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Measuring the performances of decision-making units using interval efficiencies[☆]

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Abstract

Efficiency is a relative measure because it can be measured within different ranges. The traditional data envelopment analysis (DEA) measures the efficiencies of decision-making units (DMUs) within the range of less than or equal to one. The corresponding efficiencies are referred to as the best relative efficiencies, which measure the best performances of DMUs and determine an efficiency frontier. If the efficiencies are measured within the range of greater than or equal to one, then the worst relative efficiencies can be used to measure the worst performances of DMUs and determine an inefficiency frontier. In this paper, the efficiencies of DMUs are measured within the range of an interval, whose upper bound is set to one and the lower bound is determined through introducing a virtual anti-ideal DMU, whose performance is definitely inferior to any DMUs. The efficiencies turn out to be all intervals and are thus referred to as interval efficiencies, which combine the best and the worst relative efficiencies in a reasonable manner to give an overall measurement and assessment of the performances of DMUs. The new DEA model with the upper and lower bounds on efficiencies is referred to as bounded DEA model, which can incorporate decision maker (DM) or assessor's preference information on input and output weights. A Hurwicz criterion approach is introduced and utilized to compare and rank the interval efficiencies of DMUs and a numerical example is examined using the proposed bounded DEA model to show its potential application and validity.
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1. Introduction

Data envelopment analysis (DEA), developed in [2], measures the efficiencies of decision-making units (DMUs) within the range of less than or equal to one and evaluates their performances using the best relative efficiencies. If a DMU is evaluated to have the best relative efficiency of one, then it is said to be DEA efficient; otherwise it is non-DEA efficient. DEA efficient DMUs are usually thought to perform better than non-DEA efficient DMUs.

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However, if efficiencies are measured within the range of greater than or equal to one, then the worst relative efficiencies can be used to measure the worst performances of DMUs. Contrary to the best relative efficiencies, which determine an efficiency frontier, the worst relative efficiencies of DMUs define an inefficiency frontier. If a DMU is evaluated to have the worst relative efficiency of one, then it is said to be DEA inefficient; otherwise, it is non-DEA inefficient. DEA inefficient DMUs are usually thought to perform worse than non-DEA inefficient DMUs.

From the above analyses we can see that efficiency is a relative measure. It can be measured either within the range of less than or equal to one, or within the range of greater than or equal to one. When measured within different ranges, it has different meanings. The resultant assessment conclusions are usually different. Any assessment using only one type of efficiency is obviously one-sided. Ideally, both types of efficiencies should be used at the same time to assess the performances of DMUs.

Doyle et al. [5] and Entani et al. [6] are the few persons, to the best of our knowledge, to consider and measure efficiencies from both the optimistic and the pessimistic points of view. Their models have similar structures and all have some significant drawbacks, which will be seen very clearly in our later discussion in Section 3.1.

In this paper, we reconsider the problem of performance measurement. We measure the efficiencies of DMUs within the range of an interval so that the best and the worst relative efficiencies can be measured within a unified DEA model framework. In order to determine the range of interval efficiency, a virtual anti-ideal DMU is introduced, whose performance is definitely the worst among all the DMUs. So, its best relative efficiency can be utilized as the constraint on the lower bound efficiencies of DMUs. A new DEA model with the upper and lower bounds on efficiencies is thus developed to compute the best and the worst relative efficiencies of each DMU, which constitute an interval to give an overall measurement and assessment of the performance of each DMU. Since the performances of DMUs are characterized by interval efficiencies, an appropriate ranking approach for interval numbers is needed.

The rest of the paper is organized as follows: in Section 2, we briefly introduce the basic DEA models for measuring the best and the worst relative efficiencies of DMUs. Section 3 analyses Entani et al.'s DEA models, points out their drawbacks and develops the bounded DEA model for crisp input and output data. Section 4 briefly introduces the Hurwicz criterion approach (HCA) for comparing and ranking interval efficiencies of DMUs. This is followed by a numerical example, which is provided to show the potential application of the proposed bounded DEA models. The paper is concluded in Section 6.

2. DEA models for measuring the best and the worst relative efficiencies of DMUs

2.1. CCR model for measuring the best relative efficiencies of DMUs

Assume that there are n DMUs to be evaluated, each DMU with m inputs and s outputs. We denote by $x_{ij}\,(i=1,\ldots,m)$ and $y_{rj}\,(r=1,\ldots,s)$ the values of inputs and outputs of $\mathrm{DMU}_j\,(j=1,\ldots,n)$, which are all known and positive. According to the implication of efficiency, the efficiency of DMU_j is defined as

$$\theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, \tag{1}$$

where u_r and v_i are output and input weights assigned to the rth output and the ith input, respectively. In order to determine the efficiency of DMU_j relative to the other DMUs, Charnes et al. [2] developed the following well-known CCR model, which measures the best efficiencies of DMUs within the range of less than or equal to one:

$$\begin{aligned} \text{Max} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\ \text{s.t.} \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j=1,\ldots,n, \\ & u_r, v_i \geqslant \varepsilon, \quad r=1,\ldots,s; \quad i=1,\ldots,m, \end{aligned} \tag{2}$$

where the subscript zero represents the DMU under evaluation, u_r and v_i are decision variables and ε is the non-Archimedean infinitesimal. Through Charnes–Cooper's transformation [1], the fractional programming above can be equivalently changed into the following linear programming (LP) model:

$$\begin{aligned}
\text{Max} \quad & \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m.
\end{aligned} \tag{3}$$

If there exists a set of positive weights that makes $\theta_0^* = 1$, then DMU_0 is referred to be DEA efficient (CCR-efficient); otherwise, we call it to be non-DEA efficient rather than DEA inefficient because non-DEA efficient does not necessarily mean DEA inefficient. In fact, DEA efficient and DEA inefficient are only two extreme cases. For n different DMUs, there is a total number of n LP models to be solved. Accordingly, there are n different sets of weights, which are the basis to calculate the cross-efficiency matrix [4].

