ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 319 (2006) 17–33
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Global attractivity of a positive periodic solution for a nonautonomous stage structured population dynamics with time delay and diffusion ☆

Zhengqiu Zhang *, Li Wang

Department of Applied Mathematics, Hunan University, Changsha 410082, PR China

Received 18 February 2004

Submitted by K. Gopalsamy

Abstract

By employing the continuation theorem of coincidence degree theory, the existence of a positive periodic solution for a nonautonomous stage structured population dynamics with time delay and diffusion is established. Further, by constructing a Lyapunov functional and using the result of the existence of positive periodic solution, the attractivity of a positive periodic solution for above system is obtained.

Keywords: Time delay; Stage-structures; Diffusion; Positive periodic solutions; Lyapunov functional; The continuation theorem of coincidence degree theory

1. Introduction

In the study of the population dynamics, in order to make the population models more practical and accurate, more and more realistic factors have been considered, such as stage-structure (see [1–4,12,13]), diffusion (see [2,5–7,13]), time delay (see [1,5–7,10,11]), but the models with all the factors seem to be rarely considered.

☆ The project supported by NNSF of China (10271044).
* Corresponding author.
E-mail address: z_q_zhang@sina.com.cn (Z. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.01.046

To consider all these factors, Li et al. [8] introduced the following nonautonomous population model with stage-structure, diffusion and time delay:

$$\begin{cases} x_1'(t) = \alpha_1(t)y_1(t) - r_1(t)x_1(t) - \alpha_1(t-\tau)\exp\left(-\int_{t-\tau}^{t} r_1(s)\,ds\right)y_1(t-\tau), \\ y_1'(t) = \alpha_1(t-\tau)\exp\left(-\int_{t-\tau}^{t} r_1(s)\,ds\right)y_1(t-\tau) - \beta_1(t)y_1^2(t) \\ \qquad + D_1(t)\big(y_2(t) - y_1(t)\big) + R(t)y_1(t)z(t), \\ z'(t) = \alpha_3(t)z(t) - r_3(t)z^2(t) - \theta(t)y_1(t)z(t), \\ x_2'(t) = \alpha_2(t)y_2(t) - r_2(t)x_2(t) - \alpha_2(t-\tau)\exp\left(-\int_{t-\tau}^{t} r_2(s)\,ds\right)y_2(t-\tau), \\ y_2'(t) = \alpha_2(t-\tau)\exp\left(-\int_{t-\tau}^{t} r_2(s)\,ds\right)y_2(t-\tau) - \beta_2(t)y_2^2(t) + D_2(t)\big(y_1(t) - y_2(t)\big), \end{cases} \tag{1.1}$$

where $x_i(t)$, $y_i(t)$, $i = 1, 2$, represent the immature and mature predator population densities in the path i, respectively. $z(t)$ represents the prey population density in the patch 1, and $y_1(t)$, in patch 1, is its predator. The mature population $y_i(t)$, $i = 1, 2$, can disperse between the two patches. τ denotes the length of time that predator i $(i = 1, 2)$ grow from the birth to maturity, $\alpha_i(t)$ denotes the bearing rate of immature predator in patch i, $i = 1, 2$, $\alpha_i(t-\tau)$ denotes the bearing rate of mature predator in patch i, $i = 1, 2$, $r_i(t)$ is the death rate of immature predator in patch i, $i = 1, 2$, $\beta_i(t)$ denotes the death rate of mature predator in patch i, $i = 1, 2$, $R(t)$ denotes the preying effective rate in patch 1, $D_i(t)$ is the diffusive coefficients of mature predator in patch i, $i = 1, 2$, α_3 denotes the bearing rates of prey in patch 1, r_3 denotes the death rate of prey in patch 1, θ denotes the preying rate in patch 1, $\alpha_i(t)$, $r_i(t)$ $(i = 1, 2, 3)$, $\beta_i(t)$, $D_i(t)$ $(i = 1, 2)$, $R(t)$, $\theta(t)$ are positive continuous functions on $[0, +\infty)$.

In [8], the authors first established the existence of a positive periodic solution by using a fixed point theorem (see [8]) and the persistent result drawn by them. Then obtained the sufficient condition for a unique positive periodic solution which is globally attractive by using Lyapunov functional. To state the result in [8], we make two assumptions and a notation:

(H1) all the coefficients in system (1.1) are positive continuous ω-periodic functions with $\omega > 0$;

(H2) $0 < \min\{\underline{\alpha_i},\ \underline{r_i}\ (i = 1, 2, 3),\ \underline{\beta_i},\ \underline{D_i}\ (i = 1, 2),\ \underline{R},\ \underline{\theta}\}$
$\leqslant \max\{\overline{\alpha_i},\ \overline{r_i}\ (i = 1, 2, 3), \overline{\beta_i},\ \overline{D_i}\ (i = 1, 2),\ \overline{R},\ \bar{\theta}\} < +\infty.$

For a positive continuous ω-periodic function $f(t)$, we set

$$\overline{f} = \max_{t\in[0,\omega]}\{f(t)\}, \qquad \underline{f} = \min_{t\in[0,\omega]}\{f(t)\}, \qquad \tilde{f} = \frac{1}{\omega}\int_0^{\omega} f(t)\,dt.$$

Theorem 1.1. [8] *In addition to* (H1) *and* (H2), *we assume that the following conditions hold*:

(H3) $\underline{\alpha_3} - \bar{\theta} M_{25} > 0$, *for any given* $\varepsilon_1 > 0$, $M_{25} \overset{\text{def}}{=} \frac{\bar{\alpha} e^{-\underline{r}\tau} + \overline{R} M_3}{\underline{\beta}} + \varepsilon_2$, *here* $M_3 \overset{\text{def}}{=} \frac{\overline{\alpha_3}}{\underline{r_3}} + \varepsilon_1$, *for any given* ε_1, $\bar{\alpha} = \max\{\overline{\alpha_1}, \overline{\alpha_2}\}$, $\underline{r} = \min\{\underline{r_1}, \underline{r_2}\}$, $\underline{\beta} = \min\{\underline{\beta_1}, \underline{\beta_2}\}$, $\underline{\alpha} = \min\{\underline{\alpha_1}, \underline{\alpha_2}\}$, $\overline{\beta} = \max\{\overline{\beta_1}, \overline{\beta_2}\}$, $\bar{r} = \max\{\overline{r_1}, \overline{r_2}\}$;

(H4) $\underline{\alpha_1} e^{-\overline{r_1}\tau} > \overline{D_1} - \underline{R} m_3$, $m_3 \overset{\text{def}}{=} \frac{\underline{\alpha_3} - \bar{\theta} M_{25}}{\overline{r_3}} - \varepsilon_3 > 0$, ε_3 *is a given constant which is sufficiently small*;

(H5) $\underline{\alpha_2} e^{-\overline{r_2}\tau} > \overline{D_2}$;

(H6) $2\underline{\beta_1} m_2 + \underline{D_1} - \overline{D_2} - \bar{\theta} - \overline{R} M_3 - \underline{\alpha} e^{-\underline{r}\tau} > 0$;

