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Physics Letters B

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# The second physical moment of the heavy quark vector correlator at $\mathcal{O}(\alpha_s^3)$

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## ARTICLE INFO

## Article history:

Received 14 July 2008

Received in revised form 16 September 2008

Accepted 19 September 2008

Available online 26 September 2008

Editor: B. Grinstein

## PACS:

12.38.Bx

12.38.-t

11.55.Fv

14.65.Dw

14.65.Fy

## Keywords:

Perturbative calculations

Quantum chromodynamics

Dispersion relations

Charm quarks

Bottom quarks

## ABSTRACT

The second moment of the heavy quark vector correlator at  $\mathcal{O}(\alpha_s^3)$  is presented. The implications of this result on recent determinations of the charm and bottom quark mass are discussed.

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## 1. Introduction

Correlators of quark currents are of prime interest for several phenomenological applications. Their low-energy expansions, in particular, allow for the precise determination of charm and bottom quark masses via QCD sum rules [1–5]. For this reason, heavy quark correlators have been frequently investigated in the framework of perturbation theory.

Up to  $\mathcal{O}(\alpha_s^2)$ , analytic expansions to great depth are known for the low energy region. The three-loop QCD corrections to the correlator of two vector currents were first calculated in [6]. In [7] up to seven terms in the low energy expansion were obtained. This calculation also included further currents, namely the scalar, pseudo-scalar, and axial-vector current. Recently the calculation at three-loops has been extended to moments up to  $n = 30$  for all four currents [8,9].

The moments of the vector correlator can then be used to extract the value of the masses of the charm and bottom quark from  $e^+e^-$  data in the threshold region using the  $R$ -ratio, since they are

related via a dispersion relation. A brief outline of this method is given in Section 2, which was first applied at three loops in [3].

At three loops a significant, sometimes dominant part of the error arises from the theoretical uncertainty due to higher orders, often estimated by the renormalization scale dependence. Therefore the calculation had to be taken to the four-loop level [10,11] to reach a precision comparable to or below the experimental data. The contributions from double-fermionic loop insertions of heavy and/or light quarks are known explicitly up to 30 terms in the low energy expansion [12]. The contributions due to light quark loop insertions of  $\mathcal{O}(\alpha_s^n n^{n-1})$  are known to all orders in  $\alpha_s$  [13]. Recently the lower moments were also calculated for the remaining three currents in [14].

In [4] the first moment of the vector correlator was used to extract the masses of the charm and bottom quarks. Since all but constant terms are known from renormalization group arguments, the analysis was done for up to the fourth moment, employing a conservative error estimate for the missing constant terms.

In this Letter we present the calculation of the second moment of the vector correlator and discuss its impact on the determination of the charm and bottom quark masses.

The outline of this Letter is as follows: In Section 2 we set the framework and notations used throughout the Letter. In Section 3

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we explain the details of the calculation, present the result for the second physical moment and discuss its impact on the quark mass determination. A brief summary and conclusions are given in Section 4.

## 2. Notation

The correlator  $\Pi^{\mu\nu}(q)$  of two vector currents is defined as

$$\Pi^{\mu\nu}(q) = i \int dx e^{iqx} \langle 0 | T j^\mu(x) j^\nu(0) | 0 \rangle, \quad (1)$$

with the current  $j^\mu(x) = \bar{\Psi}(x) \gamma^\mu \Psi(x)$  being composed of the heavy quark fields  $\Psi(x)$ . The function  $\Pi^{\mu\nu}(q)$  is conveniently written in the form

$$\Pi^{\mu\nu}(q) = (-q^2 g^{\mu\nu} + q^\mu q^\nu) \Pi(q^2). \quad (2)$$

It can be related to the ratio  $R(s) = \sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-)$  with the help of the dispersion relation

$$\Pi(q^2) = \frac{1}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s-q^2)}, \quad (3)$$

where the normalization  $\Pi(0) = 0$  has been adopted.

