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Separation for the biharmonic differential operator in the Hilbert space associated with the existence and uniqueness theorem

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Abstract

In this paper, we have studied the separation for the following biharmonic differential operator:

$$Au = \Delta\Delta u + V(x)u(x), \quad x \in R^n,$$

in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space H_1 and $\Delta\Delta u$ is the biharmonic differential operator, while $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator in R^n. Moreover, we have studied the existence and uniqueness of the solution of the biharmonic differential equation

$$Au = \Delta\Delta u + V(x)u(x) = f(x)$$

in the Hilbert space H, where $f(x) \in H$.

Keywords: Separation; Biharmonic differential operator; Operator potential; Hilbert space $H = L_2(R^n, H_1)$; Coercive estimate

1. Introduction

The concept of separation for differential operator was first introduced by Everitt and Giertz [6,7]. They have obtained the separation results for the Stürm–Liouville differential operator

$$Ay(x) = -y''(x) + V(x)y(x), \quad x \in R, \tag{1}$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $y(x) \in L_2(R)$ and $Ay(x) \in L_2(R)$ imply both of $y''(x)$ and $V(x)y(x) \in L_2(R)$. More fundamental results of separation of differential operator were obtained by Everitt and Giertz [8,9]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [2], Otelbaev [16], Zettle [20] and Mohamed et al. [10–15]. The separation for the differential operators with the matrix potentials was first studied by

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Bergbaev [1]. Brown [3] has shown that certain properties of positive solutions of disconjugate second-order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second-order differential operators have been studied by Brown et al. [4,5]. Mohamed et al. [13] have studied the separation property of the Stürm–Liouville differential operator

$$Ay(x) = -\big(\mu(x)y'\big)' + V(x)y(x), \quad x \in R, \tag{2}$$

in the Hilbert space $H_p(R)$ $(p = 1, 2)$, where $V \in L(l_p)$ is an operator potential which is a bounded linear operator on l_p and $\mu(x) \in C^1(R)$ is a positive continuous function on R.

Mohamed et al. [11] have studied the separation property for the linear differential operator

$$Ay(x) = (-1)^m D^{2m} y(x) + V(x)y(x), \quad x \in R, \tag{3}$$

in the Banach space $L_p(R)^l$, where $V(x)$ is an $l \times l$ positive hermitian matrix, $D^{2m} = \frac{d^{2m}}{dx^{2m}}$ is the classical differentiation of order $2m$.

Mohamed et al. [14] have studied the separation of the Schrödinger operator

$$Ay(x) = -\Delta y(x) + V(x)y(x), \quad x \in R^n, \tag{4}$$

with the operator potential $V(x) \in C^1(R^n, L(H_1))$ in the Hilbert space $L_2(R^n, H_1)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n.

Mohamed et al. [15] have studied the separation for the general second-order differential operator

$$Ay(x) = -\sum_{i,j=1}^{n} a_{ij}(x) D_i^j y(x) + V(x)y(x), \quad x \in R^n, \tag{5}$$

in the weighted Hilbert space $L_{2,k}(R^n, H_1)$ with the operator potential $V(x)$, where $a_{ij} \in C^2(R^n)$ and $D_i^j = \frac{\partial^2}{\partial x_i \partial x_j}$.

Zayed et al. [17] have obtained recent results on the separation of linear and nonlinear Schrödinger-type operators with operator potentials in Banach spaces. Furthermore, Zayed et al. [18] have studied the separation of the elliptic differential operator

$$Ay(x) = -\sum_{i,j=1}^{n} \big[D_i\big(P_{ij}(x)D_j y(x)\big) - P_{ij}(x)b_i(x)b_j(x)y(x)\big] + V(x)y(x), \tag{6}$$

in the weighted Hilbert space $L_{2,k}(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$, where $P_{ij}(x)$ and $b_i(x)$ are real-valued continuous functions while $D_i = \frac{\partial}{\partial x_i}$.

Recently, Zayed et al. [19] have studied the separation for the Laplace Beltrami differential operator in Hilbert spaces and obtained recent results on it.

The main objective of the present paper is to study the separation for the following biharmonic differential operator:

$$Au = \Delta\Delta u + V(x)u(x), \quad x \in R^n, \tag{7}$$

in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$ and $\Delta\Delta u$ is the biharmonic differential operator, while $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator in R^n. We derive also the coercive estimate for the operator (7). The existence and uniqueness of the solution of the biharmonic differential equation $Au = \Delta\Delta u + V(x)u(x) = f(x)$ in H is given.

2. Some notations

In this section we introduce the definitions that will be used in the subsequent section.