(H7) $2\underline{\beta_2} m_5 + \underline{D_2} - \overline{D_1} - \underline{\alpha} e^{-\underline{r}r} > 0$;

(H8) $\underline{r_3} - \overline{R} M_{25} > 0$,

here, $m_2 \overset{\text{def}}{=} \frac{\underline{\alpha_1} e^{-\overline{r_1}\tau} + \underline{R} m_3 - \overline{D_1}}{\overline{\beta_1}} - \varepsilon_4$, $m_5 \overset{\text{def}}{=} \frac{\underline{\alpha_2} e^{-\overline{r_2}\tau} - \overline{D_2}}{\overline{\beta_2}} - \varepsilon_5 > 0$, ε_5, ε_4 *are two given and sufficient small constants.*

Then system (1.1) *has a unique positive ω-periodic solution which is globally attractive.*

Remark (A). A biological interpretation for Theorem 1.1 is that if

(H3) the bearing rate of prey in patch 1 is far larger than preying rate of mature predator in patch 1;
(H4) the rate of diffusion of mature predator in patch 1 is much smaller than the bearing rate of immature predator in patch 1;
(H5) the rate of diffusion of mature predator in patch 2 is much smaller than the bearing rate of immature predator in patch 2;
(H6) the bearing rate of immature predator in patch 1 is larger than the preying rate in patch 1;
(H7) the bearing rate of immature predator in patch 2 is larger than preying rate in patch 2;
(H8) the death rate of prey in patch 1 is smaller than the preying effective rate in patch 1, then system (1.1), after a long time, will approach a very stable status which exclude the interference from outward forces.

In this paper, instead of using fixed theorem, we establish the existence of positive ω-periodic solutions for system (1.1) by using Mawhin's continuation theorem of coincidence degree theory. Then establish the globally attractive result of a positive ω-periodic solution for system (1.1) by constructing different Lyapunov functional and using different technique from those in [8], and the existence result of positive periodic solutions. Compared with the result obtained in [8], our result is concise and is easily verified. Therefore, we obtain new sufficient condition for the global attractivity of a positive periodic solution of system (1.1) under concise condition.

The organization of this paper is as follows. In Section 2, by using the independent subsystem method, we obtain sufficient condition for the existence of positive periodic solutions of system (1.1). In Section 3, by using the independent subsystem method, we obtain the condition of the global attractivity of a positive periodic solution of system (1.1).

2. The existence of a positive periodic solution

In this section, based on Mawhin's continuation theorem, we shall study the existence of at least one positive periodic solution of system (1.1). First, we shall make some preparations.

Let X, Z be Banach spaces, let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and let $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L / \operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof our existence theorem, we will use the continuation theorem of Gaines and Mawhin [9].

Lemma 2.1 *(Continuation theorem). Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*:

(a) $Lx \neq \lambda N(x, \lambda)$, $\forall \lambda \in (0, 1)$, $x \in \partial\Omega$;
(b) $QN(x, 0) \neq 0$, $\forall x \in \operatorname{Ker} L \cap \partial\Omega$;
(c) *Brouwer degree* $\deg_B(JQN(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$.

Let us consider the subsystem of system (1.1)

$$
\begin{cases}
y_1'(t) = \alpha_1(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_1(s)\,ds\right)y_1(t-\tau) - \beta_1(t)y_1^2(t) + D_1(t)\big(y_2(t) - y_1(t)\big) \\
\qquad\qquad + R(t)y_1(t)z(t), \\
z'(t) = \alpha_3(t)z(t) - r_3(t)z^2(t) - \theta(t)y_1(t)z(t), \\
y_2'(t) = \alpha_2(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_2(s)\,ds\right)y_2(t-\tau) - \beta_2(t)y_2^2(t) \\
\qquad\qquad + D_2(t)\big(y_1(t) - y_2(t)\big).
\end{cases}
\tag{2.1}
$$

Theorem 2.1. *Assume that* (H1) *and* (H′1) *hold, where*

(H′1) $\underline{\alpha_3} > \bar{\theta}\big(\frac{\bar{\alpha}}{\underline{\beta_1}}e^{-\underline{r}\tau} + \frac{\overline{R}\overline{\alpha_3}}{\underline{\beta_1 r_3}}\big)$.

Then system (2.1) *has at least one positive ω-periodic solution.*

Proof. Consider the system

$$\begin{cases} \dfrac{du_1(t)}{dt} = \alpha_1(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_1(s)\,ds\right)e^{u_1(t-\tau)-u_1(t)} - \beta_1(t)e^{u_1(t)} \\ \qquad\qquad + D_1(t)\left(e^{u_3(t)-u_1(t)} - 1\right) + R(t)e^{u_2(t)}, \\ \dfrac{du_2(t)}{dt} = \alpha_3(t) - r_3(t)e^{u_2(t)} - \theta(t)e^{u_1(t)}, \\ \dfrac{du_3(t)}{dt} = \alpha_2(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_2(s)\,ds\right)e^{u_3(t-\tau)-u_3(t)} - \beta_2(t)e^{u_3(t)} \\ \qquad\qquad + D_2(t)\left(e^{u_1(t)-u_3(t)} - 1\right), \end{cases} \tag{2.2}$$

where all parameters are the same as those in system (2.1). It is easy to see that if system (2.2) has an ω-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$, then $(e^{u_1^*(t)}, e^{u_2^*(t)}, e^{u_3^*(t)})^T$ is a positive ω-periodic solution of system (2.1). Therefore, for system (2.1) to have at least one positive ω-periodic solution it is sufficient that system (2.2) has at least one ω-periodic solution. To apply Lemma 2.1 to system (2.2), we first define

$$X = Z = \left\{u(t) = \left(u_1(t), u_2(t), u_3(t)\right)^T \in C\left(R, R^3\right),\ u(t+\omega) = u(t)\right\}$$

and

$$\|u\| = \left\|\left(u_1(t), u_2(t), u_3(t)\right)^T\right\| = \sum_{i=1}^{3} \max_{t\in[0,\omega]} \left|u_i(t)\right|$$

for any $u \in X$ (or Z). Then X and Z are Banach spaces with the norm $\|\cdot\|$. Let

$$N(u,\lambda) = \begin{bmatrix} \alpha_1(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_1(s)\,ds\right)e^{u_1(t-\tau)-u_1(t)} - \beta_1(t)e^{u_1(t)} \\ \quad + \lambda D_1(t)\left(e^{u_3(t)-u_1(t)} - 1\right) + \lambda R(t)e^{u_2(t)}, \\ \alpha_3(t) - r_3(t)e^{u_2(t)} - \lambda\theta(t)e^{u_1(t)}, \\ \alpha_2(t-\tau)\exp\left(-\displaystyle\int_{t-\tau}^{t} r_2(s)\,ds\right)e^{u_3(t-\tau)-u_3(t)} - \beta_2(t)e^{u_3(t)} \\ \quad + \lambda D_2(t)\left(e^{u_1(t)-u_3(t)} - 1\right) \end{bmatrix}, \quad u \in X;$$

$$Lu = u' = \frac{du(t)}{dt}, \quad Pu = \frac{1}{\omega}\int_0^{\omega} u(t)\,dt, \quad u \in X;$$

$$Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\,dt, \quad z \in Z.$$

Then it follows that

$$\operatorname{Ker} L = R^3, \qquad \operatorname{Im} L = \left\{z \in Z\colon \int_0^{\omega} z(t)\,dt = 0\right\} \quad \text{is closed in } Z,$$

$$\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L,$$

and P, Q are continuous projectors such that

$$\operatorname{Im} P = \ker L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_p(z) = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt.$$