To extract the quark masses the experimental data on the right-hand side of (3) has to be compared with the theoretical evaluation of  $\Pi(q^2)$  on the left-hand side. This is best done by comparing the corresponding Taylor series in  $q^2$ . The  $n$ th derivatives with respect to  $q^2$  at  $q^2 = 0$  define the experimental moments

$$\mathcal{M}_n^{\text{exp}} = \int ds \frac{R(s)}{s^{n+1}}, \quad (4)$$

which can be compared with the theoretical moments

$$\mathcal{M}_n^{\text{th}} = Q_q^2 \frac{9}{4} \left( \frac{1}{4\bar{m}_q^2} \right)^n \bar{C}_n. \quad (5)$$

The latter are related to the Taylor coefficients  $\bar{C}_n$  of the vacuum polarization function

$$\bar{\Pi}(q^2) = \frac{3Q_q^2}{16\pi^2} \sum_{n \geq 0} \bar{C}_n \bar{z}^n, \quad (6)$$

with  $\bar{z} = q^2 / (4\bar{m}^2)$ . Symbols carrying a bar indicate that the renormalization has been performed in the  $\overline{\text{MS}}$  scheme. The coefficients  $\bar{C}_n$  can be expanded in a power series in  $\frac{\alpha_s}{\pi}$

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1)} + \left( \frac{\alpha_s}{\pi} \right)^2 \bar{C}_n^{(2)} + \left( \frac{\alpha_s}{\pi} \right)^3 \bar{C}_n^{(3)} + \dots$$

The four-loop contribution  $\bar{C}_n^{(3)}$  can be decomposed according to the number of quark loops and colour structures as follows:

$$\begin{aligned} \bar{C}_n^{(3)} = & C_F T_F^2 n_l^2 \bar{C}_{ll,n}^{(3)} + C_F T_F^2 n_h^2 \bar{C}_{hh,n}^{(3)} + C_F T_F^2 n_l n_h \bar{C}_{lh,n}^{(3)} \\ & + C_F T_F n_l (C_A \bar{C}_{lNA,n}^{(3)} + C_F \bar{C}_{lA,n}^{(3)}) + \bar{C}_{n_f,n}^{(3)} \\ & + C_F T_F n_h (C_A \bar{C}_{hNA,n}^{(3)} + C_F \bar{C}_{hA,n}^{(3)}) + \frac{n_h}{N_C} d^{abc} d^{abc} \bar{C}_{S,n}^{(3)}. \end{aligned} \quad (7)$$

$\bar{C}_{n_f,n}^{(3)}$  contains the purely bosonic contributions, where we set the number of colours  $N_C = 3$  for simplicity, while  $\bar{C}_{S,n}^{(3)}$  denotes the contribution from singlet diagrams.  $C_F = \frac{N_C^2 - 1}{2N_C}$  and  $C_A = N_C$  are the Casimir operators of the fundamental and adjoint representation of the  $SU(N_C)$  group, respectively.  $T_F = \frac{1}{2}$  is the index of the fundamental representation.  $d^{abc}$  is the symmetric structure constant.  $n_l$  and  $n_h = 1$  denote the number of light and heavy quarks, respectively.

## 3. Calculation and results

The diagrams have been generated using QGRAF [15]. Expanding them in  $q^2$  results in four-loop tadpole integrals. Using EXP [16] they are mapped to six topologies with the maximum of nine lines. The main difficulty of the calculation lies in the reduction of the vast amount of integrals to the small set of 13 master integrals. This is done using Integration-By-Parts identities [20] together with the Laporta algorithm [21] which is efficiently implemented in the multi-threaded C++ program CRUSHER [17]. CRUSHER uses GiNAC [18] for simple algebraic manipulations and Fermat [19] for the simplification of complicated ratios of polynomials. A supplementary technique to perform the reduction to master integrals is based on the idea that self energy subgraphs of the integral can be reduced independently in order to effectively reduce the number of loops of the diagram. This can be useful because these integrals have up to two more propagator powers than integrals without an internal self energy and are therefore more cumbersome for traditional Laporta algorithm. In combination with Groebner Bases and the Mathematica package FIRE [22–24] it is also possible to calculate integrals without internal self energies. A more detailed description of the calculation techniques will be published soon [25]. In total the reduction of 1.8 million integrals was needed in order to perform the calculation, which is done using FORM [26] in combination with the MATAD [27] setup. The necessary master integrals have been calculated in [28–34]. We confirm the results for the zeroth and first moment given in [8,10,11].