Let H_1 be a separable Hilbert space with the norm $\|.\|_1$ and the scalar product $\langle .,. \rangle_1$. We introduce the Hilbert space $H = L_2(R^n, H_1)$ of all vector functions $u(x)$, $x \in R^n$ equipped with the norm

$$\|u\|^2 = \int_{R^n} \big\|u(x)\big\|_1^2 \, dx. \tag{8}$$

The symbol $\langle u, v\rangle$, where $u, v \in H$ denotes the scalar product in the Hilbert space H which is defined by

$$\langle u, v\rangle = \int_{R^n} \langle u, v\rangle_1 \, dx. \tag{9}$$

The space of all vector functions $u(x)$, $x \in R^n$ that have generalized derivatives $D^\alpha u(x)$, $\alpha \leqslant 2$ such that $u(x)$ and $D^\alpha u(x)$ belong to H is denoted by $W_2^2(R^n, H_1)$.

We say that the function $u(x) \in W_{2,\mathrm{loc}}^2(R^n, H_1)$ if for all functions $Q(x) \in C_0^\infty(R^n)$, the vector functions $Q(x)u(x) \in W_2^2(R^n, H_1)$.

3. The main results

Definition 1. The biharmonic differential operator A of the form $Au = \Delta\Delta u(x) + V(x)u(x)$, $x \in R^n$ is said to be separated in the Hilbert space H if the following statement holds: If $u(x) \in H \cap W_{2,\mathrm{loc}}^2(R^n, H_1)$ and $Au(x) \in H$ imply both of $\Delta\Delta u(x)$ and $V(x)u(x) \in H$.

The main results in this paper have been formulated as follows:

Theorem 1. *If the following conditions are satisfied for all $x \in R^n$:*

$$\left\| V_0^{-1/2}\left(\frac{\partial^2 V}{\partial x_i^2}\right) V^{-1} V u \right\| \leqslant \sigma_1 \|Vu\| \tag{10}$$

and

$$\left\| V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\| \leqslant \sigma_2 \|Vu\|, \tag{11}$$

where σ_1 and σ_2 are positive constants satisfying $\sigma_1 + 2\sigma_2 < \frac{2}{n}$ while $V_0 = \operatorname{Re} V$, then the coercive estimate

$$\|Vu\| + \|\Delta\Delta u\| + \left\| \sum_{i=1}^n V_0^{1/2}\left(\frac{\partial^2 u}{\partial x_i^2}\right) \right\| \leqslant N\|Au\| \tag{12}$$

is valid, where

$$N = 1 + 2\left[1 - \frac{n}{2\beta}(\sigma_1 + 2\sigma_2)\right]^{-1} + \left[1 - \frac{n\beta}{2}(\sigma_1 + 2\sigma_2)\right]^{-1/2}\left[1 - \frac{n}{2\beta}(\sigma_1 + 2\sigma_2)\right]^{-1/2} \tag{13}$$

is a constant independent on u(x) while β is given by

$$\frac{n}{2}(\sigma_1 + 2\sigma_2) < \beta < \frac{2}{n(\sigma_1 + 2\sigma_2)}. \tag{14}$$

That is, the biharmonic differential operator A given by (7) *is separated in the Hilbert space H.*

Proof. From the definition of the scalar product in H and by integrating by parts, we obtain

$$\left\langle \frac{\partial u}{\partial x_i}, v \right\rangle = -\left\langle u, \frac{\partial v}{\partial x_i} \right\rangle \quad \text{for all } u, v \in C_0^\infty\left(R^n\right)$$

and, consequently, we get

$$\langle Au, Vu\rangle = \langle \Delta\Delta u + Vu, Vu\rangle = \langle \Delta\Delta u, Vu\rangle + \langle Vu, Vu\rangle.$$

On setting $\Delta u = W(x)$, we have

$$\begin{aligned}\langle Au, Vu\rangle &= \langle \Delta W, Vu\rangle + \langle Vu, Vu\rangle \\ &= \left\langle \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_i^2}, Vu \right\rangle + \langle Vu, Vu\rangle \\ &= -\sum_{i=1}^{n} \left\langle \frac{\partial W}{\partial x_i}, \frac{\partial (Vu)}{\partial x_i} \right\rangle + \langle Vu, Vu\rangle \\ &= \sum_{i=1}^{n} \left\langle W, \frac{\partial}{\partial x_i}\left(V \frac{\partial u}{\partial x_i}\right)\right\rangle + \sum_{i=1}^{n} \left\langle W, \frac{\partial}{\partial x_i}\left(u \frac{\partial V}{\partial x_i}\right)\right\rangle + \langle Vu, Vu\rangle \\ &= \left\langle \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V \frac{\partial^2 u}{\partial x_i^2} \right\rangle + 2\left\langle \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle + \left\langle \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} u \frac{\partial^2 V}{\partial x_i^2} \right\rangle + \langle Vu, Vu\rangle. \end{aligned} \tag{15}$$