Thus

$$QN(u,\lambda) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega F_1(s)\,ds \\ \frac{1}{\omega}\int_0^\omega F_2(s)\,ds \\ \frac{1}{\omega}\int_0^\omega F_3(s)\,ds \end{bmatrix}$$

and

$$K_p(I-Q)N(u,\lambda) = \begin{bmatrix} \int_0^t F_1(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t F_1(s)\,ds\,dt + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_1(s)\,ds \\ \int_0^t F_2(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t F_2(s)\,ds\,dt + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_2(s)\,ds \\ \int_0^t F_3(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t F_3(s)\,ds\,dt + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_3(s)\,ds \end{bmatrix},$$

where

$$\begin{aligned} F_1(s) &= \alpha_1(s-\tau)\exp\left(-\int_{s-\tau}^{s} r_1(t)\,dt\right)e^{u_1(s-\tau)-u_1(s)} - \beta_1(s)e^{u_1(s)} \\ &\quad + \lambda D_1(s)\left(e^{u_3(s)-u_1(s)} - 1\right) + \lambda R(s)e^{u_2(s)}, \\ F_2(s) &= \alpha_3(s) - r_3(s)e^{u_2(s)} - \lambda\theta(s)e^{u_1(s)}, \\ F_3(s) &= \alpha_2(s-\tau)\exp\left(-\int_{s-\tau}^{s} r_2(t)\,dt\right)e^{u_3(s-\tau)-u_3(s)} - \beta_2(s)e^{u_3(s)} \\ &\quad + \lambda D_2(s)\left(e^{u_1(s)-u_3(s)} - 1\right). \end{aligned}$$

Obviously, QN and $K_p(I-Q)N$ are continuous. It is not difficult to show that $K_p(I-Q)N(\bar{\Omega})$ is compact for any open bounded $\Omega \subset X$ by using the Arzela–Ascoli theorem. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we reach the point where we search for an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.1). Corresponding to the operator equation $Lx = \lambda N(x,\lambda)$, $\lambda \in (0,1)$, we have

$$\begin{cases} \dfrac{du_1(t)}{dt} = \lambda\Bigg[\alpha_1(t-\tau)\exp\Bigg(-\int\limits_{t-\tau}^{t} r_1(s)\,ds\Bigg)e^{u_1(t-\tau)-u_1(t)} - \beta_1(t)e^{u_1(t)} \\ \qquad\qquad + \lambda D_1(t)\big(e^{u_3(t)-u_1(t)}-1\big) + \lambda R(t)e^{u_2(t)}\Bigg], \\ \dfrac{du_2(t)}{dt} = \lambda\big[\alpha_3(t) - r_3(t)e^{u_2(t)} - \lambda\theta(t)e^{u_1(t)}\big], \\ \dfrac{du_3(t)}{dt} = \lambda\Bigg[\alpha_2(t-\tau)\exp\Bigg(-\int\limits_{t-\tau}^{t} r_2(s)\,ds\Bigg)e^{u_3(t-\tau)-u_3(t)} - \beta_2(t)e^{u_3(t)} \\ \qquad\qquad + \lambda D_2(t)\big(e^{u_1(t)-u_3(t)}-1\big)\Bigg]. \end{cases} \tag{2.3}$$

Assume that $u = u(t) \in X$ is a solution of system (2.3) for a certain $\lambda \in (0,1)$. Because of $(u_1(t), u_2(t), u_3(t))^T \in X$, there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i = 1,2,3.$$

It is clear that

$$u_i'(\xi_i) = 0, \quad u_i'(\eta_i) = 0, \quad i = 1,2,3.$$

From this and system (2.3), we obtain

$$\alpha_1(\xi_1-\tau)\exp\Bigg(-\int\limits_{\xi_1-\tau}^{\xi_1} r_1(s)\,ds\Bigg)e^{u_1(\xi_1-\tau)-u_1(\xi_1)} - \beta_1(\xi_1)e^{u_1(\xi_1)} \\ + \lambda D_1(\xi_1)\big(e^{u_3(\xi_1)-u_1(\xi_1)}-1\big) + \lambda R(\xi_1)e^{u_2(\xi_1)} = 0, \tag{2.4}$$

$$\alpha_3(\xi_2) - r_3(\xi_2)e^{u_2(\xi_2)} - \lambda\theta(\xi_2)e^{u_1(\xi_2)} = 0, \tag{2.5}$$

$$\alpha_2(\xi_3-\tau)\exp\Bigg(-\int\limits_{\xi_3-\tau}^{\xi_3} r_2(s)\,ds\Bigg)e^{u_3(\xi_3-\tau)-u_3(\xi_3)} - \beta_2(\xi_3)e^{u_3(\xi_3)} \\ + \lambda D_2(\xi_3)\big(e^{u_1(\xi_3)-u_3(\xi_3)}-1\big) = 0, \tag{2.6}$$

and

$$\alpha_1(\eta_1-\tau)\exp\Bigg(-\int\limits_{\eta_1-\tau}^{\eta_1} r_1(s)\,ds\Bigg)e^{u_1(\eta_1-\tau)-u_1(\eta_1)} - \beta_1(\eta_1)e^{u_1(\eta_1)} \\ + \lambda D_1(\eta_1)\big(e^{u_3(\eta_1)-u_1(\eta_1)}-1\big) + \lambda R(\eta_1)e^{u_2(\eta_1)} = 0, \tag{2.7}$$

$$\alpha_3(\eta_2) - r_3(\eta_2)e^{u_2(\eta_2)} - \lambda\theta(\eta_2)e^{u_1(\eta_2)} = 0, \tag{2.8}$$

$$\alpha_2(\eta_3-\tau)\exp\Bigg(-\int\limits_{\eta_3-\tau}^{\eta_3} r_2(s)\,ds\Bigg)e^{u_3(\eta_3-\tau)-u_3(\eta_3)} - \beta_2(\eta_3)e^{u_3(\eta_3)} \\ + \lambda D_2(\eta_3)\big(e^{u_1(\eta_3)-u_3(\eta_3)}-1\big) = 0. \tag{2.9}$$

There two cases to consider for (2.4)–(2.6).

Case 1. Assume that $u_1(\xi_1) \geqslant u_3(\xi_3)$; then $u_1(\xi_1) \geqslant u_3(\xi_1)$.