Inserting the master integrals and performing the renormalization of the strong coupling constant and the mass in the  $\overline{\text{MS}}$  scheme leads to the following result for the second moment at  $\mu^2 = m^2$  as defined in Eq. (7):

$$\begin{aligned} \bar{C}_{n_f,2}^{(3)} = & + \frac{64985074258811347}{353072079360000} - \frac{2900811008}{3648645} a_5 \\ & - \frac{1662518706713}{21016195200} (24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\ & + \frac{362601376}{54729675} \log^5 2 - \frac{725202752}{10945935} \zeta_2 \log^3 2 \\ & - \frac{1684950406}{3648645} \zeta_4 \log 2 + \frac{112680551036302633}{47076277248000} \zeta_3 \\ & - \frac{26401638588211}{28021593600} \zeta_4 - \frac{164928917}{270270} \zeta_5, \\ \bar{C}_{S,2}^{(3)} = & + \frac{5881974201847}{8369115955200} \\ & + \frac{97011619}{696729600} (24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\ & + \frac{796232393699}{371960709120} \zeta_3 - \frac{745372259}{185794560} \zeta_4, \\ \bar{C}_{hNA,2}^{(3)} = & - \frac{20427854209619}{5649153269760} \\ & - \frac{31595849}{11612160} (24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\ & - \frac{29638030087837}{697426329600} \zeta_3 + \frac{968787977}{15482880} \zeta_4 + \frac{362}{63} \zeta_5, \\ \bar{C}_{lNA,2}^{(3)} = & - \frac{22559166733}{16796160000} - \frac{520999}{4354560} (24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\ & - \frac{309132631}{12902400} \zeta_3 + \frac{167529079}{5806080} \zeta_4, \\ \bar{C}_{hA,2}^{(3)} = & - \frac{3732009196157}{271593907200} \\ & - \frac{130387543}{2177280} (24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \end{aligned}$$

$$\begin{aligned}
& -\frac{5\,811\,074\,101\,069}{6\,706\,022\,400}\zeta_3 + \frac{2\,218\,910\,663}{1\,451\,520}\zeta_4, \\
\bar{C}_{IA,2}^{(3)} = & +\frac{35\,754\,300\,3871}{11\,757\,312\,000} + \frac{520\,999}{2\,177\,280}(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\
& -\frac{36\,896\,356\,307}{174\,182\,400}\zeta_3 + \frac{598\,455\,689}{2\,903\,040}\zeta_4, \\
\bar{C}_{lh,2}^{(3)} = & +\frac{9\,040\,709}{62\,705\,664} - \frac{2029}{41\,472}(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2) \\
& -\frac{12\,159\,109}{4\,644\,864}\zeta_3 + \frac{99\,421}{55\,296}\zeta_4, \\
\bar{C}_{hh,2}^{(3)} = & +\frac{1\,842\,464\,707}{646\,652\,160} - \frac{2\,744\,471}{1\,064\,448}\zeta_3, \\
\bar{C}_{ll,2}^{(3)} = & +\frac{15\,441\,973}{19\,136\,250} - \frac{32}{45}\zeta_3,
\end{aligned}$$

where Riemann's zeta function  $\zeta_n$  and the polylogarithm  $\text{Li}_n(1/2)$  are defined by

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \text{and} \quad a_n = \text{Li}_n(1/2) = \sum_{k=1}^{\infty} \frac{1}{2^k k^n}. \quad (8)$$

For completeness we also give the results for the singlet contribution to the zeroth and first moment:

$$\begin{aligned}
\bar{C}_{S,0}^{(3)} = & \frac{2411}{20160} - \frac{6779}{4480}\zeta_3 + \frac{2189}{768}\zeta_4 - \frac{5}{48}\zeta_5 \\
& - \frac{73}{576}(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2), \quad (9)
\end{aligned}$$

$$\begin{aligned}
\bar{C}_{S,1}^{(3)} = & \frac{664\,837}{2\,566\,080} - \frac{2\,017\,831}{855\,360}\zeta_3 + \frac{175}{48}\zeta_4 \\
& - \frac{739}{4320}(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2). \quad (10)
\end{aligned}$$

Numerically at  $\mu^2 = m^2$  one finds  $\bar{C}_2^{(3)}|_{n_f=3} = -3.49373 + 0.155877$  and  $\bar{C}_2^{(3)}|_{n_f=4} = -2.64381 + 0.155877$ . The second term in each of these equations corresponds to the singlet contribution.

