 4.3.** Assume (H1)–(H3) and the condition

(H4) there exists positive constants \(d_k\), such that \(\bar{I}_k(y) \leq d_k\) for each \(y \in \mathbb{R}^n\), \(k = 1, \ldots, m\), are satisfied. Then the BVP (4)–(7) has at least one solution.

**Proof.** (H1) and (H2) imply by Lemma 2.4 that \(F\) is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function \(f: \Omega \rightarrow L^1([0, T], \mathbb{R}^n)\) such that \(f(y) \in F(y)\) for all \(y \in \Omega\).

Consider the following problem:

\[ y''(t) - \lambda y(t) = f(y_t), \quad t \in [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \tag{15} \]
\[ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{16} \]
\[ \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{17} \]
\[ y(t) = \phi(t), \quad t \in [-r, 0], \]
\[ y(0) - y(T) = \mu_0, \quad y'(0) - y'(T) = \mu_1. \tag{18} \]

Transform the problem into a fixed point problem. Consider the operator \(\overline{G}: \Omega \rightarrow \Omega\) defined by

\[ \overline{G}(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\
\int_0^T M(t, s) f(y_s) ds + M(t, 0)\mu_1 + N(t, 0)\mu_0 \\
+ \sum_{k=1}^m [M(t, t_k)I_k(y(t_k)) + N(t, t_k)\bar{I}_k(y(t_k))], & t \in [0, T]. \end{cases} \]

**Remark 4.4.** Clearly from Lemma 4.2 the fixed points of \(\overline{G}\) are solutions to (4)–(7). If \(y\) is a solution of the problem (15)–(18), then \(y\) is a solution to the problem (4)–(7).

As in Theorem 3.2 we can show that \(\overline{G}\) is completely continuous. Now we prove only that the set

\[ \mathcal{E}(\overline{G}) := \{ y \in \Omega : y = \sigma \overline{G}(y), \text{ for some } 0 < \sigma < 1 \} \]
is bounded. Let $y \in \mathcal{E}(\overline{G})$. Then $y = \sigma \overline{G}(y)$ for some $0 < \sigma < 1$. Thus

$$y(t) = \sigma \left[ \int_0^T M(t, s)g(s) \, ds + M(t, 0)\mu_1 + N(t, 0)\mu_0 \right]$$

$$+ \sigma \left[ \sum_{k=1}^m [M(t, t_k)I_k(y(t_k)) + N(t, t_k)\overline{I}_k(y(t_k))] \right].$$

This implies by (H2), (H3) and (H4) that for each $t \in [0, T]$ we have

$$|y(t)| \leq \int_0^T |M(t, s)||h(s)| \, ds + |M(t, 0)||\mu_1| + |N(t, 0)||\mu_0|$$

$$+ \sum_{k=1}^m \left[ |M(t, t_k)|c_k + |N(t, t_k)|d_k \right]$$

$$\leq \sup_{(t, s) \in [0, T] \times [0, t]} |M(t, s)| \left[ \|h\|_{L^1} + |\mu_1| \sum_{k=1}^m c_k \right]$$

$$+ \sup_{(t, s) \in [0, T] \times [0, t]} |N(t, s)| \left[ |\mu_0| + \sum_{k=1}^m d_k \right] = l^*.$$

Then

$$\|y\|_{\Omega} \leq \max \left\{ \|\phi\|_{\Omega}, l^* \right\},$$

where $l^*$ depends only on the functions $h$ and $c_k, d_k$ ($k = 1, \ldots, m$). This shows that $\mathcal{E}(\overline{G})$ is bounded. As a consequence of Schaefer’s theorem we deduce that $\overline{G}$ has a fixed point $y$ which is a solution to problem (15)–(18). Then from Remark 4.4 $y$ is a solution to the problem (4)–(7). □

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