From this and (2.4), we have

$$\alpha_1(\xi_1-\tau)\exp\left(-\int_{\xi_1-\tau}^{\xi_1} r_1(s)\,ds\right)e^{u_1(\xi_1-\tau)-u_1(\xi_1)} - \beta_1(\xi_1)e^{u_1(\xi_1)} + \lambda R(\xi_1)e^{u_2(\xi_1)}$$
$$= \lambda D_1(\xi_1)\left(e^{u_3(\xi_1)-u_1(\xi_1)} - 1\right) > 0,$$

that is

$$\begin{aligned}\underline{\beta_1}e^{2u_1(\xi_1)} &\leqslant \beta_1(\xi_1)e^{2u_1(\xi_1)} \\ &\leqslant \overline{\alpha_1}e^{-\underline{r}\tau}e^{u_1(\xi_1)} + \overline{R}e^{-\underline{r}\tau}e^{u_2(\xi_1)+u_1(\xi_1)} \\ &\leqslant \overline{\alpha_1}e^{-\underline{r}\tau}e^{u_1(\xi_1)} + \overline{R}e^{u_2(\xi_2)+u_1(\xi_1)}.\end{aligned}$$

Hence

$$\underline{\beta_1}e^{u_1(\xi_1)} \leqslant \overline{\alpha_1}e^{-\underline{r}\tau} + \overline{R}e^{u_2(\xi_2)}. \tag{2.10}$$

(2.5) implies that

$$\underline{r_3}e^{u_2(\xi_2)} < r_3(\xi_2)e^{u_2(\xi_2)} < \alpha_3(\xi_2) \leqslant \overline{\alpha_3}. \tag{2.11}$$

From (2.10) and (2.11), we have

$$e^{u_1(\xi_1)} < \frac{\overline{\alpha_1}}{\underline{\beta_1}}e^{-\underline{r}\tau} + \frac{\overline{R}\,\overline{\alpha_3}}{\underline{r_3}\,\underline{\beta_1}}. \tag{2.12}$$

Thus

$$e^{u_3(\xi_3)} < \frac{\overline{\alpha_1}}{\underline{\beta_1}}e^{-\underline{r}\tau} + \frac{\overline{R}\,\overline{\alpha_3}}{\underline{r_3}\,\underline{\beta_1}}.$$

Case 2. Assume that $u_1(\xi_1) < u_3(\xi_3)$; then $u_1(\xi_3) < u_3(\xi_3)$.

From this and (2.6), we have

$$\alpha_2(\xi_3-\tau)\exp\left(-\int_{\xi_3-\tau}^{\xi_3} r_2(s)\,ds\right)e^{u_3(\xi_3-\tau)-u_3(\xi_3)} - \beta_2(\xi_3)e^{u_3(\xi_3)}$$
$$= \lambda D_2(\xi_3)\left(1 - e^{u_1(\xi_3)-u_3(\xi_3)}\right) > 0,$$

that is

$$\begin{aligned}\underline{\beta_2}e^{2u_3(\xi_3)} &\leqslant \beta_2(\xi_3)e^{2u_3(\xi_3)} \\ &< \overline{\alpha_2}(\xi_3-\tau)\exp\left(-\int_{\xi_3-\tau}^{\xi_3} r_2(s)\,ds\right)e^{u_3(\xi_3-\tau)} \\ &\leqslant \overline{\alpha_2}e^{-\underline{r}\tau}e^{u_3(\xi_3)}.\end{aligned}$$

Hence

$$e^{u_3(\xi_3)} < \frac{\overline{\alpha_2}}{\underline{\beta_2}} e^{-\underline{r}\tau}.$$

Thus

$$e^{u_1(\xi_1)} < \frac{\overline{\alpha_2}}{\underline{\beta_2}} e^{-\underline{r}\tau}.$$

From Cases 1 and 2, we obtain

$$u_1(\xi_1) < \ln\left(\frac{\overline{\alpha}}{\underline{\beta_2}} e^{-\underline{r}\tau} + \frac{\overline{R}\,\overline{\alpha_3}}{\underline{\beta_3}\,\underline{r_3}}\right) \stackrel{\text{def}}{=} d_1, \tag{2.13}$$

$$u_2(\xi_2) < \ln \frac{\overline{\alpha_3}}{\underline{r_3}} \stackrel{\text{def}}{=} d_2, \tag{2.14}$$

$$u_3(\xi_3) < \ln\left(\frac{\overline{\alpha}}{\underline{\beta}} e^{-\underline{r}\tau} + \frac{\overline{R}\,\overline{\alpha_3}}{\underline{\beta_1}\,\underline{r_3}}\right) \stackrel{\text{def}}{=} d_1. \tag{2.15}$$

There two cases to consider for (2.7)–(2.9).

Case 1. Assume that $u_1(\eta_1) \leqslant u_3(\eta_3)$; then $u_1(\eta_1) < u_3(\eta_1)$.

From this and (2.7), we have

$$\begin{aligned}
\beta_1(\eta_1)e^{u_1(\eta_1)} &= \alpha_1(\eta_1-\tau)\exp\left(-\int_{\eta_1-\tau}^{\eta_1} r_1(s)\,ds\right)e^{u_1(\eta_1-\tau)-u_1(\eta_1)} \\
&\quad + \lambda D_1(\eta_1)\left(e^{u_3(\eta_1)-u_1(\eta_1)}-1\right) + \lambda R(\eta_1)e^{u_2(\eta_1)} \\
&> \alpha_1(\eta_1-\tau)\exp\left(-\int_{\eta_1-\tau}^{\eta_1} r_1(s)\,ds\right)e^{u_1(\eta_1-\tau)-u_1(\eta_1)},
\end{aligned}$$

that is

$$\overline{\beta_1}e^{2u_1(\eta_1)} > \underline{\alpha_1}e^{-\overline{r_1}\tau}e^{u_1(\eta_1)}.$$

Hence

$$e^{u_1(\eta_1)} > \frac{\underline{\underline{\alpha_1}}}{\overline{\beta_1}}e^{-\overline{r_1}\tau}, \qquad e^{u_3(\eta_3)} > \frac{\underline{\underline{\alpha_1}}}{\overline{\beta_1}}e^{-\overline{r_1}\tau}.$$

Case 2. Assume that $u_1(\eta_1) > u_3(\eta_3)$; then $u_1(\eta_3) > u_3(\eta_3)$.