2.2. DEA model for measuring the worst relative efficiencies of DMUs

As mentioned before, efficiency is a relative measure. It can be measured within different ranges. If the efficiencies of DMUs are measured within the range of greater than or equal to one, then the following fractional programming model can be constructed to measure the worst performance of each DMU [7]:

$$\begin{aligned}
\text{Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\
\text{s.t.} \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geqslant 1, \quad j = 1, \ldots, n, \\
& u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m,
\end{aligned} \tag{4}$$

which can be further transformed into the following equivalent LP model:

$$\begin{aligned}
\text{Min} \quad & \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geqslant 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m.
\end{aligned} \tag{5}$$

Efficiencies determined by the above LP model (5) are referred to as the worst relative efficiencies. Contrary to the CCR model (3) that determines an efficiency frontier for n DMUs, model (5) determines an inefficiency frontier for them. We refer to those DMUs lying on the inefficiency frontier to be DEA inefficient, while those not lying on the inefficiency frontier to be non-DEA inefficient. Note here that non-DEA inefficient does not necessarily mean DEA efficient. They are two different concepts.

Since the best relative efficiencies measure the best performances of DMUs, while the worst relative efficiencies measure their worst performances, such two types of relative efficiencies usually lead to two distinctive assessment conclusions. Any assessment using only one type of efficiency is obviously not all-sided. Therefore, there is a clear need to combine both types of relative efficiencies and give an overall measurement and assessment of the performance of each DMU.

3. Bounded DEA models for measuring interval efficiencies of DMUs

3.1. Review of existing work

Since the best and the worst relative efficiencies are measured within different ranges, they are incomparable. Therefore, they cannot be directly used to form an efficiency interval for each DMU. In order to be able to generate an interval efficiency assessment for each DMU, Entani et al. [6] constructed the following upper and lower bounds mathematical programming model for DMU_0:

$$\begin{aligned}
&\text{Max/Min} \quad \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\max_j \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}},\\
&\text{s.t.} \quad u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m,
\end{aligned} \tag{6}$$

where the upper bound model was further transformed into the model below, which is equivalent to the standard CCR model (2) and can be solved through model (3):

$$\begin{aligned}
&\text{Max} \quad \theta_0^{\mathrm{U}} = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0}\\
&\text{s.t.} \quad \max_j \left\{ \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \right\} = 1,\\
&\qquad u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m,
\end{aligned} \tag{7}$$

while the lower bound model was converted into the following model, which cannot be replaced with an equivalent LP problem:

$$\begin{aligned}
&\text{Min} \quad \theta_0^{\mathrm{L}} = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0}\\
&\text{s.t.} \quad \max_j \left\{ \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \right\} = 1,\\
&\qquad u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m.
\end{aligned} \tag{8}$$

By assuming that $\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} = 1$ for each DEA efficient unit, Entani et al. divided the above model (8) into the following n_1 sub-optimization problems, where n_1 is the number of DEA efficient units:

$$\begin{aligned}
&\text{Min} \quad \theta_{0j}^{\mathrm{L}} = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0}\\
&\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} = 1,\\
&\qquad u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m.
\end{aligned} \tag{9}$$

For the sub-optimization problem (9), it can be simplified as the following equivalent LP model:

$$\begin{aligned} \text{Min} \quad & \theta_{0j}^{\mathrm{L}} = \sum_{r=1}^{s} u_r y_{r0} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{i0} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} = 0, \\ & u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m. \end{aligned} \tag{10}$$

Let $\theta_{0j}^{\mathrm{L}*}$ be the optimum objective function value of the above LP model (10). It is obvious that when $j = 0$, $\theta_{0j}^{\mathrm{L}*} = 1$. So, the lower bound efficiency of DMU_0 was finally determined by

$$\theta_0^{\mathrm{L}*} = 1 \wedge \min_{j \neq 0} \{\theta_{0j}^{\mathrm{L}*}\}, \tag{11}$$

where $a \wedge b = \min\{a, b\}$. Accordingly, the efficiency interval for DMU_0 is denoted as $[\theta_0^{\mathrm{L}*}, \theta_0^{\mathrm{U}*}]$, where $\theta_0^{\mathrm{U}*}$ is the optimum objective function value of the upper bound model (7).

Prior to Entani et al., Doyle et al. [5] developed the following three pairs of upper and lower bounds evaluation models:

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\max_{j \neq 0} \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}} \\ \text{s.t.} \quad & u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \end{aligned} \tag{12}$$

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\frac{1}{n-1} \sum_{j \neq 0}^{n} (\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij})} \\ \text{s.t.} \quad & u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \end{aligned} \tag{13}$$

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\min_{j \neq 0} \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}} \\ \text{s.t.} \quad & u_r, v_i \geqslant 0, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m. \end{aligned} \tag{14}$$

They all have the similar structures to models (6). So, we focus mainly on the discussion of model (6) and its solution procedure. Carefully analyzing models (8)–(10), the following drawbacks have been found:

(1) One important feature of measuring the worst relative efficiencies of DMUs is to identify DEA inefficient DMUs, which perform the worst among all DMUs from the pessimistic point of view, and to determine an inefficiency frontier so that DM or assessor can know which DMUs are DEA inefficient and which DMUs are not. But models (8)–(10) fail to do so. They can identify only one DMU with the smallest lower bound efficiency and not all DEA inefficient DMUs. Accordingly, they cannot determine the inefficiency frontier. So, much information useful to DM or assessor was lost.

(2) Models (8)–(10) use only one DMU, i.e. DMU_j as the reference set to compute the lower bound efficiency of DMU_0. So, model (10) has only two constraint conditions, which leads to only one input and one output weights to be nonzero and all the other input and output weights to be zero. That is to say, only one input and one output data of DMU_0 were effectively used and all the other input and output data were ignored when computing its lower bound efficiency. This is obviously unreasonable and unacceptable.

(3) Both models (9) and (10) can only guarantee $\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} = 1$, but they cannot guarantee that $\sum_{r=1}^{s} u_r y_{rj} / \sum_{j=1}^{m} v_i x_{ij} = 1$ is the maximum of all the efficiencies of DMUs because there exists no such a mechanism that can guarantee the efficiencies of all the other DMUs to be less than or equal to one. So, by simply assuming that $\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} = 1$ for each DEA efficient unit, the maximization condition

$\max_j \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\} = 1$ cannot be guaranteed. Therefore, models (9) and (10) may not be equivalent to model (8).

Evidently, model (6) cannot reasonably measure the worst relative efficiencies of DMUs and cannot determine the inefficiency frontier. Neither can models (12)–(14). So, in what follows, we will develop a new DEA model with the constraint of the upper and lower bounds on efficiency. For convenience and simplicity, we refer to it as bounded DEA model. The Bounded DEA model measures the performances of DMUs within the range of an interval and thus can effectively make the most of all the input and output data to measure both the best and the worst relative efficiencies of DMUs.

