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An Investigation of Bhattacharya-Type Designs

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Quasi-residual designs are balanced incomplete block designs having the parameters of a residual BIBD. For $\lambda = 1$ or 2 all quasi-residual designs are also residual designs, but a single counterexample due to Bhattacharya for the design (16, 24, 9, 6, 3) shows this not to be the case for $\lambda = 3$. We examine the block structure of this type of design, and use this information to construct eight new solutions for (16, 24, 9, 6, 3); along with the Bhattacharya design, these are the only known counterexamples for $\lambda = 3$.

The more general case with $\lambda > 3$ is mentioned briefly.

1. INTRODUCTION

A balanced incomplete block design (BIBD) with parameters (v, b, r, k, λ) is an arrangement of v varieties in b blocks such that each block contains $k < v$ distinct varieties, each variety occurs in r blocks, and every pair of varieties occurs in λ blocks. It is well known (see, for example, Ryser [8]) that

$$bk = vr \text{ and } \lambda(v - 1) = r(k - 1), \tag{1}$$

and Fisher [3] has also proved that it is necessary that $b \geqslant v$.

A symmetric BIBD has $b = v, r = k$. From such a design with parameters (v, v, k, k, λ) one may form a *residual design* by deleting one specific block and all varieties occurring in that block to leave a design with parameters $(v', b', r', k', \lambda')$, where

$$v' = v - k, \quad b' = v - 1, \quad r' = k, \quad k' = k - \lambda, \quad \lambda' = \lambda. \tag{2}$$

A *quasi-residual design* is defined in [5] as a design having the parameters of a residual design. For $\lambda = 1$ or 2 all quasi-residual designs are residual designs [2, 9]. Bhattacharya [1] showed that this is not the case for $\lambda = 3$ by giving a BIBD (16, 24, 9, 6, 3) which is not derivable by residuation from the corresponding symmetric design (25, 25, 9, 9, 3). This impossibility rests upon the fact that the Bhattacharya design (see Hall [4]) has two blocks, and only two, which intersect in four varieties, whereas it is well known (see, for example, Ryser [8]) that any pair of blocks in a symmetric design intersect in precisely λ varieties.

Quasi-residual designs which are not residual designs may be called *non-extensible*, since they cannot be extended to yield the corresponding symmetric design by reversing the process of residuation.

Little is known about non-extensible quasi-residual designs for $\lambda \geqslant 3$. Previously it was not even known if the Bhattacharya design is the only nonextensible design with parameters (16, 24, 9, 6, 3). We answer this question in the negative below by constructing eight new solutions for this design, all of which are non-extensible and are non-isomorphic to the Bhattacharya design

These designs are of further interest in that they are, along with the Bhattacharya design, the only published non-extensible quasi-residual designs with $\lambda = 3$. (Parker [7] has given examples of non-extensible designs, but the minimal λ value in this group of designs seems to be 10. There do not seem to be any examples in the literature of non-extensible designs having λ equal to any of 4, 5,..., 9.)

2. Block Intersection Properties

Let the blocks of the design be $B_1, B_2, \dots, B_b$, and denote the number of varieties in $B_i \cap B_j$ by s_{ij}. For any set of $t \leqslant b$ blocks we call the matrix $B_t = (s_{ij})$ the *block intersection*, or *structural matrix of the t blocks*. For any particular block B we also define *intersection numbers* $a_i (i = 0, 1, 2, \dots)$ by

$$a_i = \text{number of blocks intersecting } B \text{ in precisely } i \text{ varieties.}$$

Consider now quasi-residual designs with $\lambda = 3$. From relations (1) and (2) it follows that the parameters of these designs may be written as

$$v = \tfrac{1}{3}k(k+2), \quad b = \tfrac{1}{3}(k+2)(k+3), \quad r = k+3, \quad k \quad \lambda = 3. \qquad (3)$$

A study of the block intersection properties of these designs has been made in [5] and [6], and it has been shown that, for any design with parameters (3), $s_{ij} \leqslant 5$, and further, that $s_{ij} \leqslant 3$ except possibly for designs given by a small number of values of k. Among these, $k = 6$, yielding the parameter set (16, 24, 9, 6, 3), is exceptional.

