Interval systems of max-separable linear equations

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Received 22 February 2001; accepted 29 June 2001
Submitted by R.A. Brualdi

Abstract

An interval system of equations is weakly solvable if at least one of its subsystems is solvable, and it is strongly solvable if all its subsystems are solvable. We give necessary and sufficient conditions enabling efficient testing of weak and strong solvability over max-plus and max-min algebras. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 15A06; 90C48

Keywords: Max-separable linear equations; Interval systems; Weak solvability; Strong solvability

1. Introduction

Systems of max-separable linear equations arise in several branches of applied mathematics: for example max-plus algebra in the description of discrete-event dynamic systems [1,4] and max-min algebra in modelling fuzzy relations [7]. However, it is often unrealistic to expect that the entries in the coefficient-matrix and in the right-hand side could be estimated precisely. Choosing unsuitable values may lead to unsolvable systems, so methods for restoring solvability by modifying the input

* This work was supported by the Slovak Agency for Science, contract #1/7465/20 “Combinatorial Structures and Complexity of Algorithms”.
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data (see [2,5,6]) or by dropping some equations have been studied [3]. Another possibility is to replace each entry by an interval of possible values. Then we can ask about weak solvability of an interval system (whether the system is solvable for at least one choice of data in the prescribed intervals) or we can study its strong solvability, meaning that each system with data in the prescribed intervals is solvable.

The importance of interval computations in classical algebraic structures has been appreciated for a relatively long time, see e.g., the overview in the monograph [8]; however, the authors are not aware of any works concerning interval computations in extremal algebras. For interval linear systems over the field of real numbers it has been proved that checking both their weak as well as strong solvability is NP-hard [9,12]. In this paper we show that in the max-min and max-plus cases both weak and strong solvability can be tested efficiently.

2. Preliminaries

In what follows \( B = (\oplus, \otimes) \) may, unless stipulated otherwise, be taken at will to be either of two structures in which \( a \oplus b = \max\{a, b\} \): the max-min algebra in which \( B \) is the real unit interval \([0, 1]\) and \( a \otimes b = \min\{a, b\} \), and the max-plus algebra in which \( B \) is the additive group of reals and \( a \otimes b = a + b \).

The set of all \( m \times n \) matrices over \( B \) is denoted by \( B(m,n) \) and the set of all column \( m \)-vectors over \( B \) by \( B(m) \). The operations \( \otimes, \oplus \) are used for matrix multiplication formally in the same way as in the classical algebra. Let us realise that \( \oplus \) and \( \otimes \) are isotone and continuous in both algebras, which will be often used in this paper.

For a given matrix interval \( A = \langle A, \overline{A} \rangle \) with \( A, \overline{A} \in B(m,n), A \leq \overline{A} \) and a given vector interval \( b = \langle b, \overline{b} \rangle \) with \( b, \overline{b} \in B(m), b \leq \overline{b} \) the notation

\[
A \otimes x = b
\]

represents an interval system of linear max-separable equations, i.e., the family of all systems of equations of the form

\[
A \otimes x = b
\]

such that \( A \in A, b \in b \), inequalities are meant elementwise. Each system of the form (2) is said to be a subsystem of system (1) if \( A \in A \) and \( b \in b \). A subsystem is called extremal if each of its equations is either of the form \((A \otimes x)_i = \overline{b}_i\) (lower-upper, LU-equation) or \((\overline{A} \otimes x)_i = b_i\) (upper-lower, UL-equation). We say that interval system (1) has a constant matrix if \( A = \overline{A} \) and it has a constant right-hand side if \( b = \overline{b} \).

**Definition 1.** We say that system (1) is weakly solvable if at least one of its subsystems is solvable and we call system (1) strongly solvable if all its subsystems are solvable.
Definition 2. We say that a vector \( y \in \mathcal{B}(n) \) is a possible solution of system (1) if there exist \( A \in \mathcal{A} \) and \( b \in \mathcal{b} \) such that \( A \otimes y = b \).