3.2. *Bounded DEA model for crisp data*

In order to reasonably measure the interval efficiencies of DMUs, we first introduce the concept of anti-ideal DMU.

Definition 1. An anti-ideal DMU (ADMU) is a virtual DMU, which consumes the most inputs only to produce the least outputs.

Note that although the anti-ideal DMU is a virtual DMU, it may exist in practical production activities because the waste of resources is always allowed in the theory of production possibility set. According to the above definition, we denote by $x_i^{\max} (i = 1, \ldots, m)$ and $y_r^{\min} (r = 1, \ldots, s)$ the inputs and outputs of the anti-ideal DMU, respectively, where $x_i^{\max}$ is the maximum of the ith input and $y_r^{\min}$ the minimum of the rth output. They are determined by the following equations:

$$x_i^{\max} = \max_j \{x_{ij}\}, \quad i = 1, \ldots, m,$$
$$y_r^{\min} = \min_j \{y_{rj}\}, \quad r = 1, \ldots, s.$$

Since the anti-ideal DMU utilizes the most inputs to produce the least outputs, its performance is without doubt the worst among all the DMUs. So, its efficiency should be the smallest at any circumstance.

Let θ_{ADMU}^* be the best relative efficiency of the anti-ideal DMU. Then it can be determined by the following fractional programming model:

$$\begin{aligned} \text{Max} \quad & \theta_{\mathrm{ADMU}} = \frac{\sum_{r=1}^{s} u_r y_r^{\min}}{\sum_{i=1}^{m} v_i x_i^{\max}} \\ \text{s.t.} \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j = 1, \ldots, n, \\ & u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \end{aligned} \tag{15}$$

which can be solved through the following LP model:

$$\begin{aligned} \text{Max} \quad & \theta_{\mathrm{ADMU}} = \sum_{r=1}^{s} u_r y_r^{\min} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_i^{\max} = 1, \\ & u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m, \end{aligned} \tag{16}$$

where ε is the non-Archimedean infinitesimal. After θ^*_{ADMU} is determined, we know that the efficiencies of all the DMUs cannot be less than it. Therefore, we can measure the efficiencies of DMUs within the range of interval $[\theta^*_{\mathrm{ADMU}}, 1]$. The following pair of fractional programming models reflects this idea:

$$\begin{aligned}
&\text{Max/Min} \quad \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\
&\text{s.t.} \quad \theta^*_{\mathrm{ADMU}} \leqslant \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, \quad j = 1, \ldots, n, \\
&\qquad u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m,
\end{aligned} \tag{17}$$

which can be equivalently transformed into the following pair of LP models:

$$\begin{aligned}
&\text{Max/Min} \quad \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\
&\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i (\theta^*_{\mathrm{ADMU}} x_{ij}) \geqslant 0, \quad j = 1, \ldots, n, \\
&\qquad \sum_{i=1}^{m} v_i x_{i0} = 1, \\
&\qquad u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m.
\end{aligned} \tag{18}$$

Both models (17) and (18) are called bounded DEA models. Let $\theta_0^{\mathrm{U}*}$ and $\theta_0^{\mathrm{L}*}$ be the maximum and the minimum of the above objective function, respectively. Then they form an efficiency interval, denoted by $[\theta_0^{\mathrm{L}*}, \theta_0^{\mathrm{U}*}]$, which measures the best and the worst relative efficiencies of DMU_0 and its efficiency range. Repeating the above solution process for each DMU, we can obtain both the best and the worst relative efficiencies of all the DMUs and their efficiency intervals $[\theta_j^{\mathrm{L}*}, \theta_j^{\mathrm{U}*}] (j = 1, \ldots, n)$.

About the interval efficiency, $[\theta_0^{\mathrm{L}*}, \theta_0^{\mathrm{U}*}]$, we give the following definition:

Definition 2. If $\theta_0^{\mathrm{U}*} = 1$, then DMU_0 is referred to be DEA efficient; if $\theta_0^{\mathrm{L}*} = \theta^*_{\mathrm{ADMU}}$, then it is referred to be DEA inefficient; If DMU_0 is neither DEA efficient nor DEA inefficient, then we call it to be DEA unspecified.

All the DEA efficient DMUs determine an efficient production frontier (efficiency frontier), while all the DEA inefficient DMUs together define an inefficient production frontier or called the inefficiency frontier. For those DEA unspecified units, they are always enveloped by both the efficiency and the inefficiency frontiers. Note that some DMU(s) may be both DEA efficient and DEA inefficient. Such DMUs have the widest efficiency interval $[\theta^*_{\mathrm{ADMU}}, 1]$. Their evaluations in fact contain the biggest uncertainty.

3.3. Bounded DEA model with preference information on weights

Traditional DEA approach often uses so-called assurance region (AR) approach or cone-ratio method to restrict factor weights $u_r (r = 1, \ldots, s)$ and/or $v_i (i = 1, \ldots, m)$. The interested reader may refer to Cooper and Seiford [3] for details. As a matter of fact, these two approaches are also applicable to the bounded DEA models (17) and (18). Here we consider how to incorporate DM or assessor's preference information on input and output weights into the bounded DEA models.

Since $u_r (r = 1, \ldots, s)$ and $v_i (i = 1, \ldots, m)$ are factor weights with different dimensions, they are usually incomparable. To take into account DM or assessor's preference information, we first carry out scale transformation to eliminate the dimension for each output and input factor.