Equating the real parts of both sides of (15), we obtain

$$\begin{aligned}\operatorname{Re}\langle Au, Vu\rangle &= \left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_i^2} \right\rangle + 2\operatorname{Re}\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle \\ &\quad + \operatorname{Re}\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \left(\frac{\partial^2 V}{\partial x_i^2}\right) V^{-1} Vu \right\rangle + \langle Vu, Vu\rangle. \end{aligned} \tag{16}$$

Since for complex number Z, we have

$$-|Z| \leqslant \operatorname{Re} Z \leqslant |Z|, \tag{17}$$

then on using the Cauchy–Schwartz inequality, we get

$$\operatorname{Re}\langle Au, Vu\rangle \leqslant \big|\langle Au, Vu\rangle\big| \leqslant \|Au\|\|Vu\|. \tag{18}$$

Consequently, we deduce from (11), (17) and (18) that

$$\begin{aligned}\operatorname{Re}\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle &\geqslant -\left|\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle\right| \\ &\geqslant -\left\| \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2} \right\| \left\| \sum_{i=1}^{n} V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\| \\ &\geqslant -n\sigma_2 \left\| \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2} \right\| \|Vu\|. \end{aligned} \tag{19}$$

It is well known [17] that for any $\beta > 0$ and for any $y_1, y_2 \in R^n$, we have

$$|y_1||y_2| \leqslant \frac{\beta}{2}|y_1|^2 + \frac{1}{2\beta}|y_2|^2. \tag{20}$$

From (19) and (20) we find that

$$\operatorname{Re}\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i} \right\rangle \geqslant -\frac{n\sigma_2\beta}{2} \left\| \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2} \right\|^2 - \frac{n\sigma_2}{2\beta}\|Vu\|^2. \tag{21}$$

Similarly, with the aid of (10), (17), (18) and (20) we can show that

$$\operatorname{Re}\left\langle \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} V_0^{-1/2} \left(\frac{\partial^2 V}{\partial x_i^2}\right) V^{-1} Vu \right\rangle \geqslant -\frac{n\sigma_1\beta}{2} \left\| \sum_{k=1}^{n} V_0^{1/2} \frac{\partial^2 u}{\partial x_k^2} \right\|^2 - \frac{n\sigma_1}{2\beta}\|Vu\|^2. \tag{22}$$

From (16), (18), (21) and (22) we deduce that

$$\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2)\right]\left\|\sum_{k=1}^{n}V_0^{1/2}\frac{\partial^2 u}{\partial x_k^2}\right\|^2+\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]\|Vu\|^2\leqslant\|Au\|\|Vu\|. \tag{23}$$

Choosing $\frac{n}{2}(\sigma_1+2\sigma_2)<\beta<\frac{2}{n(\sigma_1+2\sigma_2)}$, we deduce from (23) that

$$\|Vu\|\leqslant\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]^{-1}\|Au\| \tag{24}$$

and

$$\left\|\sum_{k=1}^{n}V_0^{1/2}\frac{\partial^2 u}{\partial x_k^2}\right\|\leqslant\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2)\right]^{-1/2}\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]^{-1/2}\|Au\|. \tag{25}$$

Since $Au=\Delta\Delta u(x)+V(x)u(x)$, then we get

$$\|\Delta\Delta u\|\leqslant\|Au\|+\|Vu\|\leqslant\left\{1+\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]^{-1}\right\}\|Au\|. \tag{26}$$

From (24)–(26) we have the coercive estimate

$$\|Vu\|+\|\Delta\Delta u\|+\left\|\sum_{k=1}^{n}V_0^{1/2}\frac{\partial^2 u}{\partial x_k^2}\right\|\leqslant N\|Au\|, \tag{27}$$

where

$$N=1+2\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]^{-1}+\left[1-\frac{n\beta}{2}(\sigma_1+2\sigma_2)\right]^{-1/2}\left[1-\frac{n}{2\beta}(\sigma_1+2\sigma_2)\right]^{-1/2} \tag{28}$$

is a constant independent on $u(x)$.