It is shown in [5] that there are only six different possible sets of intersection numbers a_i that a block in a design with parameters (3) can have, and in [6] it is shown that, for $k = 6$, only three of these are possible. We use this fact to classify each block of any BIBD (16, 24, 9, 6, 3) as one of three types, depending on which of the three sets of intersection numbers it has. These are as follows:

Block type	*Intersection numbers*				
I	$a_0 = 1$	$a_1 = 0$	$a_2 = 18$	$a_3 = 4$	$a_4 = 0$
II	$a_0 = 0$	$a_1 = 3$	$a_2 = 15$	$a_3 = 5$	$a_4 = 0$
III	$a_0 = 0$	$a_1 = 2$	$a_2 = 18$	$a_3 = 2$	$a_4 = 1$

Two blocks intersecting in four varieties will both be of Type III, and any design possessing Type III blocks will necessarily be non-extensible. The Bhattacharya design has two, and only two, blocks of Type III; in an attempt to find other non-extensible designs let us suppose there are at least four blocks of Type III in the design. (Note that the number of blocks of Type III must be even.)

We first quote the following lemma, proved in [6].

LEMMA 1. *If* $s_{12} = 4$ *in a* BIBD (16, 24, 9, 6, 3), *then the first two rows of the structural matrix of the entire design may be taken as*

$$\begin{bmatrix} k & 4 & 1 & 3 & 1 & 3 & 2 & \cdots & 2 \\ 4 & k & 3 & 1 & 3 & 1 & 2 & \cdots & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Suppose now that, in a BIBD (16, 24, 9, 6, 3), B_1, B_2, B_3, B_4, are Type III blocks, and $s_{12} = s_{34} = 4$. We prove

LEMMA 2. *For blocks* B_1, B_2, B_3, B_4 *as given above, we may take as the structural matrix either*

$$\text{(a)}\quad S_4{}^1 = \begin{bmatrix} k & 4 & 1 & 3 \\ 4 & k & 3 & 1 \\ 1 & 3 & k & 4 \\ 3 & 1 & 4 & k \end{bmatrix} \qquad \text{or} \quad \text{(b)}\quad S_4{}^2 = \begin{bmatrix} k & 4 & 2 & 2 \\ 4 & k & 2 & 2 \\ 2 & 2 & k & 4 \\ 2 & 2 & 4 & k \end{bmatrix}.$$

Proof. From Lemma 1, $s_{13} = 1, 2$ or 3. Suppose $s_{13} = 1$. Then since $s_{12} = 4$, we must have $s_{23} = 3$. Also, $s_{32} = 3$ and then, since $s_{34} = 4$, we must have $s_{42} = 1$. Then also $s_{14} = 3$, and the structural matrix is $S_4{}^1$ above. Since we may interchange B_3 and B_4 the same result obviously holds if $s_{13} = 3$.

If $s_{13} = 2$, a similar argument gives the structural matrix as $S_4{}^2$.

Using this knowledge of block intersections in (16, 24, 9, 6, 3) designs, we now proceed to the construction of some non-extensible such designs.

3. Construction of Designs

We are considering designs which have at least four blocks of Type III. In view of Lemma 2 we may distinguish two cases:

(i) Some four blocks of the design have structural matrix $S_4{}^1$.

(ii) No four blocks of the design have structural matrix $S_4{}^1$.

First we consider

Case (i). Suppose B_1, B_2, B_3, B_4 have structural matrix $S_4{}^1$. We prove

Lemma 3. *If* B_1, B_2, B_3, B_4 *in a* BIBD (16, 24, 9, 6, 3) *have structural matrix* $S_4{}^1$, *then they may be represented as follows*:

$$\begin{aligned} B_1 &: \xi\ \theta_1\theta_2\theta_3a_1a_2 \\ B_2 &: \xi\ \theta_1\theta_2\theta_3b_1b_2 \\ B_3 &: \xi\ \psi_1\psi_2\psi_3b_1b_2 \\ B_4 &: \xi\ \psi_1\psi_2\psi_3a_1a_2 \end{aligned}$$

Proof. Since $s_{12} = 4$, we may write B_1 and B_2 as indicated. Since $s_{23} = 3$ and $s_{13} = 1$, b_1 and b_2 must both occur in B_3, as well as one of $\xi, \theta_1, \theta_2, \theta_3$, say ξ. For if b_1 and b_2 do not both occur in B_3 then two or more of $\xi, \theta_1, \theta_2, \theta_3$ must, implying $s_{13} > 1$.