Let

$$\tilde{y}_{rj} = \frac{y_{rj}}{\max_j \{y_{rj}\}} = \frac{y_{rj}}{y_r^{\max}}, \quad r = 1, \ldots, s; \quad j = 1, \ldots, n, \tag{19}$$

$$\tilde{x}_{ij} = \frac{x_{ij}}{\max_j \{x_{ij}\}} = \frac{x_{ij}}{x_i^{\max}}, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n. \tag{20}$$

The scale-transformed input and output data are of no dimensions and are all within the range of [0, 1]. Since DEA model has the property of unit-invariance, the use of scale transformation to input and output data does not change the efficiencies of DMUs. Therefore, we have

$$\theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} = \frac{\sum_{r=1}^{s} \tilde{u}_r \tilde{y}_{r0}}{\sum_{i=1}^{m} \tilde{v}_i \tilde{x}_{i0}}, \tag{21}$$

where $\tilde{u}_r (r = 1, \ldots, s)$ and $\tilde{v}_i (i = 1, \ldots, m)$ are the factor weights corresponding to the scale-transformed output and input data. They have no dimensions and are thus comparable. They can be utilized to express the DM or assessor's preference on outputs and inputs. According to the relative importance between outputs and inputs, DM or assessor may provide various types of preference information on outputs and inputs such as $\tilde{u}_{r_1} \geqslant \tilde{u}_{r_2}$, $\tilde{v}_{i_1} \geqslant \tilde{v}_{i_2}$, $\tilde{u}_{r_3} = \tilde{u}_{r_4}$, $\tilde{v}_{i_3} = \tilde{v}_{i_4}$, $\alpha \leqslant \tilde{u}_{r_5}/\tilde{u}_{r_6} \leqslant \beta$, $\gamma \leqslant \tilde{v}_{i_5}/\tilde{v}_{i_6} \leqslant \delta$,and so on. Substituting (19) and (20) into (21), we have

$$\theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} = \frac{\sum_{r=1}^{s} \tilde{u}_r \tilde{y}_{r0}}{\sum_{i=1}^{m} \tilde{v}_i \tilde{x}_{i0}} = \frac{\sum_{r=1}^{s} (\tilde{u}_r / y_r^{\max}) y_{r0}}{\sum_{i=1}^{m} (\tilde{v}_i / x_i^{\max}) x_{i0}} \tag{22}$$

from which we know that

$$\tilde{u}_r = u_r y_r^{\max}, \quad r = 1, \ldots, s, \tag{23}$$

$$\tilde{v}_i = v_i x_i^{\max}, \quad i = 1, \ldots, m. \tag{24}$$

These are two very important formulae, which show that the factor weights $u_r (r = 1, \ldots, s)$ and $v_i (i = 1, \ldots, m)$ multiplied by the maxima of output and input data can be used to express DM or assessor's preference. For example, DM or assessor's preference information mentioned above can be equivalently expressed as $u_{r_1} y_{r_1}^{\max} \geqslant u_{r_2} y_{r_2}^{\max}$, $v_{i_1} x_{i_1}^{\max} \geqslant v_{i_2} x_{i_2}^{\max}$, $u_{r_3} y_{r_3}^{\max} = u_{r_4} y_{r_4}^{\max}$, $v_{i_3} x_{i_3}^{\max} = v_{i_4} x_{i_4}^{\max}$, $\alpha \leqslant u_{r_5} y_{r_5}^{\max} / u_{r_6} y_{r_6}^{\max} \leqslant \beta$, $\gamma \leqslant v_{i_5} x_{i_5}^{\max} / v_{i_6} x_{i_6}^{\max} \leqslant \delta$. Such preference information on factor weights can be easily incorporated into the bounded DEA model.

Let

$$A^+ = \{u = (u_r) \mid u_{r_1} y_{r_1}^{\max} \geqslant u_{r_2} y_{r_2}^{\max}, u_{r_3} y_{r_3}^{\max} = u_{r_4} y_{r_4}^{\max}, \alpha \leqslant u_{r_5} y_{r_5}^{\max} / u_{r_6} y_{r_6}^{\max} \leqslant \beta\}, \tag{25}$$

$$A^- = \{v = (v_i) \mid v_{i_1} x_{i_1}^{\max} \geqslant v_{i_2} x_{i_2}^{\max}, v_{i_3} x_{i_3}^{\max} = v_{i_4} x_{i_4}^{\max}, \gamma \leqslant v_{i_5} x_{i_5}^{\max} / v_{i_6} x_{i_6}^{\max} \leqslant \delta\}. \tag{26}$$

Then the bounded DEA model with the preference information on weights can be expressed as follows:

$$\begin{aligned}
\text{Max/Min} \quad & \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i (\theta_{\text{ADMU}}^* x_{ij}) \geqslant 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& (u_r) \in A^+, \\
& (v_i) \in A^-, \\
& u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m,
\end{aligned} \tag{27}$$

where

$$\theta^*_{\mathrm{ADMU}} = \mathrm{Max} \sum_{r=1}^{s} u_r y_r^{\min} \tag{28}$$

$$\begin{aligned} \text{s.t.}\quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_i^{\max} = 1, \\ & (u_r) \in A^+, \\ & (v_i) \in A^-, \\ & u_r, v_i \geqslant \varepsilon, \quad r = 1, \ldots, s; \quad i = 1, \ldots, m. \end{aligned} \tag{29}$$

4. Hurwicz criterion approach for comparing and ranking interval efficiencies

In interval efficiency assessment, since the final efficiency score for each DMU is characterized by an interval, a simple yet practical ranking approach is thus needed for comparing and ranking the efficiencies of different DMUs. A few approaches have already been developed to compare and rank interval numbers, but they all have some shortcomings.

Here, we choose Hurwicz criterion approach (HCA) as the approach for comparing and ranking interval efficiencies. The main reason why we choose it is because the comparisons and rankings based on the best and the worst relative efficiencies of DMUs are only two special cases of HCA. Therefore, DM or assessor may choose different levels of optimism to carry out the best relative efficiency analysis, or the worst relative efficiency analysis or an overall assessment. The approach is detailed as follows.

Let $A_i = [a_i^{\mathrm{L}}, a_i^{\mathrm{R}}] = \langle m(A_i), w(A_i)\rangle (i = 1, \ldots, n)$ be the interval efficiencies of n DMUs, where $m(A_i) = \frac{1}{2}(a_i^{\mathrm{R}} + a_i^{\mathrm{L}})$ and $w(A_i) = \frac{1}{2}(a_i^{\mathrm{R}} - a_i^{\mathrm{L}})$ are their midpoints (centers) and widths. Since the values of interval efficiencies are uncertain, they are naturally associated with the decision making under uncertainty. Here, interval efficiencies are viewed as decision alternatives. A common feature between interval efficiencies and alternatives in decision making under uncertainty is that the probabilities for them to take values are unknown or cannot be determined. The difference between them is that interval efficiency has an infinite amount of values (states) within an interval, while an alternative in decision making under uncertainty has only a finite number of states of nature (values). Their common feature decides that the decision-making approaches under uncertainty should be able to be used to compare and rank interval efficiencies.

