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# Modified algebraic Bethe ansatz for XXZ chain on the segment – II – general cases

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**Abstract**

The spectral problem of the Heisenberg XXZ spin-$\frac{1}{2}$ chain on the segment is investigated within a modified algebraic Bethe ansatz framework. We consider in this work the most general boundaries allowed by integrability. The eigenvalues and the eigenvectors are obtained. They are characterised by a set of Bethe roots with cardinality equal to $N$, the length of the chain, and which satisfies a set of Bethe equations with an additional term.



## 1. Introduction

In this work, we study the spectral problem of the Heisenberg XXZ spin-$\frac{1}{2}$ chain on the segment, namely

$$H = \epsilon\,\sigma_1^z + \kappa^-\,\sigma_1^- + \kappa^+\,\sigma_1^+ + \sum_{k=1}^{N-1}\left(\sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y + \Delta \sigma_k^z \otimes \sigma_{k+1}^z\right)$$
$$+ \nu\,\sigma_N^z + \tau^-\,\sigma_N^- + \tau^+\,\sigma_N^+, \qquad (1.1)$$

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where $N$ is the length of the chain and $\sigma_i^{\pm,x,y,z}$ are the standard Pauli matrices, which act non-trivially on the site $i$ of the quantum space $\mathcal{H} = \otimes_{i=1}^{N}\mathbb{C}^2$. The theory is characterised by the anisotropy parameter $\Delta = \frac{q+q^{-1}}{2}$, where $q$ is generic, and by left $\{\epsilon, \kappa^{\pm}\}$ and right $\{\nu, \tau^{\pm}\}$ boundary couplings that we consider generic.

The possibility of exactly solving the Hamiltonian (1.1) is due to the fact that it can be embedded into the quantum inverse scattering framework [33]. Indeed, the spin-$\frac{1}{2}$ chain on the segment can be obtained from the logarithmic derivative of the double-row transfer matrix of the lattice six vertex model with reflected ends. The commutativity of the double-row transfer matrix, which allows one to diagonalise the Hamiltonian and the transfer matrix in a common basis, follows from the so-called reflection equations [12], in addition to the Yang–Baxter equation.

The complete characterisation of the spectrum of the double-row transfer matrix is, however, a challenging problem, which has been investigated by many authors, either from a Bethe ansatz (BA) point of view [28,29,7,17,16,36,14,15,6,31,32,9,24] or from alternative approaches [2,22,30,1,20,26], see for instance [4] for more historical details. The main difficulty arises from the breaking of the $U(1)$-symmetry, here induced by general boundary configurations allowed by the reflection equations. Let us also mention that, besides its intrinsic interest, the solution of this problem also finds applications in other areas such as out-of-equilibrium statistical physics, high energy physics, condensed matter, mathematical physics, among others.

In a previous paper [4], one of the authors considered the Hamiltonian (1.1) with triangular boundaries, i.e., $\kappa^+ = \tau^+ = 0$ or $\kappa^- = \tau^+ = 0$, from the so-called modified algebraic Bethe ansatz (MABA) approach. This method was first implemented in the case of the XXX spin-$\frac{1}{2}$ chain on the segment with the most general boundary conditions [5] and recently applied to the totally asymmetric exclusion process [13]. By means of this method, the algebraic Bethe ansatz framework [34,33] can now be used for quantum integrable models with finite number of degree of freedom and which are not invariant by the $U(1)$-symmetry.

Here, we consider the Heisenberg XXZ spin-$\frac{1}{2}$ chain on the segment with generic boundary parameters. We obtain constructively the Bethe vectors, the eigenvalues (or equivalently the functional Baxter T-Q equation) and the Bethe equations. We recover the Baxter T-Q equation with the new additional term discovered in [9,24]. The new results are the construction of the associated Bethe vectors which characterise the eigenstates of the Hamiltonian (1.1) as well as the off-shell action of the transfer matrix on it.

We also consider the limiting cases with left general and right upper triangular boundaries, i.e. $\tau^+ = 0$, and the case where the boundary parameters satisfy certain constraints [28,29,21], which can be obtained by requesting the vanishing of extra unwanted terms in the off-shell action of the transfer matrix on the Bethe vectors.