We then denote the other three varieties in B_3 by ψ_1, ψ_2, ψ_3. By an argument similar to the above, B_4 must contain a_1, a_2 and hence, since $s_{34} = 4$, B_4 also contains $\xi, \theta_1, \theta_2, \theta_3$.

Consider blocks B_1, B_2, B_3, B_4 as in the above lemma, and consider the occurrence of the varieties $\xi, \theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3$ in the remaining 20 blocks of the design.

Noting that every variety must occur nine times and that every variety pair must occur three times, and bearing in mind the result of Lemma 1, a little reflection establishes that the above varieties must occur in one of the following three forms, which we may call *design skeletons.* (We identify arrangements which are isomorphic under any permutation of $\theta_1, \theta_2, \theta_3$ or ψ_1, ψ_2, ψ_3.)

	D_1	D_2	D_3
B_1 :	$\xi\ \theta_1\theta_2\theta_3a_1a_2$	$\xi\ \theta_1\theta_2\theta_3a_1a_2$	$\xi\ \theta_1\theta_2\theta_3a_1a_2$
B_2 :	$\xi\ \theta_1\theta_2\theta_3b_1b_2$	$\xi\ \theta_1\theta_2\theta_3b_1b_2$	$\xi\ \theta_1\theta_2\theta_3b_1b_2$
B_3 :	$\xi\ \psi_1\psi_2\psi_3b_1b_2$	$\xi\ \psi_1\psi_2\psi_3b_1b_2$	$\xi\ \psi_1\psi_2\psi_3b_1b_2$
B_4 :	$\xi\ \psi_1\psi_2\psi_3a_1a_2$	$\xi\ \psi_1\psi_2\psi_3a_1a_2$	$\xi\ \psi_1\psi_2\psi_3a_1a_2$
B_5 :	$\xi\ \theta_1\psi_1$	$\xi\ \theta_1\psi_1$	$\xi\ \theta_1\psi_1$
B_6 :	$\xi\ \theta_2\psi_2$	$\xi\ \theta_2\psi_2$	$\xi\ \theta_2\psi_2$
B_7 :	$\xi\ \theta_3\psi_3$	$\xi\ \theta_2\psi_3$	$\xi\ \theta_3\psi_3$
B_8 :	$\theta_1\theta_2\psi_1\psi_2$	$\theta_1\theta_2\psi_1\psi_2$	$\theta_1\theta_2\psi_1\psi_3$
B_9 :	$\theta_1\theta_3\psi_1\psi_3$	$\theta_1\theta_3\psi_2\psi_3$	$\theta_1\theta_3\psi_2\psi_3$
B_{10} :	$\theta_2\theta_3\psi_2\psi_3$	$\theta_2\theta_3\psi_1\psi_3$	$\theta_2\theta_3\psi_1\psi_2$
B_{11} :	ξ	ξ	ξ
B_{12} :	ξ	ξ	ξ
B_{13} :	$\theta_1\psi_2$	$\theta_1\psi_1$	$\theta_1\psi_1$
B_{14} :	$\theta_1\psi_2$	$\theta_1\psi_2$	$\theta_1\psi_2$
B_{15} :	$\theta_1\psi_3$	$\theta_1\psi_3$	$\theta_1\psi_2$
B_{16} :	$\theta_1\psi_3$	$\theta_1\psi_3$	$\theta_1\psi_3$
B_{17} :	$\theta_2\psi_1$	$\theta_2\psi_1$	$\theta_2\psi_1$
B_{18} :	$\theta_2\psi_1$	$\theta_2\psi_2$	$\theta_2\psi_2$
B_{19} :	$\theta_2\psi_3$	$\theta_2\psi_3$	$\theta_2\psi_3$
B_{20} :	$\theta_2\psi_3$	$\theta_2\psi_3$	$\theta_2\psi_3$
B_{21} :	$\theta_3\psi_1$	$\theta_3\psi_1$	$\theta_3\psi_1$
B_{22} :	$\theta_3\psi_1$	$\theta_3\psi_1$	$\theta_3\psi_1$
B_{23} :	$\theta_3\psi_2$	$\theta_3\psi_2$	$\theta_3\psi_2$
B_{24} :	$\theta_3\psi_2$	$\theta_3\psi_2$	$\theta_3\psi_3$