Let α be DM or assessor's level of optimism $(0 \leqslant \alpha \leqslant 1)$, then Hurwicz decision criterion selects the interval efficiency with the highest weighted average value as the most preferred one, namely,

$$\max_i \{\alpha \max(A_i) + (1 - \alpha) \min(A_i)\} = \max_i \{\alpha a_i^{\mathrm{R}} + (1 - \alpha) a_i^{\mathrm{L}}\}.$$

When $\alpha = 1$, 0 or 0.5, the above Hurwicz decision criterion becomes the maximax, maximin and equally likely criteria, respectively, which are widely used in decision making under uncertainty. Based on the above analysis, we give the following definition for comparing and ranking interval efficiencies.

Definition 3. Let $A_i = [a_i^{\mathrm{L}}, a_i^{\mathrm{R}}] = \langle m(A_i), w(A_i)\rangle$ be interval efficiency and α be DM or assessor's level of optimism $(0 \leqslant \alpha \leqslant 1)$. Then Hurwicz index value of A_i is defined as

$$\begin{aligned} H(A_i) &= \alpha \max(A_i) + (1 - \alpha) \min(A_i) = \alpha a_i^{\mathrm{R}} + (1 - \alpha) a_i^{\mathrm{L}} \\ &= m(A_i) + (2\alpha - 1) w(A_i). \end{aligned}$$

The parameter α may be understood as DM or assessor's attitude towards risk. For $\alpha > 0.5$, the DM or assessor is said to be optimistic and risk-seeking. Those with $\alpha = 0.5$ are referred to be risk-neutral. If $\alpha < 0.5$, the DM or assessor is pessimistic and risk-averse.

It is evident that the bigger the Hurwicz index value, the better the interval efficiency. If $H(A) > H(B)$, then A is said to be superior to B. The interval efficiency with the biggest Hurwicz index value is the most preferred one and should be ranked at the first place. With the help of Hurwicz index values, a complete ranking order for all the interval efficiencies can be made. We call such a ranking approach Hurwicz criterion approach (HCA).

About HCA, we have the following two properties:

Property 1. *Let* $A=[a_{\mathrm{L}}, a_{\mathrm{R}}]$ *and* $B=[b_{\mathrm{L}}, b_{\mathrm{R}}]$ *be two interval efficiencies. If* $a_{\mathrm{L}} \leqslant b_{\mathrm{L}}$ *and* $a_{\mathrm{R}} \leqslant b_{\mathrm{R}}$, *then* $H(A) \leqslant H(B)$.

Property 2. *Let* $A=[a_{\mathrm{L}}, a_{\mathrm{R}}]=\langle m(A), w(A)\rangle$ *and* $B=[b_{\mathrm{L}}, b_{\mathrm{R}}]=\langle m(B), w(B)\rangle$ *be two interval efficiencies. If A is included in B, i.e.* $a_{\mathrm{L}} \geqslant b_{\mathrm{L}}$ *but* $a_{\mathrm{R}} \leqslant b_{\mathrm{R}}$, *then*

(1) $H(A)-H(B) \leqslant m(A)-m(B)$ *if* $\alpha>0.5$;
(2) $H(A)-H(B)=m(A)-m(B)$ *if* $\alpha=0.5$;
(3) $H(A)-H(B) \geqslant m(A)-m(B)$ *if* $\alpha<0.5$.

Properties 1 and 2 show that if two interval efficiencies are not nested, then the one with the higher lower and upper bounds is better; if one interval efficiency is included in another, then the ranking order depends not only on their centers but also on DM or assessor's level of optimism, i.e. attitude towards risk. For example, $A=[0.8, 0.9]$ and $B=[0.6, 1.0]$ are two nested interval efficiencies with $m(A)=0.85$ and $m(B)=0.8$. If $\alpha=0.8$, then $H(A)=0.88<9.2=H(B)$. So, $A \prec B$. But if $\alpha=0.4$, then $H(A)=0.84>0.76=H(B)$. So, $A \succ B$.

Since the ranking order generated by HCR depends on DM or assessor's level of optimism, it is desirable to conduct a sensitivity analysis to α so that DM or assessor can know how stable his/her ranking is when his/her level of optimism varies. The following theorem shows this.

Theorem 1. *Let* $A_i=[a_i^{\mathrm{L}}, a_i^{\mathrm{R}}]$ $(i=1, \ldots, n)$ *be a set of interval efficiencies. For a given level of optimism,* α_0, *if the ranking is* $A_{i_1} \succ A_{i_2} \succ \cdots \succ A_{i_n}$, *then there exists an interval for level of optimism,* α, *which is determined by*

$$(\alpha_{\mathrm{L}}, \alpha_{\mathrm{R}}) \cap [0, 1],$$

where

$$\alpha_{\mathrm{L}}=\max_j\left\{\left.\frac{a_{i_{j+1}}^{\mathrm{L}}-a_{i_j}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}}-a_{i_j}^{\mathrm{L}}\right)-\left(a_{i_{j+1}}^{\mathrm{R}}-a_{i_{j+1}}^{\mathrm{L}}\right)\right|}\,\right|\left(a_{i_j}^{\mathrm{R}}-a_{i_j}^{\mathrm{L}}\right)-\left(a_{i_{j+1}}^{\mathrm{R}}-a_{i_{j+1}}^{\mathrm{L}}\right)>0\right\},$$

$$\alpha_{\mathrm{R}}=\min_j\left\{\left.\frac{a_{i_j}^{\mathrm{L}}-a_{i_{j+1}}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}}-a_{i_j}^{\mathrm{L}}\right)-\left(a_{i_{j+1}}^{\mathrm{R}}-a_{i_{j+1}}^{\mathrm{L}}\right)\right|}\,\right|\left(a_{i_j}^{\mathrm{R}}-a_{i_j}^{\mathrm{L}}\right)-\left(a_{i_{j+1}}^{\mathrm{R}}-a_{i_{j+1}}^{\mathrm{L}}\right)<0\right\}.$$

When α *varies within the above interval, the ranking among the interval efficiencies remains unchanged.*