This paper is organised as follows. In Section 2, the basic properties of the quantum group $U_q(\widehat{sl_2})$ and of its coideal sub-algebra are reminded. Next, in Section 3, we recall the dynamical gauge transformations which allow the construction of the dynamical operators and of the dynamical transfer matrix. In Section 4, the representation theory, necessary for the implementation of the MABA, is considered. Then, in Section 5, we develop the modified algebraic Bethe ansatz and in Section 6 we give some limit cases of our result. Finally, we conclude in Section 7.

## 2. XXZ chain on the segment in the reflection algebra formalism

Quantum integrable models can be constructed from quantum groups and their coideal sub-algebras, the so-called reflection algebras. For the XXZ spin chain on the segment, one consider the quantum group $U_q(\widehat{sl_2})$ and its reflection algebras [12,33]. Here we give only the needed relations, more details can be found for instance in [4]. The fundamental object in this context is the so-called $R$-matrix, which acts on $V_a \otimes V_b$,[1] and it is given by,

$$R_{ab}(u) = \begin{pmatrix} b(qu) & 0 & 0 & 0 \\ 0 & b(u) & 1 & 0 \\ 0 & 1 & b(u) & 0 \\ 0 & 0 & 0 & b(qu) \end{pmatrix}, \quad b(u) = \frac{u-u^{-1}}{q-q^{-1}}, \tag{2.1}$$

and which satisfies the quantum Yang–Baxter equation that acts on $V_a \otimes V_b \otimes V_c$,

$$R_{ab}(u_a/u_b)R_{ac}(u_a/u_c)R_{bc}(u_b/u_c) = R_{bc}(u_b/u_c)R_{ac}(u_a/u_c)R_{ab}(u_a/u_b). \tag{2.2}$$

In order to consider spin chains on the segment, we also need the so-called $K^-$-matrix and its dual, the $K^+$-matrix, which are given by,[2]

$$K^-(u) = \begin{pmatrix} k^-(u) & \tau^2 c(u) \\ \tilde{\tau}^2 c(u) & k^-(u^{-1}) \end{pmatrix}, \quad k^-(u) = \nu_- u + \nu_+ u^{-1}, \quad c(u) = u^2 - u^{-2}, \tag{2.3}$$

$$K^+(u) = \begin{pmatrix} k^+(qu) & \tilde{\kappa}^2 c(qu) \\ \kappa^2 c(qu) & k^+(q^{-1}u^{-1}) \end{pmatrix}, \quad k^+(u) = \epsilon_+ u + \epsilon_- u^{-1}, \tag{2.4}$$

where $\{\epsilon_\pm, \kappa, \tilde{\kappa}\}$ and $\{\nu_\pm, \tau, \tilde{\tau}\}$ are generic parameters. They are the most general solutions [18] of the reflection equation [12] and of the dual reflection equation [33], that acts on $V_a \otimes V_b$,

$$R_{ab}(u_1/u_2)K_a^-(u_1)R_{ab}(u_1u_2)K_b^-(u_2) = K_b^-(u_2)R_{ab}(u_1u_2)K_a^-(u_1)R_{ab}(u_1/u_2), \tag{2.5}$$

$$\begin{aligned} &R_{ab}(u_2/u_1)K_a^+(u_1)R_{ab}(q^{-2}u_1^{-1}u_2^{-1})K_b^+(u_2) \\ &\quad = K_b^+(u_2)R_{ab}(q^{-2}u_1^{-1}u_2^{-1})K_a^+(u_1)R_{ab}(u_2/u_1). \end{aligned} \tag{2.6}$$

The $R$-matrix (2.1) and the $K^-$-matrix (2.3) allow the construction of the so-called double-row monodromy matrix, that acts on $V_a \otimes \mathcal{H}$, given by,

$$\begin{aligned} K_a(u) &= R_{a1}(u/v_1)\dots R_{aN}(u/v_N)K_a^-(u)R_{aN}(uv_N)\dots R_{a1}(uv_1), && (2.7) \\ &= \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) + \frac{1}{b(qu^2)}\mathscr{A}(u) \end{pmatrix}_a, && (2.8) \end{aligned}$$

with operator entries in the auxiliary space and denoted by $\{\mathscr{A}(u), \mathscr{B}(u), \mathscr{C}(u), \mathscr{D}(u)\}$. Each of these operators act on the quantum space $\mathcal{H} = V_1 \otimes \cdots \otimes V_N$. This representation is useful in the construction of the eigenstates by means of the $\mathscr{B}(u)$ operator. The parameters $v_i$ in (2.7) are called inhomogeneities.