For each of D_1, D_2, D_3 it is possible to place a_1, a_2, b_1, b_2 in blocks B_{11} to B_{24} (in view of Lemma 1, none of a_1, a_2, b_1, b_2 can appear in B_5 ,..., B_{10}) in various ways so as to satisfy the basic BIBD requirements and also satisfy Lemma 1. Finally, once this has been accomplished, we may attempt to adjoin five new varieties s, p, q, x, y to the skeletons in order to give designs (16, 24, 9, 6, 3).

Using a trial-and-error approach six non-isomorphic (16, 24, 9, 6, 3) designs were found; two designs were derived from each of skeletons D_1, D_2 and D_3 above. These designs, labeled D_{11}, D_{12}, D_{21}, D_{22}, D_{31} and D_{32}, are given and discussed in the next section.

CASE (ii): In this case no four blocks have structural matrix S_4^1, so we consider B_1, B_2, B_3, B_4 with structural matrix S_4^2.

Using a method of construction similar to the above, two solutions of this type were found. A good deal more trial and error was involved in this case, since fewer restrictions can be set in the placement of varieties here than for case (i). The designs found are labeled E_1 and E_2, and are also given and discussed below.

No attempt was made in either case to be exhaustive, due to the prohibitive number of possibilities to be considered; it is quite possible there are more of these designs to be found.

4. Features of the Designs

The eight new designs found are given below. For ease of transcription the varieties are written as 1, 2,..., 16, as given by the mapping

$$\begin{pmatrix} \xi & \theta_1 & \theta_2 & \theta_3 & \psi_1 & \psi_2 & \psi_3 & a_1 & a_2 & b_1 & b_2 & s & p & q & x & y \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{pmatrix}$$

All of the designs are clearly non-extensible, since each contains pairs of blocks intersecting in four varieties.

If two designs are isomorphic the number of blocks of Types I, II, and III, respectively, must be the same for each design, since relabeling of varieties and blocks does not change the block intersections.

Each of these designs is then non-isomorphic to the Bhattacharya design, since the latter has only two blocks of Type III, whereas each of these designs has at least six Type III blocks.

The type of each block in the designs is given beside the block to facilitate finding whether the designs are all non-isomorphic. The only possibly isomorphic designs are those with the same block structure. It can be shown fairly easily that the pairs of designs below with the same block structure, namely D_{11} and D_{12}, D_{21} and D_{22}, are nonisomorphic, but we will not give the proofs here.

The designs are as follows:

D_{11}

1, 2, 3, 4, 8, 9	(III)	2, 4, 5, 7, 12, 14	(I)	3, 5, 8, 10, 14, 16	(III)
1, 2, 3, 4, 10, 11	(III)	3, 4, 6, 7, 13, 14	(I)	3, 5, 9, 11, 14, 16	(III)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 8, 11, 12, 15	(III)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 9, 10, 12, 15	(III)
1, 2, 5, 12, 15, 16	(I)	2, 6, 8, 10, 14, 15	(III)	4, 5, 8, 10, 13, 15	(III)
1, 3, 6, 13, 15, 16	(I)	2, 6, 9, 11, 14, 15	(III)	4, 5, 9, 11, 13, 15	(III)
1, 4, 7, 14, 15, 16	(I)	2, 7, 8, 11, 13, 16	(III)	4, 6, 8, 11, 12, 16	(III)
2, 3, 5, 6, 12, 13	(I)	2, 7, 9, 10, 13, 16	(III)	4, 6, 9, 10, 12, 16	(III)