The following theorem has its analogy in the real case for nonnegative solutions as the Oettli–Prager theorem, see [10].

Theorem 1. A vector \( y \in \mathcal{B}(n) \) is a possible solution of system (1) if and only if \( A \otimes y \leq b \) and \( A \otimes y \geq b \).

Proof. Let us consider the product \((A \otimes y)_i\) as a function (determined by \( y \)) of \( n \) variables \( a_{i1}, a_{i2}, \ldots, a_{in} \). This is an isotone continuous function and so the image of the \( n \)-dimensional interval \((a_{i1}, \bar{a}_{i1}) \times (a_{i2}, \bar{a}_{i2}) \times \cdots \times (a_{in}, \bar{a}_{in})\) is the interval \( I_i = ((A \otimes y)_i, (\bar{A} \otimes y)_i) \) on the real line. So \( y \) is a possible solution if and only if for each \( i \) the intervals \( I_i \) and \( b_i = (b_i, \bar{b}_i) \) intersect (it suffices to take for the \( i \)th right-hand side an arbitrary \( b_i \in I_i \cap b_i \) and for the coefficients of the \( i \)th equation its preimage), which is equivalent to the condition stated in the theorem. \( \square \)

For the study of systems of max-separable linear equations it is necessary to realise that max-min as well as max-plus are examples of algebras in which left-multiplications \( m_a : x \rightarrow a \otimes x \) are residuated, which implies that each inequality \( a \otimes x \leq b \) has a maximum solution \( x^*(a, b) \). Max-separable linear functions are then also residuated, and a system of the form (2) is solvable if and only if the vector \( x^*(A, b) \), called the principal solution and defined by

\[
x_j^*(A, b) = \min_i \{b_i; a_{ij} > b_i\}
\] (3)

(where by definition \( \min \emptyset = 1 \)) for the max-min case and

\[
x_j^*(A, b) = \min_i \{b_i - a_{ij}\}
\] (4)

for the max-plus case, is a solution (for the details see e.g. [4,5]). Inequality \( A \otimes x^*(A, b) \leq b \) holds always and residuation theory implies the following assertion, which is a reformulation of Corollary 2 of [5].

Theorem 2. Let \( A \in \mathcal{B}(m, n), b \in \mathcal{B}(m) \) be given. Let us denote \( d^* = A \otimes x^*(A, b) \). Then \( d^* \leq b \) and for each \( d \leq b \) solvability of \( A \otimes x = d \) implies \( d \leq d^* \).

In what follows we shall use the following easily proved property of the principal solution.

Lemma 1. Let \( A \in \mathcal{B}(m, n), b, d \in \mathcal{B}(m) \) be such that \( b \leq d \). Then \( x^*(A, b) \leq x^*(A, d) \).

For systems with nonconstant matrix over the max-min algebra we shall use the following terminology taken from [2].
Definition 3. We say that a matrix $D \in \mathcal{B}(m, n)$ is closer than a matrix $A \in \mathcal{B}(m, n)$ to a vector $b \in \mathcal{B}(m)$ if

$$a_{ij} \geq d_{ij} \geq b_i \quad \text{or} \quad a_{ij} \leq d_{ij} \leq b_i$$

holds for all indices $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We write $A \vdash D \dashv b$.

In [2] the following result was proved:

Lemma 2. Let $\mathcal{B}$ be the max-min algebra, $A, D \in \mathcal{B}(m, n)$, $b \in \mathcal{B}(m)$ and $A \vdash D \dashv b$. If $A \otimes x = b$ is solvable, then $D \otimes x = b$ is solvable too.

3. Weak solvability

Theorem 3. An interval system (1) with a constant matrix $A = \underline{A} = \overline{A}$ is weakly solvable if and only if

$$A \otimes x^*(A, \overline{b}) \geq b. \quad (5)$$

Proof. If inequality (5) holds, then a right-hand side leading to a solvable subsystem is equal to $A \otimes x^*(A, \overline{b})$, since $A \otimes x^*(A, \overline{b}) \leq \overline{b}$ due to Theorem 2.