Proof. By Definition 1, the ranking, $A_{i_1} \succ A_{i_2} \succ \cdots \succ A_{i_n}$, can be equivalently expressed using Hurwicz index values as follows:

$$H(A_{i_1})>H(A_{i_2})>\cdots>H(A_{i_n})$$

or

$$H(A_{i_j})-H(A_{i_{j+1}})>0, \quad j=1, \ldots, n-1.$$

That is

$$\alpha a_{i_j}^{\mathrm{R}}+(1-\alpha) a_{i_j}^{\mathrm{L}}-\alpha a_{i_{j+1}}^{\mathrm{R}}-(1-\alpha) a_{i_{j+1}}^{\mathrm{L}}>0, \quad j=1, \ldots, n-1,$$

which can be further expressed as

$$\alpha\left[\left(a_{i_j}^{\mathrm{R}}-a_{i_{j+1}}^{\mathrm{R}}\right)-\left(a_{i_j}^{\mathrm{L}}-a_{i_{j+1}}^{\mathrm{L}}\right)\right]+\left(a_{i_j}^{\mathrm{L}}-a_{i_{j+1}}^{\mathrm{L}}\right)>0, \quad j=1, \ldots, n-1.$$

It is evident that if

$$\left(a_{i_j}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{R}}\right) - \left(a_{i_j}^{\mathrm{L}} - a_{i_{j+1}}^{\mathrm{L}}\right) > 0,$$

then

$$\alpha > \frac{a_{i_{j+1}}^{\mathrm{L}} - a_{i_j}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right)\right|}, \quad j = 1, \ldots, n-1;$$

otherwise,

$$\alpha < \frac{a_{i_j}^{\mathrm{L}} - a_{i_{j+1}}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right)\right|}, \quad j = 1, \ldots, n-1.$$

These two inequalities can be further written as

$$\alpha > \alpha_{\mathrm{L}} = \max_j \left\{ \frac{a_{i_{j+1}}^{\mathrm{L}} - a_{i_j}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right)\right|} \,\middle|\, \left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right) > 0 \right\},$$

$$\alpha < \alpha_{\mathrm{R}} = \min_j \left\{ \frac{a_{i_j}^{\mathrm{L}} - a_{i_{j+1}}^{\mathrm{L}}}{\left|\left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right)\right|} \,\middle|\, \left(a_{i_j}^{\mathrm{R}} - a_{i_j}^{\mathrm{L}}\right) - \left(a_{i_{j+1}}^{\mathrm{R}} - a_{i_{j+1}}^{\mathrm{L}}\right) < 0 \right\}.$$

Since α can only take values between 0 and 1, the final interval for α can be expressed as $(\alpha_{\mathrm{L}}, \alpha_{\mathrm{R}}) \cap [0, 1]$. □

From the above introductions, it is quite clear that HCR provides a flexible way of comparing and ranking interval efficiencies. It allows for considering DM or assessor's level of optimism and can conduct a sensitivity analysis. The information obtained through the sensitivity analysis can help DM or assessor know how stable his/her ranking is if his/her level of optimism varies.

5. Numerical example

We now examine a numerical example using the bounded DEA model to illustrate its application in real-world performance measurement.

Consider a performance-measurement problem with 10 DMUs, each DMU with one input and two outputs. The data set is taken from Entani et al. [6] and shown in Table 1, where the input is normalized to one for simplicity.

The best- and the worst-relative efficiencies of each DMU are calculated by models (3) and (5), respectively, and the results are recorded in the second and the third columns of Table 2. Models are implemented in an MS-Excel worksheet and are solved by using the Excel Solver. The non-Archimedean infinitesimal is set to be $\varepsilon = 10^{-10}$.

From the angle of the best relative efficiency, $\mathrm{DMU_A}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_J}$ are all evaluated to be DEA efficient. They together determine an efficiency frontier, which is shown in Fig. 1. Their performances are usually thought to be better than any other DMUs that are evaluated to be non-DEA efficient. The performances of those non-DEA efficient DMUs are rated to be $\mathrm{DMU_G} \sim \mathrm{DMU_J} \succ \mathrm{DMU_H} \succ \mathrm{DMU_C} \succ \mathrm{DMU_F} \succ \mathrm{DMU_D} \succ \mathrm{DMU_B}$, where the symbol '$\sim$' means 'be indifferent to', while the symbol '$\succ$' represents 'be superior to'.

However, when the DMUs are evaluated from the viewpoint of the worst relative efficiencies, $\mathrm{DMU_A}$, $\mathrm{DMU_B}$, $\mathrm{DMU_F}$ and $\mathrm{DMU_J}$ are all evaluated to be DEA inefficient. They together define an inefficiency frontier, which is also shown in Fig. 1. Their performances are usually thought to be worse than any other DMUs that are evaluated to be non-DEA inefficient. The performances of those non-DEA inefficient DMUs are rated to be $\mathrm{DMU_G} \succ \mathrm{DMU_E} \succ \mathrm{DMU_C} \succ \mathrm{DMU_I} \succ \mathrm{DMU_D} \succ \mathrm{DMU_H}$.

It is obvious that the above two assessment results are different. Especially for $\mathrm{DMU_A}$ and $\mathrm{DMU_J}$, when they are evaluated using their best performances, they are both rated to be DEA efficient, which means they perform better

Table 1
Data for 10 DMUs with one input and two outputs

DMU	Input (X_1)	Output 1 (Y_1)	Output 2 (Y_2)
A	1	1	8
B	1	2	3
C	1	2	6
D	1	3	3
E	1	3	7
F	1	4	2
G	1	4	5
H	1	5	2
I	1	6	2
J	1	7	1
Anti-ideal DMU	1	1	1

Table 2
Relative efficiencies for the 10 DMUs with one input and two outputs

DMU	The best efficiency	The worst efficiency	Interval efficiency	
			Entani et al.'s model	Bounded DEA model
A	1.0000	1.0000	[0.1428, 1.0000]	[0.2174, 1.0000]
B	0.5217	1.0000	[0.2857, 0.5217]	[0.2174, 0.5217]
C	0.8235	1.2308	[0.2857, 0.8235]	[0.2676, 0.8235]
D	0.6522	1.1250	[0.3750, 0.6522]	[0.2446, 0.6522]
E	1.0000	1.6923	[0.4285, 1.0000]	[0.3679, 1.0000]
F	0.6957	1.0000	[0.2500, 0.6957]	[0.2174, 0.6957]
G	0.9565	1.7500	[0.5714, 0.9565]	[0.3804, 0.9565]
H	0.8261	1.1000	[0.2500, 0.8261]	[0.2391, 0.8261]
I	0.9565	1.2000	[0.2500, 0.9565]	[0.2609, 0.9565]
J	1.0000	1.0000	[0.1250, 1.0000]	[0.2174, 1.0000]

