Operators $J^*J$ and Nonlinear Hammerstein Equations

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1. INTRODUCTION

Cesari and Kannan [2] have recently established, along the lines of Cesari’s alternative method, the existence of solutions to a nonlinear differential equation of the type $Eu = Nu$, where $E$ is a linear differential operator that may have a nontrivial nullspace and $N$ is a monotone operator defined on all of the underlying Hilbert space. They utilize the theory of monotone operators to solve both the auxiliary and bifurcation equations that are associated with $Eu = Nu$ by the alternative method, and they treat the auxiliary equation as a nonlinear Hammerstein equation of the type $u + KNu = f$, where $K$ is a positive operator related to $E$, usually the generalized inverse of $L = -E$.

In [4] Gustafson and Sather consider a nonlinear differential equation of the same type. They solve the associated auxiliary equation by using the square root decomposition of the linear operator $K$, an idea that has been used by Vainberg and Lavrent’ev [5]. This technique permits nonlinearities $N$ that are not everywhere defined but instead satisfy the condition that the range $\mathcal{R}(K^{1/2})$ is contained in the domain $\mathcal{D}(N)$.

In this paper we study a general Hammerstein equation that is of the same type as these auxiliary equations. Let $S$ be a real Hilbert space with inner product $(u,v)$ and norm $\|u\|$, let $J: S \to S$ be a bounded linear operator, and let $K = J^*J$. We consider the Hammerstein equation

$$u + KNu = f,$$

where $N$ is a nonlinear operator in $S$ with $\mathcal{R}(J^*) \subseteq \mathcal{D}(N)$. Operators of the form $K = J^*J$ have been studied by many authors in connection with a wide
variety of problems ranging from the numerical solution of differential
equations to problems in elasticity. For functional analytic properties of
operators $J^*J$, one is referred to Kato [6].

We present in Section 2 two existence theorems for Eq. (1). The conditions
imposed on the nonlinear operator $N$ are more general than monotonicity,
and these results are particularly useful in the study of perturbations of
nonlinear boundary value problems at resonance. Examples are given in
Section 3 to indicate the application of these theorems to existence problems
for nonlinear differential equations. The detailed development of these ideas
can be seen in Kannan and Locker [5] and Dunninger and Locker [3].

2. EXISTENCE THEOREMS

It is well known that the operator $K = J^*J$ is positive, self-adjoint, and

$$
\| K \| = \| J \|^2 = \| J^* \|^2. \tag{2}
$$

Also, for $u \in S$ we have

$$
\| Ku \|^2 = \| J^*Ju \|^2 \leq \| J^* \|^2 \| Ju \|^2 = \| K \| (Ku, u),
$$

and hence,

$$
\| K \|^{-1} \| Ku \|^2 \leq (Ku, u) \quad \text{for all } u \in S. \tag{3}
$$

Assume $\mathcal{D}(N)$ is a subspace of $S$, and fix an element $f \in \mathcal{D}(N)$. If $u \in \mathcal{D}(N)$
is a solution of Eq. (1), then setting $w = u - f$ we have

$$
w + J^*JN(w + f) = 0.
$$

This implies that $w \in \mathcal{R}(J^*J)$, and hence, there exists $v \in \mathcal{R}(J)$ such that

$$
w = J^*v \quad \text{and} \quad J^*v + J^*JN(J^*v + f) = 0.
$$

But $\mathcal{R}(J) \subseteq \mathcal{R}(J) = \mathcal{N}(J^*J)^{-1}$, and hence, $J^*$ is $1-1$ on $\mathcal{R}(J)$. Thus, the
last equation reduces to

$$
v + JN(J^*v + f) = 0. \tag{4}
$$

Hence, if $u \in \mathcal{D}(N)$ is a solution of (1), then $u = J^*v + f$, where $v \in S$, is a solution of (4). It follows immediately that Eq. (1) is (uniquely) solvable
if and only if Eq. (4) is (uniquely) solvable.

**Theorem 1.** Let $S$ be a real Hilbert space, let $J : S \to S$ be a bounded
linear operator, and let $K = J^*J$. Assume $\mathcal{D}(N)$ is a subspace of $S$ and $N: \mathcal{D}(N) \to S$ is a nonlinear operator that satisfies the conditions:

(i) $\mathcal{D}(J^*) \subset \mathcal{D}(N)$.