That is the biharmonic differential operator A given by (7) is separated in the Hilbert space H. Hence the proof of Theorem 1 is completed. □

Theorem 2. *If the biharmonic differential operator A given by* (7) *is separated in the Hilbert space H and if there are positive functions $t(x)\in C^1(R^n)$, $\psi(x)\in C^1(R^n)$ such that*

$$\left\|t^{-1}(x)\left(\frac{\partial t(x)}{\partial x_i}\right)V_0^{-1/2}\right\|\leqslant 2\sqrt{\rho_1}, \tag{29}$$

$$\left\|\psi^{-1}(x)\left(\frac{\partial \psi(x)}{\partial x_i}\right)V_0^{-1/2}\right\|\leqslant 2\sqrt{\rho_2}, \tag{30}$$

$$\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial u}{\partial x_i}\right)\right\|\leqslant 2\sqrt{\rho_3}\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|, \tag{31}$$

where $0<\rho_1+\rho_2+\rho_3<\frac{\beta}{2n}$, where β is defined by (20) *and $V_0=\operatorname{Re}V$. Then the biharmonic differential equation*

$$Au=\Delta\Delta u+V(x)u(x)=f(x), \tag{32}$$

where $f(x)\in H$ has a unique solution in the Hilbert space H.

Proof. First, we prove that the homogeneous biharmonic differential equation

$$Au=\Delta\Delta u+V(x)u(x)=0 \tag{33}$$

has only the zero solution $u(x)=0$ for all $x\in R^n$. To this end we assume that $t(x)$ and $\psi(x)$ are positive functions belonging to $C^1(R^n)$. Then on setting $\Delta u(x)=W(x)$, we have

$$\begin{aligned}\langle Vu, t\psi u\rangle &= \langle -\Delta\Delta u, t\psi u\rangle \\ &= -\sum_{i=1}^{n}\left\langle \frac{\partial^2 W}{\partial x_i^2}, t\psi u\right\rangle \\ &= \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \frac{\partial}{\partial x_i}(t\psi u)\right\rangle\end{aligned}$$

and, consequently, we get

$$\langle Vu, t\psi u\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial W}{\partial x_i}, \psi u\frac{\partial t}{\partial x_i}\right\rangle. \tag{34}$$

Equating the real parts of both sides of (34), we get

$$\begin{aligned}\langle V_0 u, t\psi u\rangle &= \left\langle t^{1/2}\psi^{1/2}V_0^{1/2}u, t^{1/2}\psi^{1/2}V_0^{1/2}u\right\rangle \\ &= \sum_{i=1}^{n}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle + \sum_{i=1}^{n}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, \psi u\frac{\partial t}{\partial x_i}\right\rangle.\end{aligned} \tag{35}$$

On the other hand, we find that

$$\begin{aligned}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, \psi u\frac{\partial t}{\partial x_i}\right\rangle &= \mathrm{Re}\left\langle t^{1/2}\psi^{1/2}\frac{\partial W}{\partial x_i}, t^{1/2}\psi^{1/2}\left[t^{-1}\left(\frac{\partial t}{\partial x_i}\right)V_0^{-1/2}\right]V_0^{1/2}u\right\rangle \\ &\leqslant \left\|t^{1/2}\psi^{1/2}\frac{\partial W}{\partial x_i}\right\|\left\|t^{1/2}\psi^{1/2}\left[t^{-1}\left(\frac{\partial t}{\partial x_i}\right)V_0^{-1/2}\right]V_0^{1/2}u\right\|,\end{aligned} \tag{36}$$

$$\begin{aligned}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle &= \mathrm{Re}\left\langle t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right), t^{1/2}\psi^{1/2}\left[\psi^{-1}\left(\frac{\partial \psi}{\partial x_i}\right)V_0^{-1/2}\right]V_0^{1/2}u\right\rangle \\ &\leqslant \left\|t^{1/2}\psi^{1/2}\frac{\partial W}{\partial x_i}\right\|\left\|t^{1/2}\psi^{1/2}\left[\psi^{-1}\left(\frac{\partial \psi}{\partial x_i}\right)V_0^{-1/2}\right]V_0^{1/2}u\right\|\end{aligned} \tag{37}$$

and

$$\begin{aligned}\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle &= \mathrm{Re}\left\langle t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right), t^{1/2}\psi^{1/2}\left(\frac{\partial u}{\partial x_i}\right)\right\rangle \\ &\leqslant \left\|t^{1/2}\psi^{1/2}\frac{\partial W}{\partial x_i}\right\|\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial u}{\partial x_i}\right)\right\|.\end{aligned} \tag{38}$$