D_{12}

1, 2, 3, 4, 8, 9	(III)	2, 4, 5, 7, 12, 14	(I)	3, 5, 9, 10, 14, 16	(III)
1, 2, 3, 4, 10, 11	(III)	3, 4, 6, 7, 13, 14	(I)	3, 5, 9, 11, 14, 15	(III)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 8, 10, 12, 16	(III)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 8, 11, 12, 15	(III)
1, 2, 5, 12, 15, 16	(I)	2, 6, 8, 10, 14, 15	(III)	4, 5, 8, 10, 13, 16	(III)
1, 3, 6, 13, 15, 16	(I)	2, 6, 8, 11, 14, 16	(III)	4, 5, 8, 11, 13, 15	(III)
1, 4, 7, 14, 15, 16	(I)	2, 7, 9, 10, 13, 15	(III)	4, 6, 9, 10, 12, 15	(III)
2, 3, 5, 6, 12, 13	(I)	2, 7, 9, 11, 13, 16	(III)	4, 6, 9, 11, 12, 16	(III)

D_{21}

1, 2, 3, 4, 8, 9	(III)	2, 4, 6, 7, 13, 14	(II)	3, 6, 8, 10, 14, 15	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 5, 7, 12, 14	(II)	3, 5, 9, 11, 14, 16	(II)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 8, 11, 13, 16	(II)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 9, 10, 12, 15	(II)
1, 2, 5, 12, 15, 16	(II)	2, 5, 8, 10, 14, 16	(II)	4, 5, 8, 10, 13, 15	(III)
1, 3, 6, 13, 15, 16	(II)	2, 6, 9, 11, 14, 15	(II)	4, 5, 9, 11, 13, 15	(III)
1, 4, 7, 14, 15, 16	(I)	2, 7, 8, 11, 12, 15	(II)	4, 6, 8, 11, 12, 16	(III)
2, 3, 5, 6, 12, 13	(I)	2, 7, 9, 10, 13, 16	(II)	4, 6, 9, 10, 12, 16	(III)

D_{22}

1, 2, 3, 4, 8, 9	(III)	2, 4, 6, 7, 12, 14	(II)	3, 5, 8, 11, 14, 15	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 5, 7, 13, 14	(II)	3, 6, 8, 10, 14, 16	(II)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 9, 10, 12, 15	(III)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 9, 11, 12, 16	(III)
1, 2, 5, 12, 15, 16	(II)	2, 5, 9, 10, 14, 16	(II)	4, 5, 8, 10, 12, 15	(II)
1, 3, 6, 13, 15, 16	(II)	2, 6, 9, 11, 14, 15	(II)	4, 5, 9, 11, 13, 16	(II)
1, 4, 7, 14, 15, 16	(I)	2, 7, 8, 10, 13, 16	(III)	4, 6, 8, 11, 12, 16	(II)
2, 3, 5, 6, 12, 13	(I)	2, 7, 8, 11, 13, 15	(III)	4, 6, 9, 10, 13, 15	(II)

D_{31}

1, 2, 3, 4, 8, 9	(III)	2, 4, 6, 7, 12, 13	(II)	3, 6, 8, 10, 14, 15	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 5, 6, 12, 14	(II)	3, 5, 9, 11, 13, 15	(II)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 8, 11, 12, 16	(III)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 9, 10, 12, 16	(III)
1, 2, 5, 12, 15, 16	(II)	2, 5, 8, 10, 13, 16	(II)	4, 7, 8, 11, 13, 15	(II)
1, 3, 6, 13, 15, 16	(II)	2, 6, 9, 11, 12, 15	(II)	4, 5, 8, 10, 12, 15	(II)
1, 4, 7, 14, 15, 16	(II)	2, 6, 8, 11, 14, 16	(II)	4, 5, 9, 11, 14, 16	(II)
2, 3, 5, 7, 13, 14	(II)	2, 7, 9, 10, 14, 15	(II)	4, 6, 9, 10, 13, 16	(II)