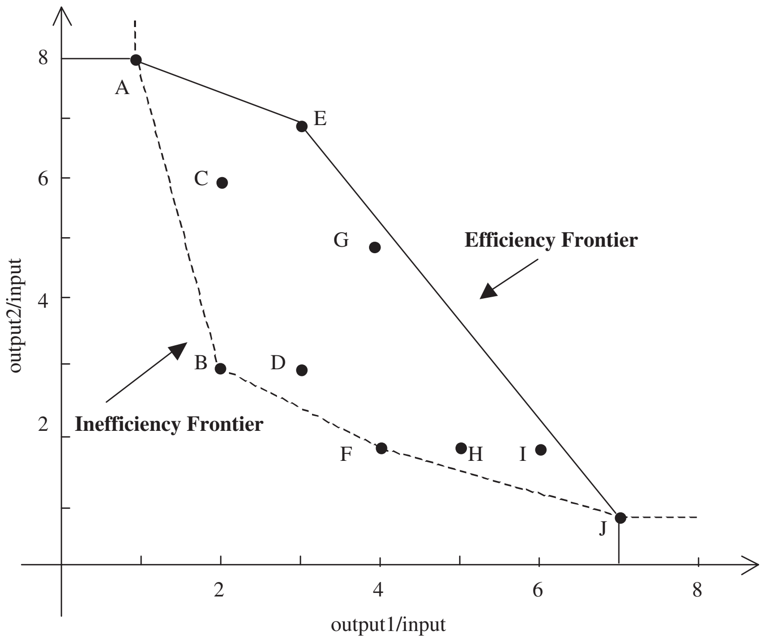

Fig. 1. Efficiency and inefficiency frontiers determined by the bounded DEA model.

than any other DMUs. However, when they are evaluated using their worst performances, they are both rated to be DEA inefficient, which means they perform worse than any other DMUs. Such two assessment results are evidently opposite. Any assessment conclusion drawn from using only the best or the worst relative efficiencies is one-sided and unreliable.

In order to give an overall assessment of each DMU from both the optimistic and the pessimistic points of view, Entani et al. used model (6) developed by themselves to measure the interval efficiency of each DMU. The results are reported in the fourth column of Table 2, from which it can be seen very clearly that their model only successfully identified one DEA inefficient DMU, i.e. $\mathrm{DMU_J}$, which has the smallest lower bound efficiency, but failed to identify the other three DEA inefficient DMUs. So, the inefficient production frontier cannot be determined by their approach.

Since there are three DMUs, i.e. $\mathrm{DMU_A}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_J}$ that are identified to be DEA efficient, in order to determine the lower bound efficiencies of DMUs, three LP models need to be solved for each DMU. Take $\mathrm{DMU_A}$ for example. In order to calculate its lower bound efficiency, the following three LP models need to be solved:

$$(\mathrm{LP1}):\quad \theta_{\mathrm{AA}}^{\mathrm{L}*}=\mathrm{Min}\, u_1+8u_2$$

$$\text{s.t.}\begin{cases} v_1=1,\\ u_1+8u_2-v_1=0,\\ u_1,u_2,v_1\geqslant 0.\end{cases}$$

$$(\mathrm{LP2}):\quad \theta_{\mathrm{AE}}^{\mathrm{L}*}=\mathrm{Min}\, u_1+8u_2$$

$$\text{s.t.}\begin{cases} v_1=1,\\ 3u_1+7u_2-v_1=0,\\ u_1,u_2,v_1\geqslant 0.\end{cases}$$

$$(\mathrm{LP3}):\quad \theta_{\mathrm{AJ}}^{\mathrm{L}*}=\mathrm{Min}\, u_1+8u_2$$

$$\text{s.t.}\begin{cases} v_1=1,\\ 7u_1+u_2-v_1=0,\\ u_1,u_2,v_1\geqslant 0.\end{cases}$$

Each of the above three LP models keeps only one of three DEA efficient DMUs continuing to be DEA efficient. The solutions to the above three LP models are as follows:

$$\theta_{\mathrm{AA}}^{\mathrm{L}*}=1,\quad u_1^*=0,\quad u_2^*=1/8\quad\text{and}\quad v_1^*=1,$$

$$\theta_{\mathrm{AE}}^{\mathrm{L}*}=1/3,\quad u_1^*=1/3,\quad u_2^*=0\quad\text{and}\quad v_1^*=1,$$

$$\theta_{\mathrm{AJ}}^{\mathrm{L}*}=1/7,\quad u_1^*=1/7,\quad u_2^*=0\quad\text{and}\quad v_1^*=1.$$

So, the final lower bound efficiency of $\mathrm{DMU_A}$ is determined by

$$\theta_A^{\mathrm{L}*}=\min\{1,1/3,1/7\}=0.1428.$$

From the above three sets of input and output weights, it can be seen that only one output (either output 1 or output 2) is effectively used in the computation of lower bound efficiency. Special attention has been paid to the second set of factor weights, i.e. $v_1^*=1, u_1^*=1/3, u_2^*=0$, from which we have the following efficiencies for $\mathrm{DMU_F}$ through $\mathrm{DMU_J}$:

$$\theta_{\mathrm{F}}=\theta_{\mathrm{G}}=\tfrac{4}{3},\quad \theta_{\mathrm{H}}=\tfrac{5}{3},\quad \theta_{\mathrm{I}}=2\quad\text{and}\quad \theta_{\mathrm{J}}=\tfrac{7}{3}.$$

They are all greater than one. Such results obviously contradict the assumption that $\max_j\{\sum_{r=1}^{s}u_r y_{rj}/\sum_{i=1}^{m}v_i x_{ij}\}=1$. So, Entani et al.'s solution approach is in fact defective.