From this and (2.9), we have

$$\begin{aligned}
\beta_2(\eta_3)e^{u_3(\eta_3)} &= \alpha_2(\eta_3-\tau)\exp\left(-\int_{\eta_3-\tau}^{\eta_3} r_2(s)\,ds\right)e^{u_3(\eta_3-\tau)-u_3(\eta_3)} \\
&\quad + \lambda D_2(\eta_3)\left(e^{u_1(\eta_3)-u_3(\eta_3)}-1\right)
\end{aligned}$$

$$> \alpha_2(\eta_3 - \tau)\exp\left(-\int\limits_{\eta_3-\tau}^{\eta_3} r_2(s)\,ds\right)e^{u_3(\eta_3-\tau)-u_3(\eta_3)},$$

that is

$$\overline{\beta_2}e^{2u_3(\eta_3)} > \underline{\alpha_2}e^{-\overline{r_2}\tau}e^{u_3(\eta_3)}.$$

Hence

$$e^{u_3(\eta_3)} > \frac{\underline{\alpha_2}}{\overline{\beta_2}}e^{-\overline{r_2}\tau}, \qquad e^{u_1(\eta_1)} > \frac{\underline{\alpha_2}}{\overline{\beta_2}}e^{-\overline{r_2}\tau}.$$

From Cases 1 and 2, we have

$$u_1(\eta_1) > \ln\left(\frac{\underline{\alpha}}{\overline{\beta_2}}e^{-\bar{r}\tau}\right) \stackrel{\text{def}}{=} p_1, \tag{2.16}$$

$$u_3(\eta_3) > \ln\left(\frac{\underline{\alpha}}{\overline{\beta_2}}e^{-\bar{r}\tau}\right) \stackrel{\text{def}}{=} p_1. \tag{2.17}$$

(2.8) implies that

$$\overline{r_3}e^{u_2(\eta_3)} > \underline{\alpha_3} - \bar{\theta}e^{u_1(\xi_1)} > \underline{\alpha_3} - \bar{\theta}\left(\frac{\bar{\alpha}e^{-\underline{r}\tau}}{\underline{\beta}} + \frac{\overline{R}\overline{\alpha_3}}{\underline{\beta_1}\,\underline{r_3}}\right).$$

Thus

$$u_2(\eta_2) > \ln\frac{1}{\overline{r_3}}\left[\underline{\alpha_3} - \bar{\theta}\left(\frac{\bar{\alpha}}{\underline{\beta}}e^{-\bar{r}\tau} + \frac{\overline{R}\overline{\alpha_3}}{\underline{\beta_1}\,\underline{r_3}}\right)\right] \stackrel{\text{def}}{=} p_2. \tag{2.18}$$

From (2.13)–(2.15) and (2.16)–(2.18), we have for $\forall t \in R$,

$$\left|u_1(t)\right| \leqslant \max\left\{|d_1|, |p_1|\right\} \stackrel{\text{def}}{=} R_1,$$

$$\left|u_2(t)\right| \leqslant \max\left\{|d_2|, |p_2|\right\} \stackrel{\text{def}}{=} R_3,$$

$$\left|u_3(t)\right| < \max\left\{|d_1|, |p_1|\right\} \stackrel{\text{def}}{=} R_3.$$

Clearly, R_i $(i = 1, 2, 3)$ are independent of λ. Denote $M = \sum_{i=1}^{3} R_i + R_0$; here R_0 is taken sufficiently large such that the solution $(\alpha^*, \beta^*, \mu^*)^T$ of the following system:

$$\begin{cases} \widetilde{A} - \tilde{\beta}_1 e^{\alpha} = 0, \\ \tilde{\alpha}_3 - \tilde{r}_3 e^{\beta} = 0, \\ \tilde{\beta} - \tilde{\beta}_2 e^{\mu} = 0 \end{cases}$$

satisfies $\|(\alpha^*, \beta^*, \mu^*)^T\| < M$, where

$$\widetilde{A} = \frac{1}{\omega}\int\limits_0^{\omega} \alpha_1(t)\exp\left(-\int\limits_{t-\tau}^{t} r_1(s)\,ds\right)dt$$

and

$$\widetilde{B} = \frac{1}{\omega}\int\limits_0^{\omega} \alpha_2(t)\exp\left(-\int\limits_{t-\tau}^{t} r_2(s)\,ds\right)dt.$$

Now we take $\Omega = \{u = (u_1(t), u_2(t), u_3(t))^T \in X\colon \|u\| < M\}$. This satisfies condition (a) of Lemma 2.1. When $u \in \partial \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, u is a constant vector in R^3 with $\sum_{i=1}^{3} |u_i| = M$. Therefore

$$QN(u,0) = \begin{bmatrix} \widetilde{A} - \tilde{\beta}_1 e^{u_1} \\ \tilde{\alpha}_3 - \tilde{r}_3 e^{u_2} \\ \tilde{\beta} - \tilde{\beta}_2 e^{u_3} \end{bmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, we will prove that condition (c) of Lemma 2.1 is satisfied.

Since the system of algebraic equations:

$$\begin{cases} \widetilde{A} - \tilde{\beta}_1 x = 0, \\ \tilde{\alpha}_3 - \tilde{r}_3 y = 0, \\ \tilde{\beta} - \tilde{\beta}_2 z = 0 \end{cases}$$

has a unique solution $(x^*, y^*, z^*)^T$ which satisfies:

$$x^* = \frac{\widetilde{A}}{\tilde{\beta}_1}, \qquad y^* = \frac{\tilde{\alpha}_3}{\tilde{r}_3}, \qquad z^* = \frac{\widetilde{B}}{\tilde{\beta}_2},$$

then

$$\begin{aligned} &\deg\big(JQN(u,0), \Omega \cap \operatorname{Ker} L, (0,0,0)^T\big) \\ &\quad = \deg\big(\big(\widetilde{A} - \tilde{\beta}_1 e^{u_1}, \tilde{\alpha}_3 - \tilde{r}_3 e^{u_2}, \widetilde{B} - \tilde{\beta}_2 e^{u_3}\big)^T, \Omega \cap \operatorname{Ker} L, (0,0,0)^T\big) \\ &\quad = \operatorname{sign} \begin{vmatrix} -\tilde{\beta}_1 x^* & 0 & 0 \\ 0 & -\tilde{r}_3 y^* & 0 \\ 0 & 0 & -\tilde{\beta}_2 z^* \end{vmatrix} \\ &\quad = \operatorname{sign}(-\tilde{\beta}_1 \tilde{r}_3 \tilde{\beta}_2 x^* y^* z^*) = -1. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Assume that* (H1) *and* (H1′) *hold. Then system* (1.1) *has at least one positive* ω*-periodic solution.*

Proof. From the first and the fourth equation of system (1.1), we obtain

$$\begin{aligned} x_i(t) = {}& \exp\left(-\int_0^t r_i(s)\,ds\right)\left[x_i(0) - \int_{-\tau}^{0} \alpha_i(s) y_i(s) \exp\left(\int_0^s r_i(\theta)\,d\theta\right) ds\right] \\ &+ \int_0^\tau \alpha_i(t-u) y_i(t-u) \exp\left(\int_t^{t-u} r_i(\theta)\,d\theta\right) du, \quad i = 1,2. \end{aligned}$$

Let $(y_1^*(t), z^*(t), y_2^*(t))^T$ be a positive ω-periodic solution of system (2.1), then we get

$$\begin{aligned} x_i^*(t) = {}& \exp\left(-\int_0^t r_i(s)\,ds\right)\left[x_i^*(0) - \int_{-\tau}^{0} \alpha_i(s) y_i^*(s) \exp\left(\int_0^s r_i(\theta)\,d\theta\right) ds\right] \\ &+ \int_0^\tau \alpha_i(t-u) y_i^*(t-u) \exp\left(\int_t^{t-u} r_i(\theta)\,d\theta\right) du, \quad i = 1,2. \end{aligned}$$