D_{32}

1, 2, 3, 4, 8, 9	(III)	2, 4, 6, 7, 12, 14	(II)	3, 5, 8, 10, 14, 16	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 5, 6, 13, 14	(II)	3, 6, 9, 11, 12, 16	(II)
1, 5, 6, 7, 10, 11	(III)	1, 8, 9, 12, 13, 14	(III)	3, 7, 9, 10, 12, 15	(II)
1, 5, 6, 7, 8, 9	(III)	1, 10, 11, 12, 13, 14	(III)	3, 7, 8, 11, 14, 15	(II)
1, 2, 5, 12, 15, 16	(II)	2, 5, 9, 11, 14, 15	(II)	4, 5, 9, 10, 13, 15	(II)
1, 3, 6, 13, 15, 16	(II)	2, 6, 8, 11, 13, 15	(II)	4, 5, 8, 11, 12, 16	(II)
1, 4, 7, 14, 15, 16	(II)	2, 6, 9, 10, 14, 16	(II)	4, 6, 8, 10, 12, 15	(II)
2, 3, 5, 7, 12, 13	(II)	2, 7, 8, 10, 13, 16	(II)	4, 7, 9, 11, 13, 16	(II)

E_1

1, 2, 3, 4, 8, 9	(III)	2, 4, 5, 7, 12, 14	(I)	3, 5, 8, 10, 14, 16	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 6, 7, 13, 14	(I)	3, 5, 9, 11, 14, 15	(II)
1, 5, 6, 7, 8, 10	(III)	1, 8, 11, 12, 13, 14	(III)	3, 7, 8, 11, 12, 15	(II)
1, 5, 6, 7, 9, 11	(III)	1, 9, 10, 12, 13, 14	(III)	3, 7, 9, 10, 12, 16	(II)
1, 2, 5, 12, 15, 16	(I)	2, 6, 8, 9, 14, 16	(II)	4, 5, 8, 10, 13, 15	(II)
1, 3, 6, 13, 15, 16	(I)	2, 6, 10, 11, 14, 15	(II)	4, 5, 9, 11, 13, 16	(II)
1, 4, 7, 14, 15, 16	(I)	2, 7, 8, 9, 13, 15	(II)	4, 6, 8, 11, 12, 16	(II)
2, 3, 5, 6, 12, 13	(I)	2, 7, 10, 11, 13, 16	(II)	4, 6, 9, 10, 12, 15	(II)

E_2

1, 2, 3, 4, 8, 9	(III)	2, 4, 6, 7, 12, 14	(II)	3, 5, 9, 11, 14, 16	(II)
1, 2, 3, 4, 10, 11	(III)	3, 4, 5, 7, 13, 14	(II)	3, 6, 8, 10, 14, 15	(II)
1, 5, 6, 7, 8, 10	(III)	1, 8, 11, 12, 13, 14	(III)	3, 7, 8, 11, 12, 16	(II)
1, 5, 6, 7, 9, 11	(III)	1, 9, 10, 12, 13, 14	(III)	3, 7, 9, 10, 12, 15	(II)
1, 2, 5, 12, 15, 16	(II)	2, 5, 8, 10, 14, 16	(II)	4, 5, 8, 9, 13, 15	(II)
1, 3, 6, 13, 15, 16	(II)	2, 6, 9, 11, 14, 15	(II)	4, 5, 10, 11, 12, 15	(II)
1, 4, 7, 14, 15, 16	(I)	2, 7, 8, 11, 13, 15	(II)	4, 6, 8, 9, 12, 16	(II)
2, 3, 5, 6, 12, 13	(I)	2, 7, 9, 10, 13, 16	(II)	4, 6, 10, 11, 13, 16	(II)

5. Summary and Comments

Eight new solutions for the quasi-residual BIBD (16, 24, 9, 6, 3) have been given. These are all non-extensible, and along with a solution for the same design due to Bhattacharya, are the only known examples of non-extensible quasi-residual BIBD's with $\lambda = 3$. It is also believed they are the only known non-extensible solutions with $\lambda < 10$.

It would be of interest if other such designs could be found, either for $\lambda = 3$, $k > 6$, or for $\lambda > 3$. Other interesting problems concerning these designs arise, and these will be discussed for the case $\lambda = 3$ in [6].

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