Conversely, let $A \otimes x = b$ be a solvable subsystem of (1) for some $b \in [b, \overline{b})$. Then Lemma 1 implies

$$A \otimes x^*(A, \overline{b}) \geq A \otimes x^*(A, b) = b \geq b.$$ 

□

A condition for weak solvability of systems with a nonconstant matrix and a constant right-hand side can be stated formally and identically for both max-min and max-plus, using a ‘canonical’ matrix of an interval system. However, the definitions of canonical matrices are different in the two cases.

Definition 4. Let $\mathcal{B}$ be the max-min algebra, let $A, \underline{A} \in \mathcal{B}(m, n)$, $A \leq \underline{A}$ and a constant right-hand side $b \in \mathcal{B}(m)$ be given. We call the matrix $A(b)$ the canonical matrix of system $A \otimes x = b$ if

$$a_{ij}(b) = \begin{cases} b_i & \text{if } a_{ij} \leq b_i \leq \overline{a}_{ij}, \\ \underline{a}_{ij} & \text{if } b_i < \underline{a}_{ij}, \\ \overline{a}_{ij} & \text{if } b_i > \overline{a}_{ij}. \end{cases}$$

Lemma 3. Let $A \otimes x = b$ be any interval system over the max-min algebra with a constant right-hand side $b$. Then $x^*(A(b), b) \geq x^*(A, b)$ for any subsystem $A \otimes x = b$. 
Proof. Since the canonical matrix fulfills \( a_{ij}(b) > b_i \) only if \( a_{ij} > b_i \), we have for each \( j \):
\[
\{ i; \ a_{ij}(b) > b_i \} \subseteq \{ i; \ a_{ij} > b_i \}
\]
and so
\[
\min\{b_i; \ a_{ij}(b) > b_i\} \geq \min\{b_i; \ a_{ij} > b_i\}. \quad \square
\]

Definition 5. Let \( \mathcal{B} \) be the max-plus algebra, let \( A, \bar{A} \in \mathcal{B}(m, n) \), \( A \leq \bar{A} \), and \( b \in \mathcal{B}(m) \) be given. The canonical matrix \( A(b) \) of system \( A \otimes x = b \) is defined by the following algorithm:

Algorithm CanonicalMatrix:

for each \( j \) do
begin
\( a_j(b) = \max_i\{a_{ij} - b_i\} \);
for each \( i \) do if \( \bar{a}_{ij} \geq a_j(b) + b_i \) then
\( a_{ij}(b) = a_j(b) + b_i \)
else \( a_{ij}(b) = \bar{a}_{ij} \);
end

Lemma 4. Let \( A \otimes x = b \) be any interval system over max-plus algebra with a constant right-hand side. Then
(a) \( \max_i\{a_{ij}(b) - b_i\} = a_j(b) \) for each \( j \),
(b) \( x_j^*(A(b), b) = -a_j(b) \) for each \( j \),
(c) for any subsystem \( A \otimes x = b \) we have \( x^*(A, b) \leq x^*(A(b), b) \),
(d) if there exists a solvable subsystem, then \( A(b) \otimes x = b \) is solvable too.

Proof. (a) Due to Line 2 of the Algorithm,
\[
a_j(b) = \max_i\{a_{ij} - b_i\} \leq \max_i\{\bar{a}_{ij} - b_i\}. \tag{6}
\]
Suppose we have inequality \( a_j(b) > a_{ij}(b) - b_i \), or equivalently \( a_{ij}(b) < a_j(b) + b_i \) for all \( i \). This is possible only if \( a_{ij}(b) = \bar{a}_{ij} < a_j(b) + b_i \) for all \( i \). Thus \( a_j(b) > \bar{a}_{ij} - b_i \) for all \( i \), which is impossible due to (6).