Let

$$x_i^*(0) = \int_{-\tau}^{0} \alpha_i(s) y_i^*(s) \exp\left(\int_0^s r_i(\theta)\, d\theta \right) ds,$$

then

$$x_i^*(t) = \int_0^{\tau} \alpha_i(t-u) y_i^*(t-u) \exp\left(\int_t^{t-u} r_i(\theta)\, d\theta \right) du.$$

Therefore, $(x_i^*(t), y_1^*(t), z^*(t), x_2^*(t), y_2^*(t))^T$ with $x_i^*(0) = \int_{-\tau}^0 \alpha_i(s) y_i^*(s) \exp(\int_0^s r_i(\theta)\, d\theta)\, ds$ is a positive ω-periodic solution of system (1.1). This completes the proof of Theorem 2.2. □

3. Global attractivity of a positive periodic solution

In this section, by constructing a Lyapunov functional, we derive sufficient condition for the global attractivity of a positive periodic solution of system (1.1).

Lemma 3.1. [10, P_4, Lemma 1.2.2, Barbalat's lemma] *Let f be a nonnegative function defined in $[0, +\infty)$ such that f is integrable on $[0, +\infty)$ and uniformly continuous on $[0, +\infty)$. Then* $\lim_{t \to +\infty} f(t) = 0$.

Theorem 3.1. *In addition to the conditions in Theorem* 2.1*, we assume further that system* (2.1) *satisfies*

(H′2) $\theta + \frac{\overline{D}_2}{e^{p_1}} + \frac{\bar{\alpha} e^{-\bar{r}\tau}}{e^{p_1}} < \underline{\beta}_1$;
(H′3) $\overline{R} < \underline{r}_3$;
(H′4) $\frac{\overline{D}_1}{e^{p_1}} + \frac{\bar{\alpha} e^{-\bar{r}\tau}}{e^{p_1}} < \underline{\beta}_2$.

Then system (2.1) *has a unique positive ω-periodic solution which attracts all positive solutions of system* (2.1).

Remark (B). A biological interpretation for Theorem 3.1 is that if the death rate of mature predator in patch 1 is larger than the its preying rate in patch 1, the preying effective rate in patch 1 is smaller than the death rate of prey in patch 1 and the death rate of mature predator in patch 2 is larger than the bearing rate of immature predator in patches 1 and 2, then system (2.1), after a long time, will approach a very stable status which exclude the interference from outward forces.

Proof of Theorem 3.1. By Theorem 2.1, system (2.1) has at least one positive ω-periodic solution, say $(y_1^*(t), z^*(t), y_2^*(t))^T$. Then for $t \geqslant 0$, from the proof of Theorem 2.1, we have

$$e^{p_1} < y_1^*(t) < e^{d_1}, \qquad e^{p_2} < z^*(t) < e^{d_2}, \qquad e^{p_1} < y_2^*(t) < e^{d_1}.$$

Suppose that $(y_1(t), z(t), y_2(t))^T$ is a positive solution of system (2.1) with the initial conditions

$$\begin{cases} y_1(s) = \phi_1(s) \geqslant 0, & s \in [-\tau, 0],\ \phi_1(0) > 0, \\ z(s) = \phi_2(s) \geqslant 0, & s \in [-\tau, 0],\ \phi_2(0) > 0, \\ y_2(s) = \phi_3(s) \geqslant 0, & s \in [-\tau, 0],\ \phi_3(0) > 0. \end{cases}$$

Consider the following Lyapunov functional defined by

$$V(t)=\sum_{i=1}^{2}\left|\ln y_i-\ln y_i^*\right|+|\ln z-\ln z^*| \\ +\bar{\alpha}e^{-\bar{r}\tau}\int_{t-\tau}^{t}\left[\frac{|y_1(s)-y_1^*(s)|+|y_2(s)-y_2^*(s)|}{e^{p_1}}\right]ds, \quad \text{for } t\geqslant 0.$$

Apparently $V(t)\geqslant 0$ for $t\geqslant 0$.

By calculating the right upper derivative of $V(t)$ along the solution of (2.1), we have

$$\begin{aligned}
D^+V(t)={}&\sum_{i=1}^{2}\operatorname{sign}\left(y_i-y_i^*\right)\left(\frac{y_i'}{y_i}-\frac{y_i'^*}{y_i^*}\right)+\operatorname{sign}(z-z^*)\left(\frac{z'}{z}-\frac{z'^*}{z^*}\right)\\
&+\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t)-y_1^*(t)\right|+\left|y_2(t)-y_2^*(t)\right|\right)\\
&-\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t-\tau)-y_1^*(t-\tau)\right|+\left|y_2(t-\tau)-y_2^*(t-\tau)\right|\right)\\
={}&\operatorname{sign}\left(y_1-y_1^*\right)\left\{\alpha_1(t-\tau)\exp\left(-\int_{t-\tau}^{t}r_1(s)\,ds\right)\left(\frac{y_1(t-\tau)}{y_1(t)}-\frac{y_1^*(t-\tau)}{y_1^*(t)}\right)\right.\\
&\left.-\beta_1(t)\left(y_1(t)-y_1^*(t)\right)+D_1(t)\left(\frac{y_2(t)}{y_1(t)}-\frac{y_2^*(t)}{y_1^*(t)}\right)+R(t)\left(z(t)-z^*(t)\right)\right\}\\
&+\operatorname{sign}\left(y_2-y_2^*\right)\left\{\alpha_2(t-\tau)\exp\left(-\int_{t-\tau}^{t}r_2(s)\,ds\right)\left(\frac{y_2(t-\tau)}{y_2(t)}-\frac{y_2^*(t-\tau)}{y_2^*(t)}\right)\right.\\
&\left.-\beta_2(t)\left(y_2(t)-y_2^*(t)\right)+D_2(t)\left(\frac{y_1(t)}{y_2(t)}-\frac{y_1^*(t)}{y_2^*(t)}\right)\right\}\\
&+\operatorname{sign}(z-z^*)\left\{-r_3(t)\left(z(t)-z^*(t)\right)-\theta(t)\left(y_1(t)-y_1^*(t)\right)\right\}\\
&+\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t)-y_1^*(t)\right|+\left|y_2(t)-y_2^*(t)\right|\right)\\
&-\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t-\tau)-y_1^*(t-\tau)\right|+\left|y_2(t-\tau)-y_2^*(t-\tau)\right|\right)\\
\leqslant{}&\alpha_1(t-\tau)\exp\left(-\int_{t-\tau}^{t}r_1(s)\,ds\right)A_1(t)-\beta_1(t)\left|y_1(t)-y_1^*(t)\right|+A_2(t)\\
&+R(t)\left|z(t)-z^*(t)\right|+\alpha_2(t-\tau)\exp\left(-\int_{t-\tau}^{t}r_2(s)\,ds\right)A_3(t)\\
&-\beta_2(t)\left|y_2(t)-y_2^*(t)\right|+A_4(t)-r_3(t)\left|z(t)-z^*(t)\right|+\theta(t)\left|y_1(t)-y_1^*(t)\right|
\end{aligned}$$