As a contrast, we now utilize the bounded DEA model (18) developed in this paper to re-evaluate the problem. To do so, we first define the anti-ideal DMU, which is shown in the last row of Table 1. Its best relative efficiency is found to

Table 3
Hurwicz index values and ranking orders for the 10 DMUs under three different levels of optimism

DMU	$\alpha = 1$		$\alpha = 0.5$		$\alpha = 0$	
	Hurwicz	Rank	Hurwicz	Rank	Hurwicz	Rank
A	1.0000	1	0.6087	3	1.0000	7
B	0.5217	10	0.3696	10	1.0000	7
C	0.8235	7	0.5455	6	1.2308	3
D	0.6522	9	0.4484	9	1.1250	5
E	1.0000	1	0.6840	1	1.6923	2
F	0.6957	8	0.4566	8	1.0000	7
G	0.9565	4	0.6684	2	1.7500	1
H	0.8261	6	0.5326	7	1.1000	6
I	0.9565	4	0.6087	3	1.2000	4
J	1.0000	1	0.6087	3	1.0000	7

be $\theta^*_{\mathrm{ADMU}} = 0.2174$ by running model (16). Running model (18) for each DMU, we get the interval efficiencies of the 10 DMUs, which are presented in the last column of Table 2, from which it can be seen very clearly that the bounded DEA model not only identifies the three DEA efficient DMUs correctly, but also identifies the four DEA inefficient DMUs fully. The identified DEA efficient units are $\mathrm{DMU_A}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_J}$. $\mathrm{DMU_A}$, $\mathrm{DMU_B}$, $\mathrm{DMU_F}$ and $\mathrm{DMU_J}$ are the four identified DEA inefficient DMUs. Such assessment results are fully consistent with the results obtained by the traditional CCR model (3) and the worst relative efficiency model (5).

Although $\mathrm{DMU_A}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_J}$ are all evaluated to be DEA efficient, due to the differences in lower bound efficiencies, their performances are in fact not the same. Through comparing their lower bound efficiencies, we find that $\mathrm{DMU_E} \succ \mathrm{DMU_A} \sim \mathrm{DMU_J}$. As such, $\mathrm{DMU_A}$, $\mathrm{DMU_B}$, $\mathrm{DMU_F}$ and $\mathrm{DMU_J}$ are all rated to be DEA inefficient, due to their differences in upper bound efficiency, their performances are not the same either. Through comparing their upper bound efficiencies, we may arrive at the conclusion that $\mathrm{DMU_A} \sim \mathrm{DMU_J} \succ \mathrm{DMU_F} \succ \mathrm{DMU_B}$. The remaining five DMUs belong to DEA unspecified units. They are all enveloped by the efficient and inefficient production frontiers.

In this example, both $\mathrm{DMU_A}$ and $\mathrm{DMU_J}$ are evaluated to be DEA efficient and DEA inefficient. This phenomenon shows that the two different production frontiers simultaneously pass through these two specific DMUs (see Fig. 1). Usually, DEA-efficient units perform well, but this does not mean each DEA-efficient unit is the best. As such, DEA inefficient units usually perform poor, but not every DEA-inefficient unit performs the worst. So, when a DMU is both DEA efficient and DEA inefficient, it is likely to mean that the DMU is neither the best nor the worst.

In order to give an overall ranking of the performances of the 10 DMUs, DM or assessor's level of optimism is usually required. For a given level of optimism, α, we can calculate the Hurwicz index value of each DMU, based on which a ranking order can be generated. Table 3 shows the ranking orders for the 10 DMUs under three different levels of optimism. For $\alpha = 1$, the DM or assessor is absolutely optimistic. He/She considers only the best performance of each DMU. In this situation, the ranking is $\mathrm{DMU_A} \sim \mathrm{DMU_E} \sim \mathrm{DMU_J} \succ \mathrm{DMU_G} \sim \mathrm{DMU_I} \succ \mathrm{DMU_H} \succ \mathrm{DMU_C} \succ \mathrm{DMU_F} \succ \mathrm{DMU_D} \succ \mathrm{DMU_{10}}$. For $\alpha = 0$, the DM or assessor is absolutely pessimistic. He/She considers only the worst performances of DMUs. In this situation, he/she ranks the 10 DMUs as $\mathrm{DMU_G} \succ \mathrm{DMU_E} \succ \mathrm{DMU_C} \succ \mathrm{DMU_I} \succ \mathrm{DMU_D} \succ \mathrm{DMU_H} \succ \mathrm{DMU_A} \sim \mathrm{DMU_B} \sim \mathrm{DMU_F} \sim \mathrm{DMU_J}$. For a neutral DM or assessor, his/her level of optimism is 0.5. In this situation, the ranking order for the 10 DMUs is $\mathrm{DMU_E} \succ \mathrm{DMU_G} \succ \mathrm{DMU_A} \sim \mathrm{DMU_I} \sim \mathrm{DMU_J} \succ \mathrm{DMU_C} \succ \mathrm{DMU_H} \succ \mathrm{DMU_F} \succ \mathrm{DMU_D} \succ \mathrm{DMU_B}$. The above three levels of optimism are three extreme situations. In real performance-rating applications, the level of optimism, α, is decided by DM or assessor.

6. Concluding remarks

The performances of DMUs can be measured within different ranges of efficiencies. The traditional DEA approach measures them using the best relative efficiency within the range of less than or equal to one. They can also be measured using the worst relative efficiency within the range of greater than or equal to one. The best and the worst relative efficiencies measure the performances of DMUs from the optimistic and the pessimistic points of view, respectively.

Neither of them, however, can give an overall assessment of the performances of DMUs. So, there is a clear need to combine them to give an overall assessment of the performances of DMUs. Through introducing an anti-ideal DMU, we have shown in this paper that the best and the worst relative efficiencies can be unified within the framework of so-called bounded DEA model, which measures the performances of DMUs using interval efficiencies within the range of an interval. DM or assessor's preference information on input and output weights can also be incorporated into the bounded DEA model easily and conveniently.

Compared with Entani et al.'s model, the bounded DEA model developed in this paper has some attractive advantages. First of all, it can identify DEA efficient and inefficient DMUs correctly and fully. DEA efficient DMUs form an efficiency frontier, while DEA inefficient DMUs define an inefficiency frontier. All the DEA unspecified DMUs are enveloped by both frontiers. Next, the bounded DEA model can make the most of all input and output data in the process of calculating both the upper and lower bound efficiencies of each DMU. So, both the upper and lower bound efficiencies are reasonably determined. Last but not least, the bounded DEA model only needs to solve $(2n + 1)$ LP problems. One is solved to determine the best relative efficiency of the anti-ideal DMU. The other $2n$ LP problems are solved to compute the upper and lower bounds efficiencies of n DMUs, respectively. The computational burden is substantially reduced.

Since interval efficiencies measure the performances of DMUs more comprehensively than the traditional DEA efficiency, they are expected to have widely potential applications in the future. It is worthwhile mentioning here that the bounded DEA model developed in this paper is input-oriented, but can be extended to other situations such as output-oriented, BCC as well as additive DEA models easily. It can also be extended to model interval input and output data. Due to the limitation of space, it is omitted in this paper.

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