On the other hand, if for some \( i \) inequality \( a_j(b) < a_{ij}(b) - b_i \) holds, then \( a_{ij}(b) = \bar{a}_{ij} > a_j(b) + b_i \). This means that in this case the condition from Line 3 of the Algorithm holds, implying \( a_{ij}(b) = a_j(b) + b_i \), which contradicts the assumption.

(b) According to (4), the principal solution of the system \( A(b) \otimes x = b \) is
\[
x_j^*(A(b), b) = \min_i\{b_i - a_{ij}(b)\} = -\max_i\{a_{ij}(b) - b_i\} = -a_j(b).
\]
(c) We have
\[
x_j^*(A, b) = \min_i\{b_i - a_{ij}\} = -\max_i\{a_{ij} - b_i\}
\leq -\max_i\{a_{ij} - b_i\} = -a_j(b)
\]
and (b) implies the desired inequality.

(d) Properties of the principal solution imply \( A(b) \otimes x^*(A(b), b) \leq b \). Let us now take a solvable subsystem \( A \otimes x = b \) and suppose that in row \( k \) equality has been achieved in the term \( a_{kj} \otimes x^*_j(A, b) = b_k \). Then

\[
\begin{align*}
a_{kj} + \min_i \{b_i - a_{ij}\} &= b_k, \\
\min_i \{b_i - a_{ij}\} &= b_k - a_{kj},
\end{align*}
\]

which implies

\[
\begin{align*}
aj(b) &= \max_i \{a_{ij} - b_i\} \leq \max_i \{a_{ij} - b_i\} = -\min_i \{b_i - a_{ij}\} = a_{kj} - b_k.
\end{align*}
\]

But in this case \( a_{kj} - b_k \geq a_{kj} - b_k \geq a_j(b) \). Hence for \( j, k \) the condition in Line 2 of algorithm Canonical Matrix is fulfilled and so \( a_{kj}(b) = a_j(b) + b_k \). Then in system \( A(b) \otimes x = b \) in row \( k \) in position \( j \) we have

\[
\begin{align*}
a_{kj}(b) + x^*_j(A(b), b) &= (a_j(b) + b_k) + (-a_j(b)) = b_k,
\end{align*}
\]

which means that the canonical system is solvable too. □

**Theorem 4.** An interval system (1) with a constant right-hand side \( b = \underline{b} = \overline{b} \) is weakly solvable if and only if its subsystem \( A(b) \otimes x = b \) is solvable.

**Proof.** The ‘if’ implication is trivial, since for both max-min and max-plus the canonical matrix \( A(b) \) belongs to interval \( A \). For the ‘only if’ case suppose that a subsystem \( A \otimes x = b \) for some \( A \in (A, \overline{A}) \) is solvable. Then the result for the max-plus case is implied by Lemma 4. In the max-min case we realise that \( A \vdash A(b) \vdash b \). Hence \( A(b) \otimes x = b \) is solvable by Lemma 2. □

**Theorem 5.** An interval system (1) is weakly solvable if and only if

\[
A(\overline{b}) \otimes x^*(A(\overline{b}), \overline{b}) \geq \underline{b}. \tag{7}
\]

**Proof.** If inequality (7) holds, then a solvable subsystem is the one with \( A = A(\overline{b}) \) and \( b = A(\overline{b}) \otimes x^*(A(\overline{b}), \overline{b}) \).

Conversely, if there exists a solvable subsystem \( A \otimes x = b \), then

\[
\begin{align*}
A \otimes x^*(A(\overline{b}), \overline{b}) &\geq A \otimes x^*(A, \overline{b}) \geq A \otimes x^*(A, b) = b \geq \underline{b}
\end{align*}
\]

due to Lemmas 1, 3 and 4(c).