(ii) There exists a real number $p$ with $0 < p < \|K\|^{-1}$ such that

$$(Nu - Nv, u - v) \geq -p \|u - v\|^2$$

for all $u, v \in \mathcal{D}(N)$.

(iii) $N$ is hemicontinuous, i.e., $N$ is continuous from each line segment in $\mathcal{D}(N)$ to the weak topology on $S$.

Then for each $f \in \mathcal{D}(N)$ Eq. (1) has a unique solution $u \in \mathcal{D}(N)$.

Proof. Let $F: S \to S$ be the operator defined by

$$Fv = v + JN(J^*v + f), \quad v \in S.$$ 

By the above remarks it is sufficient to show that the equation $Fv = 0$ is uniquely solvable in $S$.

Clearly $F$ is hemicontinuous by (iii). Also, for $u, v \in S$ we have by (ii) and (2) that

$$(Fu - Fv, u - v) = \|u - v\|^2 + (JN(J^*u + f) - JN(J^*v + f), u - v)$$

$$\geq \|u - v\|^2 - p \|J^*u - J^*v\|^2$$

$$\geq (1 - p \|K\|)\|u - v\|^2.$$ 

Hence, $F$ is strongly monotone on $S$. Applying the theorem of Minty [7], it follows that $Fv = 0$ is uniquely solvable in $S$, which concludes the proof.

Remark 1. We can establish the unicity directly from (3) and (ii). Indeed, if $u_1 \in \mathcal{D}(N), u_2 \in \mathcal{D}(N)$ are solutions of (1), then

$$u_1 - u_2 = -K(Nu_1 - Nu_2)$$

and

$$\|K\|^{-1}\|u_1 - u_2\|^2 = \|K\|^{-1}\|KNu_1 - KNu_2\|^2$$

$$\leq (KNu_1 - KNu_2, Nu_1 - Nu_2)$$

$$= -(u_1 - u_2, Nu_1 - Nu_2)$$

$$\leq p \|u_1 - u_2\|^2.$$ 

This implies $u_1 = u_2$.

Remark 2. If we start with a positive self-adjoint operator $K$, then we can always obtain the decomposition $K = J^*J$ by choosing $J = J^* = K^{1/2}$, and the theorem can be used when we know $\mathcal{D}(K^{1/2})$. This situation occurs in the papers of Gustafson and Sather [4] and Vainberg and Lavrent’ev [8]. On the other hand, if $K$ is the generalized inverse of a linear differential operator $L$ of order $2n$, then $L$ may have a natural decomposition of the form
TT*, where T is a differential operator of order n. In this case the decomposition $K = J^*J$ is automatically induced, and we know that $\mathcal{D}(J^*)$ is a subset of the Sobolev space $H^n$. Thus, we can take $\mathcal{D}(N) = H^n$ and condition (i) is satisfied. This also permits nonlinearities $N$ involving the derivatives of $u$. Examples of this second situation are given in the next section.

**THEOREM 2.** Let $S$ be a real Hilbert space, let $J: S \to S$ be a compact linear operator, and let $K = J^*J$. Let $p$ be a real number with $0 \leq p < \| K \|^{-1}$, let $\mathcal{D}(M)$ be a subspace of $S$, and let $M: \mathcal{D}(M) \to S$ be a nonlinear operator. Let $N$ be the nonlinear operator defined by $\mathcal{D}(N) = \mathcal{D}(M)$, $Nu = -pu + Mu$, and assume that:

(i) $\mathcal{D}(J^*) \subseteq \mathcal{D}(N) = \mathcal{D}(M)$.

(ii) There exists a real number $\gamma > 0$ such that $\| Mu \| \leq \gamma$ for all $u \in \mathcal{D}(M)$.

(iii) $M$ is continuous.

Then for each $f \in \mathcal{D}(N)$, Eq. (1) has at least one solution $u \in \mathcal{D}(N)$.