Extracting the charm and bottom quark mass from the second moment using the input data given in [4] with the new value of  $C_2^{(3)}$  leads to a shift of  $-3$  MeV for  $m_c$  and  $-2$  MeV for  $m_b$  and yields

$$\begin{aligned}
m_c(3 \text{ GeV}) &= 0.976(16) \text{ GeV} \quad \text{and} \\
m_b(10 \text{ GeV}) &= 3.607(19) \text{ GeV}. \quad (11)
\end{aligned}$$

This can be converted to the values at  $m_c$  and  $m_b$ ,  $m_c(m_c) = 1.277(16)$  GeV and  $m_b(m_b) = 4.162(19)$  GeV, respectively.

The final results for the quark masses given in [4] are  $m_b(m_b) = 4.164(25)$  GeV and  $m_c(m_c) = 1.286(13)$  GeV, respectively. In case of  $m_c$  the first moment was used at  $\mathcal{O}(\alpha_s^3)$  accuracy. For  $m_b$  the second moment, which was known only up to  $\mathcal{O}(\alpha_s^2)$  at that time, was chosen. In the latter case the logarithms at  $\mathcal{O}(\alpha_s^3)$  calculated by means of renormalization group methods were included and the error estimate was based on the missing constant term. Although this estimate was based on plausible arguments only a real calculation could prove its validity. Removing the 6 MeV error, which arises from the estimated term in case of the  $b$  quark, the total error of  $m_b$  is reduced by  $\sim 25\%$ . In order  $\alpha_s^3$  the perturbative error is practically negligible and the remaining 19 MeV error arises from the experimental uncertainty and from the value of  $\alpha_s$ . At present this is the most precise determination of the bottom quark mass.

As already discussed in [4], different moments weight the experimental results from larger and smaller  $s$  values differently. Therefore it is important to compare the obtained quark masses from several moments to test the self-consistency of the method and the stability of the results. Because of sparse and poor experimental data in the continuum region above 4.8 GeV (for  $m_c$ ) and

11.2 GeV (for  $m_b$ ), the data for  $R(s)$  were replaced by perturbative QCD in the analysis. This region can be suppressed by using higher moments, which is especially important in the case of  $m_b$  where the first moment, which was already under full theoretical control at order  $\alpha_s^3$  in [4], receives a large contribution from the region above 11.2 GeV. The situation is significantly better for the second moment, which is now also fully under control from the theory side. For the determination of  $m_c$  the first and the second moment are of equal reliability and the consistency between the two results for  $m_c(3 \text{ GeV})$ , namely 0.986(13) GeV and 0.976(16) GeV, is remarkable. On the other hand for higher moments non-perturbative effects increase (especially for  $m_c$ ) leading to larger theoretical uncertainties. For these reasons we think that for  $m_b$  the second or maybe third moment are best suited for the mass determination, while for  $m_c$  the first and second moment are preferred.

Apart from the application discussed above, the higher moments evaluated above have been used recently for quark mass determinations from lattice simulations [5] and for the reconstruction of the full  $q^2$  dependence of the vacuum polarization at  $\mathcal{O}(\alpha_s^3)$  [35].

#### 4. Summary and conclusion

We have presented the second physical moment in the low energy expansion of the heavy quark vector correlator at four-loop order, including the singlet contribution. Although this contribution only causes a rather small shift in the quark masses obtained from the second moment the error is reduced significantly. The values remain in good agreement with those extracted using the first moment.

#### Acknowledgements

We thank K.G. Chetyrkin, J.H. Kühn and M. Steinhauser for helpful discussions and cross checks. We also like to thank A.V. Smirnov for providing us with the FIRE package and extending it for our purposes and for interesting discussions.

This work was supported by the Deutsche Forschungsgemeinschaft through the SFB/TR-9 ‘‘Computational Particle Physics’’. A.M. and Ph.M. were supported by the Graduiertenkolleg ‘‘Hochenergiephysik und Teilchenastrophysik’’. A.M. thanks the Landesgraduiertenförderung for support.

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