To finish the proof we need the implication:

\[
\begin{align*}
\text{if } a_{ij} \otimes x^*_j(A(\overline{b}), \overline{b}) \geq \underline{b}_l, \quad \text{then } a_{ij}(\overline{b}) \otimes x^*_j(A(\overline{b}), \overline{b}) \geq \underline{b}_l.
\end{align*}
\]

In max-min algebra, to prove (8) it is sufficient to prove that if \( a_{ij} \geq \underline{b}_j \), then \( a_{ij}(\overline{b}) \geq \underline{b}_j \). But \( a_{ij}(\overline{b}) = \overline{b}_l \geq \underline{b}_l \) with the only exception when \( a_{ij}(\overline{b}) = \overline{a}_{ij} \) due to \( \overline{b}_l > \overline{a}_{ij} \); but in that case we have \( a_{ij}(\overline{b}) = \overline{a}_{ij} \geq \underline{a}_{ij} \geq \underline{b}_j \).
In max-plus algebra (8) means:

\[ a_{ij} - a_j(\vec{b}) \geq b_i, \quad \text{then} \quad a_{ij}(\vec{b}) - a_j(\vec{b}) \geq b_j. \]

This implication is trivial both when \( a_{ij}(\vec{b}) = a_j(\vec{b}) + b_i \) as well as when \( a_{ij}(\vec{b}) = \bar{a}_{ij} \). This finishes the proof. \( \square \)

4. Strong solvability

Over the field of real numbers, Rohn in [11] has shown that all subsystems of an interval linear system have a nonnegative solution if and only if all its extremal subsystems have a nonnegative solution. For a system with \( m \) equations this leads to testing \( 2^m \) systems, which does not provide an efficiently verifiable condition. Later, it has been shown that testing strong solvability is really NP-hard [9,12]. In the max-min and max-plus cases, solvability of all extremal subsystems is also necessary and sufficient for strong solvability. However, the number of subsystems required to be tested in this case is substantially smaller, in fact equals to \( m, \) as we show next.

**Theorem 6.** System (1) is strongly solvable if and only if all its extremal subsystems with exactly one LU equation are solvable.

**Proof.** The ‘only if’ implication is trivial. For the converse implication suppose that there exists an unsolvable subsystem of (1) of the form \( A \otimes x = b. \) Unsolvability of this system is equivalent to the inequality

\[ A \otimes x^*(A, b) \neq b. \] \hspace{1cm} (9)

Let us suppose that the inequality in (9) has occurred in the \( i \)th equation. This means that

\[ \bigoplus_{j=1}^{n} \left[ a_{ij} \otimes x_j^*(A, b) \right] < b_i, \] \hspace{1cm} (10)

i.e., for each index \( j \)

\[ a_{ij} \otimes x_j^*(A, b) < b_i. \] \hspace{1cm} (11)

In the max-min algebra inequality (11) means that at least one of the following cases has occurred: either

\[ a_{ij} < b_i \] \hspace{1cm} (12)

or

\[ x_j^*(A, b) < b_i. \] \hspace{1cm} (13)

In the first case we also have \( a_{ij} < \bar{b}_i. \) So inequality (10) will still be valid if we replace the \( i \)th equation by its corresponding LU-extremal equation. For the second
case we realise that \( x^*_j = \min_k \{ b_k; a_{kj} > b_k \} \), see (3). If (13) holds, then \( x^*_j(A, b) = b_k \) for some \( b_k < b_i \), and so (10) will be maintained if we replace all the equations except the \( i \)th one by their UL-extremal equivalents.

For the max-plus algebra inequality (11) means (see (4))

\[
aij + \min_k \{ bk - akj \} < bi.
\]

So the minimum in \( \min_k \{ bk - akj \} \) has not been achieved in row \( i \), which means that \( \min_{k \neq i} \{ bk - akj \} = \min_k \{ bk - akj \} \).