**Proof.** As in Theorem 1 it is sufficient to solve

$$Fv = v + JN(J^*v + f) = 0,$$

which can be rewritten as

$$v + J(F_0v + F_1v) = 0,$$

where

$$F_0v = -pj^*v, \quad F_1v = -pf + M(J^*v + f).$$

Clearly $F_0$ is odd and 1-homogeneous, i.e., $F_0(\xi v) = \xi F_0(v)$ for $\xi > 0$, and $\| F_1v \| \| v \| \to 0$ as $\| v \| \to \infty$. Also, if $v + JF_0v = 0$, then

$$\| v \| = p \| JF_1v \| \leq p \| K \| \| v \|,$$

implying $v = 0$. By Theorem 1.6 in Browder [1] it follows that $Fv = 0$ has at least one solution $v \in S$. This completes the proof.

**Remark 3.** In Theorem 2 the continuity of $M$ can be replaced by requiring the continuity of $F_1$ on $S$. This is important for applications to nonlinear differential equations.

### 3. EXAMPLES

Consider the nonlinear periodic boundary value problem

$$-u'' - \lambda u + g(u) = \beta(t), \quad 0 \leq t \leq 2\pi, \tag{5}$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{6}$$
where $\lambda$ is a real number with $0 \leq \lambda < 1$, $g$ is a continuous real-valued function defined on all the real line, and $\beta$ is a function belonging to the real Hilbert space $S = L^2[0, 2\pi]$.

Let $L$ be the linear differential operator in $S$ defined by

$$D(L) = \{ u \in H^2[0, 2\pi] \mid u(0) = u(2\pi), u'(0) = u'(2\pi) \}, \quad Lu = -u'',$$

and let $N$ be the nonlinear operator in $S$ defined by

$$D(N) = H^1[0, 2\pi], \quad Nu = -\lambda u + g(u) - \beta.$$

Then the boundary value problem (5)-(6) reduces to the operator equation

$$Lu + Nu = 0.$$  \hfill (7)

We are going to describe the auxiliary and bifurcation equations that correspond to (7) and show that the auxiliary equation is solvable by Theorems 1 or 2 if $g$ is either monotone increasing or bounded.

Let $S_0 = \mathcal{N}(L)$, which consists of the constant functions, and let $P : S \to S_0$ be the orthogonal projection onto $S_0$. If we set

$$H = [L \mid D(L) \cap \mathcal{N}(L)]^{-1},$$

then the linear operator $K = H(I - P)$ is the generalized inverse of $L$. Moreover, $\|K\| = 1$ since 1 is the smallest positive eigenvalue of $L$. The auxiliary equation for (7) is precisely the Hammerstein equation (1), which is to be solved for $u \in D(N)$ as $f$ varies over the subspace $S_0$, and the bifurcation equation is the equation

$$PN[I + KN]^{-1}f \equiv 0$$  \hfill (8)

with $f \in S_0$.

To decompose the operators $L$, $H$, and $K$, let $T$ be the linear differential operator in $S$ defined by

$$D(T) = \{ u \in H^1[0, 2\pi] \mid u(0) = u(2\pi) \}, \quad Tu = u',$$

let $J_1 = [T \mid D(T) \cap \mathcal{N}(T)]^{-1}$, and let $J = J_1(I - P)$. It follows that $T^* = -T$, $J_1^* = -J_1$, $J^* = -J$, $L = TT^*$, $H = J_1^*J_1$, and $K = J^*J$. Also, $D(f^*) = D(T) \cap \mathcal{N}(T)^\perp \subset D(N)$.

**Case 1.** Assume that $g$ is monotone increasing. Then for $u, v \in D(N)$ we have

$$[g(u(t)) - g(v(t))] [u(t) - v(t)] \geq 0 \quad \text{for } 0 \leq t \leq 2\pi,$$

and hence, $(Nu - Nv, u - v) \geq -\lambda \|u - v\|^2$. Applying Theorem 1 with
$\rho = \lambda$, we conclude that the auxiliary equation (1) is uniquely solvable for each $f \in S_0$. In fact, this result is true for all $\lambda < 0$. Indeed, $N$ is strongly monotone for $\lambda < 0$, and we apply Theorem 1 with $\rho = 0$.

Case 2. Assume $g$ is bounded, say $|g(t)| \leq \gamma_0$ for $-\infty < t < \infty$. Clearly $\|g(u) - \beta\| \leq (2\pi)^{1/2} \gamma_0 + \|\beta\| = \gamma$ for all $u \in \mathcal{D}(N)$. It follows from Theorem 2 with $\rho = \lambda$ that the auxiliary equation (1) has at least one solution for each $f \in S_0$.

**References**