**Remark 2.1.** It is also possible to use another family of operators, $\{\hat{\mathscr{A}}(u), \mathscr{B}(u), \mathscr{C}(u), \hat{\mathscr{D}}(u)\}$, which allow to construct the Bethe vectors of the transfer matrix from the $\mathscr{C}(u)$ operator,

$$K_a(u) = \begin{pmatrix} \hat{\mathscr{A}}(u) + \frac{1}{b(qu^2)}\hat{\mathscr{D}}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \hat{\mathscr{D}}(u) \end{pmatrix}_a. \tag{2.9}$$

[1] In this work, we only use two-dimensional complex vector spaces, i.e., $V = \mathbb{C}^2$.

[2] For convenience, we use squared off-diagonal parameters of the $K$-matrices when compared with the notation in [4].

Bringing together the dual $K^+$-matrix (2.4) and the double-row monodromy matrix (2.8), one can construct the double-row transfer matrix,

$$t(u) = tr_a(K_a^+(u)K_a(u)) = \phi(u)k^+(u)\mathscr{A}(u) + k^+(q^{-1}u^{-1})\mathscr{D}(u) + c(qu)\Big(\kappa^2\mathscr{B}(u) + \tilde{\kappa}^2\mathscr{C}(u)\Big), \tag{2.10}$$

where

$$\phi(u) = \frac{b(q^2u^2)}{b(qu^2)}, \tag{2.11}$$

which is the operator whose spectrum will be investigated.

**Remark 2.2.** In a similar way, we can use the operators $\{\hat{\mathscr{A}}(u), \hat{\mathscr{D}}(u)\}$ to write the transfer matrix as,

$$t(u) = k^+(qu)\hat{\mathscr{A}}(u) + \phi(u)k^+(u^{-1})\hat{\mathscr{D}}(u) + c(qu)\Big(\kappa^2\mathscr{B}(u) + \tilde{\kappa}^2\mathscr{C}(u)\Big). \tag{2.12}$$

The transfer matrix, since the Yang–Baxter and reflection equation are satisfied, constitutes a family of commuting operators, if evaluated at different spectral parameters [33], i.e. $[t(u), t(v)] = 0$. For this reason, $t(u)$ can be regarded as a generating function of the conserved charges of the model. In this context, the Hamiltonian (1.1) is one of these charges and it is related to the (homogeneous) double-row transfer matrix by means of the relation,

$$H = \frac{q-q^{-1}}{2}\frac{d}{du}\ln(t(u))\Big|_{u=1, v_i=1} - \left(N\,\frac{q+q^{-1}}{2} + \frac{(q-q^{-1})^2}{2(q+q^{-1})}\right). \tag{2.13}$$

In terms of the boundary parameters of the $K$-matrices (2.3), (2.4), the couplings of the Hamiltonian (1.1) are expressed as,

$$\epsilon = \frac{(q-q^{-1})}{2}\frac{(\epsilon_+ - \epsilon_-)}{(\epsilon_+ + \epsilon_-)}, \quad \kappa^- = \frac{2(q-q^{-1})}{(\epsilon_+ + \epsilon_-)}\kappa^2, \quad \kappa^+ = \frac{2(q-q^{-1})}{(\epsilon_+ + \epsilon_-)}\tilde{\kappa}^2, \tag{2.14}$$

$$\nu = \frac{(q-q^{-1})}{2}\frac{(\nu_- - \nu_+)}{(\nu_+ + \nu_-)}, \quad \tau^- = \frac{2(q-q^{-1})}{(\nu_+ + \nu_-)}\tilde{\tau}^2, \quad \tau^+ = \frac{2(q-q^{-1})}{(\nu_+ + \nu_-)}\tau^2. \tag{2.15}$$

It is convenient to introduce a new parametrisation for the boundary parameters, namely

$$\nu_- = i\tilde{\tau}\tau\big(\mu/\tilde{\mu} + \tilde{\mu}/\mu\big), \quad \nu_+ = i\tilde{\tau}\tau\big(\mu\tilde{\mu} + 1/(\mu\tilde{\mu})\big), \tag{2.16}$$

$$\epsilon_- = i\tilde{\kappa}\kappa\big(\xi/\tilde{\xi} + \tilde{\xi}/\xi\big), \quad \epsilon_+ = i\tilde{\kappa}\kappa\big(\xi\tilde{\xi} + 1/(\tilde{\xi}\xi)\big). \tag{2.17}$$