With the aid of (20) and (29)–(31) the inequalities (36)–(38) take the forms:

$$\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, t\psi\frac{\partial u}{\partial x_i}\right\rangle \leqslant \frac{\beta}{2}\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right)\right\|^2 + \frac{2}{\beta}\rho_3\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|^2, \tag{39}$$

$$\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, tu\frac{\partial \psi}{\partial x_i}\right\rangle \leqslant \frac{\beta}{2}\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right)\right\|^2 + \frac{2}{\beta}\rho_2\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|^2 \tag{40}$$

and

$$\mathrm{Re}\left\langle \frac{\partial W}{\partial x_i}, \psi u\frac{\partial t}{\partial x_i}\right\rangle \leqslant \frac{\beta}{2}\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right)\right\|^2 + \frac{2}{\beta}\rho_1\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|^2. \tag{41}$$

From (35) and (39)–(41) we have the following inequality:

$$\left[1 - \frac{2n}{\beta}(\rho_1+\rho_2+\rho_3)\right]\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|^2 \leqslant \frac{3\beta}{2}\sum_{i=1}^{n}\left\|t^{1/2}\psi^{1/2}\left(\frac{\partial W}{\partial x_i}\right)\right\|^2. \tag{42}$$

By choosing $W(x)$ to be a constant for all $x \in R^n$, then if $\frac{2n}{\beta}(\rho_1+\rho_2+\rho_3) < 1$ we have

$$0 < \left[1 - \frac{2n}{\beta}(\rho_1+\rho_2+\rho_3)\right]\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|^2 \leqslant 0. \tag{43}$$

From (8) and (43) we obtain

$$0 < \left[1 - \frac{2n}{\beta}(\rho_1+\rho_2+\rho_3)\right]\int_{R^n}\left\|t^{1/2}\psi^{1/2}V_0^{1/2}u\right\|_1^2\,dx \leqslant 0. \tag{44}$$

Now, the inequality (44) holds only for $u(x) \equiv 0$. This proves that $u(x) = 0$ is the only solution of Eq. (33).

Furthermore, it is easy to check that the linear manifold $N = \{f\colon\ Au = f \text{ for all } u \in C_0^\infty(R^n)\}$ is dense everywhere in H. So, we can construct the sequence of vector functions $\{y_r\} \in C_0^\infty(R^n)$, where $\|y_r\| \neq 0$ for all r such that

$$\|Ay_r - f\| \to 0 \quad \text{as } r \to \infty \text{ for all } f \in H.$$

On using the coercive estimate (12), we find that

$$\left\|V(y_p - y_r)\right\| + \left\|\Delta\Delta(y_p - y_r)\right\| + \left\|\sum_{i=1}^n V_0^{1/2}\frac{\partial^2}{\partial x_i^2}(y_p - y_r)\right\| \leqslant N\left\|A(y_p - y_r)\right\|,$$

where $u = y_p - y_r$, $p, r = 1, 2, \ldots.$

As $p, r \to \infty$ we get the fundamental sequences $\{Vy_r\}$, $\{\Delta\Delta y_r\}$, $\{\sum_{i=1}^n V_0^{1/2}\frac{\partial^2}{\partial x_i^2}y_r\}$ in H. Then there exist vector functions μ_0, μ_1, μ_2 in H such that $\|Vy_r - \mu_0\|$, $\|\Delta\Delta y_r - \mu_1\|$ and $\|\sum_{i=1}^n V_0^{1/2}\frac{\partial^2 y_r}{\partial x_i^2} - \mu_2\| \to 0$ as $r \to \infty$.

Hence the sequences $\{Vy_r\}$, $\{\Delta\Delta y_r\}$ and $\{\sum_{i=1}^n V_0^{1/2}\frac{\partial^2 y_r}{\partial x_i^2}\}$ are bounded in H. This implies that as $r \to \infty$,

$$y_r \to V^{-1}\mu_0 = y, \qquad \Delta\Delta y_r \to \Delta\Delta y, \qquad \sum_{i=1}^n V_0^{1/2}\frac{\partial^2 y_r}{\partial x_i^2} \to \sum_{i=1}^n V_0^{1/2}\frac{\partial^2 y}{\partial x_i^2}.$$

Hence for a given $f \in H$ there exists $y \in H \cap W_{2,\text{loc}}^2(R^n, H_1)$ such that $Ay = f$.

Suppose that $\tilde{y}$ is another solution of the equation $Au = f$, then $A(y - \tilde{y}) = 0$ but $Au = 0$ has only the zero solution, then $y = \tilde{y}$ and the uniqueness is proved. Hence the proof of Theorem 2 is completed. □

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