Now if we replace the \( i \)th equation by LU and the other ones by their UL equivalents, then the principal solution of the new system \( A' \otimes x = b' \) will be

\[
x^*_j(A', b') = \min \left\{ \min_{k \neq i} \{ bk - \bar{a}_{kj} \}, \bar{b}_i - a_{ij} \right\},
\]

so thanks to (14) we also have in equation \( i \) of the new system

\[
a_{ij} + x^*_j(A', b') < \bar{b}_i
\]

and so the new system is unsolvable. \( \square \)

5. Examples

In this section we shall illustrate the main results of this paper by one example in the max-min and one in the max-plus algebra.

Example 1. In the max-min algebra let us consider

\[
A = \begin{pmatrix}
(0.3, 0.6) & (0.2, 0.4) & (0.4, 0.7) \\
(0.2, 0.5) & (0.5, 0.7) & (0.3, 0.5) \\
(0.1, 0.8) & (0.3, 0.5) & (0.8, 0.9)
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
(0.4, 0.6) \\
(0.4, 0.5) \\
(0.3, 0.7)
\end{pmatrix}
\]

Since \( \bar{b} = (0.6, 0.5, 0.7)^T \) the canonical matrix \( A(\bar{b}) \) will be

\[
A(\bar{b}) = \begin{pmatrix}
0.6 & 0.4 & 0.6 \\
0.5 & 0.5 & 0.5 \\
0.7 & 0.5 & 0.8
\end{pmatrix}.
\]

So \( x^*(A(\bar{b}), \bar{b}) = (1, 1, 0.7)^T \) and since \( A(\bar{b}) \otimes x^*(A(\bar{b}), \bar{b}) = \bar{b} \), this interval system is weakly solvable.

The extremal subsystem where only the first equation is LU is

\[
A = \begin{pmatrix}
0.3 & 0.2 & 0.4 \\
0.5 & 0.7 & 0.5 \\
0.8 & 0.5 & 0.9
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
0.6 \\
0.4 \\
0.3
\end{pmatrix}
\]

with \( x^*(A, b) = (0.3, 0.3, 0.3)^T \) and since \( A \otimes x^*(A, b) = (0.3, 0.3, 0.3)^T < (0.6, 0.4, 0.3)^T \), the given interval system is not strongly solvable.
Example 2. For the max-plus algebra let us take
\[
A = \begin{pmatrix}
(2, 6) & (3, 8) & (4, 6) \\
(1, 4) & (6, 6) & (5, 7) \\
(4, 5) & (3, 7) & (6, 7)
\end{pmatrix}
\text{ and } \ b = \begin{pmatrix}
(4, 7) \\
(2, 6) \\
(5, 8)
\end{pmatrix}.
\]
Here, \(\vec{b} = (7, 6, 8)^T\) and so \(a_1(\vec{b}) = -4, a_2(\vec{b}) = 0, a_3(\vec{b}) = -1\). Thus the canonical matrix \(A(\vec{b})\) computed by algorithm CanonicalMatrix will be
\[
A = \begin{pmatrix}
3 & 7 & 6 \\
2 & 6 & 5 \\
4 & 7 & 7
\end{pmatrix}.
\]
We compute \(x^*(A(\vec{b}), \vec{b}) = (4, 0, 1)^T\) and since this is a solution of the system \(A(\vec{b}) \otimes x = \vec{b}\), the interval systems is weakly solvable.

The extremal subsystem with first equation being LU and the other ones UL is
\[
A = \begin{pmatrix}
2 & 3 & 4 \\
4 & 6 & 7 \\
5 & 7 & 7
\end{pmatrix}
\text{ and } \ b = \begin{pmatrix}
7 \\
2 \\
5
\end{pmatrix}.
\]
Here, \(x^*(A, b) = (-2, -4, -5)^T\) and since \(A \otimes x^*(A, b) = (0, 2, 3)^T < (7, 2, 5)^T\), this interval system is not strongly solvable.

6. Conclusion

In the classical linear algebra many problems connected with interval linear systems are NP-hard. However, the existence of the maximum solution of a linear system of equations over the max-plus and max-min algebra ensures that the same problems become easy in these structures, as can be seen in this paper.

References