$$+\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t)-y_1^*(t)\right|+\left|y_2(t)-y_2^*(t)\right|\right)$$
$$-\frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\left(\left|y_1(t-\tau)-y_1^*(t-\tau)\right|+\left|y_2(t\tau)-y_2^*(t\tau)\right|\right),$$

where

$$A_1(t)=\begin{cases}\frac{y_1^*(t-\tau)}{y_1^*(t)}-\frac{y_1(t-\tau)}{y_1(t)}, & y_1(t)<y_1^*(t),\\ \frac{y_1(t-\tau)}{y_1(t)}-\frac{y_1^*(t-\tau)}{y_1^*(t)}, & y_1(t)>y_1^*(t),\\ 0, & y_1(t)=y_1^*(t),\end{cases}$$

$$A_2(t)=\begin{cases}D_1(t)\left(\frac{y_2^*(t-\tau)}{y_1^*(t)}-\frac{y_2(t-\tau)}{y_1(t)}\right), & y_1(t)<y_1^*(t),\\ D_1(t)\left(\frac{y_2(t)}{y_1(t)}-\frac{y_2^*(t)}{y_1^*(t)}\right), & y_1(t)>y_1^*(t),\\ 0, & y_1(t)=y_1^*(t),\end{cases}$$

$$A_3(t)=\begin{cases}\frac{y_2^*(t-\tau)}{y_2^*(t)}-\frac{y_2(t-\tau)}{y_2(t)}, & y_2(t)<y_2^*(t),\\ \frac{y_2(t-\tau)}{y_2(t)}-\frac{y_2^*(t-\tau)}{y_2^*(t)}, & y_2(t)>y_2^*(t),\\ 0, & y_2(t)=y_2^*(t),\end{cases}$$

$$A_4(t)=\begin{cases}D_2(t)\left(\frac{y_1^*(t)}{y_2^*(t)}-\frac{y_1(t)}{y_2(t)}\right), & y_2(t)<y_2^*(t),\\ D_1(t)\left(\frac{y_1(t)}{y_2(t)}-\frac{y_1^*(t)}{y_2^*(t)}\right), & y_2(t)>y_2^*(t),\\ 0, & y_2(t)=y_2^*(t).\end{cases}$$

There are the following three cases to consider for the estimate of $A_1(t)$:

(i) if $y_1(t)>y_1^*(t)$, then
$$A_1(t)<\frac{y_1(t-\tau)-y_1^*(t-\tau)}{y_1^*(t)}\leqslant\frac{|y_1(t-\tau)-y_1^*(t-\tau)|}{y_1^*(t)};$$
(ii) if $y_1(t)<y_1^*(t)$, then
$$A_1(t)<\frac{y_1^*(t-\tau)-y_1(t-\tau)}{y_1^*(t)}\leqslant\frac{|y_1^*(t-\tau)-y_1(t-\tau)|}{y_1^*(t)};$$
(iii) if $y_1(t)=y_1^*$, then
$$A_1(t)=0<\frac{|y_1(t-\tau)-y_1^*(t-\tau)|}{y_1^*(t)}.$$

From (i)–(iii), we have
$$A_1(t)<\frac{|y_1^*(t-\tau)-y_1(t-\tau)|}{y_1^*(t)}<\frac{|y_1^*(t-\tau)-y_1(t-\tau)|}{e^{p_1}}.$$
Using the same estimate as that for $A_3(t)$, we have
$$A_3(t)<\frac{|y_2^*(t-\tau)-y_2(t-\tau)|}{y_2^*(t)}<\frac{|y_2^*(t-\tau)-y_2(t-\tau)|}{e^{p_1}}.$$

There are three cases to consider for the estimate of $A_2(t)$:

(a) if $y_1(t) < y_1^*(t)$, then

$$A_2(t) < D_1(t)\left(\frac{y_2^*(t)}{y_1^*(t)} - \frac{y_2(t)}{y_1^*(t)}\right) = D_1(t)\frac{y_2^*(t) - y_2(t)}{y_1^*(t)} < D_1(t)\frac{|y_2^*(t) - y_2(t)|}{y_1^*(t)};$$

(b) if $y_1(t) > y_1^*(t)$, then

$$A_2(t) < D_1(t)\frac{y_2(t) - y_2^*(t)}{y_1^*(t)} < D_1(t)\frac{|y_2(t) - y_2^*(t)|}{y_1^*(t)};$$

(c) if $y_1(t) = y_1^*(t)$, then

$$A_2(t) = 0 < D_1(t)\frac{|y_2(t) - y_2^*(t)|}{y_1^*(t)}.$$

Hence

$$A_2(t) < D_1(t)\frac{|y_2(t) - y_2^*(t)|}{y_1^*(t)} < \overline{D_1}\frac{|y_2(t) - y_2^*(t)|}{e^{p_1}}.$$

Therefore

$$D^+V(t) < -\left(\underline{\beta}_1 - \bar{\theta} - \frac{\overline{D}_2}{e^{p_1}} - \frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\right)\left|y_1(t) - y_1^*(t)\right| - (\underline{r_3} - \overline{R})\left|z(t) - z^*(t)\right|$$
$$- \left(\underline{\beta}_2 - \frac{\overline{D}_1}{e^{p_1}} - \frac{\bar{\alpha}e^{-\underline{r}\tau}}{e^{p_1}}\right)\left|y_2(t) - y_2^*(t)\right|, \quad t \geqslant 0. \tag{3.1}$$

It follows from conditions (H′2)–(H′4) in Theorem 3.1 that there exists a constant $\alpha^* > 0$ such that

$$D^+V(t) < -\alpha^*\left(\sum_{i=1}^{2}\left|y_i(t) - y_1^*(t)\right| + \left|z(t) - z^*(t)\right|\right), \quad t \geqslant 0. \tag{3.2}$$

Integrating on both sides of inequality (3.2) leads to

$$V(t) + \alpha^*\int_0^t\left(\sum_{i=1}^{2}\left|y_i(s) - y_i^*(s)\right| + \left|z(s) - z^*(s)\right|\right)ds \leqslant V(0) < +\infty, \quad t \geqslant 0,$$

which implies that

$$\sum_{i=1}^{2}\left(\left|y_i(t) - y_i^*(t)\right| + \left|z(t) - z^*(t)\right|\right) \in L^1[0, +\infty)$$

and

$$\sum_{i=1}^{2}\left(\left|\ln y_i(t) - \ln y_i^*(t)\right|\right) + \left|\ln z(t) - \ln z^*(t)\right| < V(t) \leqslant V(0) < +\infty, \quad t > 0. \tag{3.3}$$

From the boundedness of $y_i^*(t)$ $(i = 1, 2)$ and $z_i^*(t)$ and inequality (3.3), it follows that $y_i(t)$ $(i = 1, 2)$ and $z(t)$ are bounded for $t \geqslant 0$. From the boundedness of $y_i(t)$ $(i = 1, 2)$, $z(t)$ and system (2.1), it follows that $y_i(t) - y_i^*(t)$ $(i = 1, 2)$, $z(t) - z^*(t)$ and $(y_i(t) - y_i^*(t))'$, $(z(t) - z^*(t))'$ remain bounded on $[0, +\infty)$. Hence $\sum_{i=1}^{2}(|y_i(t) - y_i^*(t)| + |z(t) - z^*(t)|)$ is uniformly continuous. By Lemma 3.1, it follows that

$$\lim_{t\to+\infty}\left(\sum_{i=1}^{2}\left|y_i(t) - y_i^*(t)\right| + \left|z(t) - z^*(t)\right|\right) = 0.$$

Therefore

$$\lim_{t\to+\infty}\big(y_i(t) - y_i^*(t)\big) = 0 \quad (i = 1, 2)$$

and

$$\lim_{t\to\infty}\left|z(t) - z^*(t)\right| = 0.$$

This implies that system (2.1) has a positive ω-periodic solution which attracts all positive solutions of system (2.1). The proof is completed. □

Theorem 3.2. *In addition to the conditions in Theorem* 2.2*, we assume that system* (1.1) *satisfies* (H′2)–(H′4)*. Then system* (1.1) *has a unique positive ω-periodic solution which attracts all positive solutions of system* (1.1).

Proof. By Theorem 2.2, system (1.1) has at least one positive ω-periodic solution, say, $(x_i^*(t), y_1^*(t), z^*(t), x_2^*(t), y_2^*(t))^T$. Let $(x_1(t), y_1(t), z(t), x_2(t), y_2(t))^T$ be a positive solution of system (1.1). Then from the proof of Theorem 3.1, we have

$$\lim_{t\to+\infty}\left|y_i(t) - y_i^*(t)\right| = 0, \quad i = 1, 2, \qquad \lim_{t\to+\infty}\big(z(t) - z^*(t)\big) = 0.$$

New we will prove $\lim_{t\to+\infty}|x_i(t) - x_i^*(t)| = 0$, $i = 1, 2$. Since

$$\begin{aligned} x_i(t) = {} & \exp\left(-\int_0^t r_i(s)\,ds\right)\left[x_i(0) - \int_{-\tau}^{0}\alpha_i(s)y_i(s)\exp\left(\int_0^s r_i(\theta)\,d\theta\right)ds\right] \\ & + \int_0^{\tau}\alpha_i(t-u)y_i(t-u)\exp\left(\int_t^{t-u} r_i(\theta)\,d\theta\right)du, \quad i = 1, 2, \end{aligned}$$

$$\begin{aligned} x_i^*(t) = {} & \exp\left(-\int_0^t r_i(s)\,ds\right)\left[x_i^*(0) - \int_{-\tau}^{0}\alpha_i(s)y_i^*(s)\exp\left(\int_0^s r_i(\theta)\,d\theta\right)ds\right] \\ & + \int_0^{\tau}\alpha_i(t-u)y_i^*(t-u)\exp\left(\int_t^{t-u} r_i(\theta)\,d\theta\right)du, \quad i = 1, 2, \end{aligned}$$

then

$$x_i(t) - x_i^*(t) = \big(x_i(0) - x_i^*(0)\big)\exp\left(-\int_0^t r_i(s)\,ds\right)$$

$$
\begin{aligned}
&- \exp\left(-\int_0^t r_i(s)\,ds\right)\left[\int_{-\tau}^0 \alpha_i(s)\big(y_i(s)-y_i^*(s)\big)\exp\left(\int_0^s r_i(\theta)\,d\theta\right)ds\right] \\
&+ \int_0^\tau \alpha_i(t-u)\big(y_i(t-u)-y_i^*(t-u)\big)\exp\left(\int_t^{t-u} r_i(\theta)\,d\theta\right)du \\
\overset{\text{def}}{=}\ & f(t)+g(t).
\end{aligned}
$$

Since

$$
\begin{aligned}
&f(t) \leqslant \big(x_i(0)-x_i^*(0)\big)e^{-\underline{r_i}t} \to 0, \quad \text{as } t\to+\infty, \\
&g(t) \leqslant e^{-\underline{r_i}(t-s)}\left(\int_{-\tau}^0 \overline{\alpha_i}\big|y_i(s)-y_i^*(s)\big|\,ds\right) + \overline{\alpha_i}\int_0^\tau \big|y_i(t-u)-y_i^*(t-u)\big|e^{-\underline{r_i}\tau}\,du \to 0, \\
&\quad \text{as } t\to+\infty,
\end{aligned}
$$

thus $x_i(t)-x_i^*(t)\to 0$, as $t\to+\infty$. This completes the proof of Theorem 3.2. □

References

[1] W.G. Aiell, H.I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with stage-dependent time delay, SIAM J. Appl. Math. 52 (1992) 855–869.

[2] S. Liu, et al., A nonautonomous stage-structured single species model with diffusion, Ann. Differential Equations 16 (2) (2000) 153–168 (in Chinese).

[3] K.G. Magnusson, Oscillations in a stage structured predator–prey system with cannibalism, in: L. Chen (Ed.), Advanced Topics in Biomathematics, Proceedings of the International Conference on Mathematical Biology 1997, ICMB'97, Hangzhou, China, May 1997, World Scientific, Singapore, 1998, pp. 195–200.

[4] W. Wang, L. Chen, A predator–prey system with stage-structured for predator, Comput. Math. Appl. 33 (1997) 83–91.

[5] X. Song, L. Chen, Harmless delays and global attractivity for nonautonomous predator–prey system with diffusion, Comput. Math. Appl. 39 (2000) 33–42.

[6] Z. Zhang, Z. Wang, Periodic solution for a two-species nonautonomous competition Lotka–Volterra patch system with time delay, J. Math. Anal. Appl. 265 (2002) 38–48.

[7] Z. Zhang, Z. Wang, Periodic solution of two-species ratio-dependent predator–prey system with time delay in a two-patch environment, ANZIAM J. 45 (2003) 233–244.

[8] F. Li, et al., Permanence, periodic and almost periodic solutions of nonautonomous stage structured population dynamics with time delay and diffusion, Ann. Differential Equations 19 (4) (2003) 505–516.

[9] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer-Verlag, Berlin, 1977.

[10] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Math. Appl., vol. 74, Kluwer Academic, Dordrecht, 1992.

[11] Z. Zhang, Z. Wang, The existence of a periodic solution for a generalized prey–predator system with delay, Math. Proc. Cambridge Philos. Soc. 137 (2004) 475–486.

[12] Z. Zhang, Periodic solution of a predator–prey system with stage-structures for predator and prey, J. Math. Anal. Appl. 302 (2005) 291–305.

[13] Z. Zhang, S. Zeng, Periodic solution of a nonautonomous stage-structured single species model with diffusion, Quart. Appl. Math. 63 (2) (2005) 277–289.