The advantage of this parametrisation is that it brings the $q$-determinant [33] to a factorized structure, i.e.,

$$\mathrm{Det}_q\{K^-(u)\} = tr_{12}\Big(P_{12}^- K_1^-(u)R_{12}(qu^2)K_2^-(qu)\Big) = b(u^2)\tilde{k}^-(qu)\tilde{k}^-(q^{-1}u^{-1}), \tag{2.18}$$

$$\mathrm{Det}_q\{K^+(u)\} = tr_{12}\Big(P_{12}^- K_2^+(qu)R_{12}(q^{-3}u^{-2})K_1^+(u)\Big) = b(q^{-4}u^{-2})\tilde{k}^+(qu)\tilde{k}^+(q^{-1}u^{-1}), \tag{2.19}$$

where we introduce useful boundary functions given by,

$$\tilde{k}^-(u) = i\tilde{\tau}\tau(\mu u + \mu^{-1}u^{-1})(\tilde{\mu}^{-1}u + \tilde{\mu}u^{-1}),$$
$$\tilde{k}^+(u) = i\tilde{\kappa}\kappa(\tilde{\xi}u + \tilde{\xi}^{-1}u^{-1})(\xi^{-1}u + \xi u^{-1}). \tag{2.20}$$

**Remark 2.3.** The parametrisations (2.16), (2.17) are respectively invariant by a $Z^2 \times Z^2$ symmetry, namely

- For (2.16) by the transformation $\mu \to \tilde{\mu}$ and $\tilde{\mu} \to \mu$ or $\tilde{\mu} \to 1/\tilde{\mu}$ and $\mu \to 1/\mu$.
- For (2.17) by the transformation $\xi \to \tilde{\xi}$ and $\tilde{\xi} \to \xi$ or $\tilde{\xi} \to 1/\tilde{\xi}$ and $\xi \to 1/\xi$.

## 3. Dynamical gauge transformation of the transfer matrix

In the case of the XXX spin-$\frac{1}{2}$ chain on the segment, due to the $SU(2)$ invariance, similarity transformations can be used to map the original problem with general boundaries to another one with a left general and a right diagonal boundaries or to a left lower triangular and right upper triangular boundaries. Both cases can be considered within the MABA framework [5,4].

For the XXZ case, the $SU(2)$ invariance is broken by the presence of the anisotropy parameter and one has to consider a more intricate transformation [7]. We recall that this transformation was introduced by Baxter in [3] to study the XYZ spin chain on the circle (see also [19]) and then used in the context of the XXZ spin chain on the segment in [7]. As we shall demonstrate, the local gauge transformation allows one to bring the dynamical double-row transfer matrix to a dynamical lower–upper (or upper–lower) structure.

In this section, after defining the gauge transformation, we introduce the dynamical operators and the dynamical transfer matrix. Next, the important multiple commutation relations for the MABA are also derived.

### 3.1. Gauge vectors

Following [7], we introduce the covariant vectors and the contravariant vectors,

$$X(u,m) = \begin{pmatrix} \alpha q^{-m} u^{-1} \\ 1 \end{pmatrix}, \quad Y(u,m) = \begin{pmatrix} \beta q^{m} u^{-1} \\ 1 \end{pmatrix}, \tag{3.1}$$

$$\tilde{X}(u,m) = \frac{qu}{\gamma_{m-1}} \begin{pmatrix} -1, & \alpha q^{-m} u^{-1} \end{pmatrix}, \quad \tilde{Y}(u,m) = \frac{qu}{\gamma_{m+1}} \begin{pmatrix} 1, & -\beta q^{m} u^{-1} \end{pmatrix}, \tag{3.2}$$

where $\alpha$ and $\beta$ are arbitrary complex parameters and $m$ is an integer which characterises the dynamical algebra produced by the gauge transformation (see e.g. [36,37,21] for more details). They satisfy the scalar products,

$$\begin{aligned} &\tilde{X}(u,m)X(u,m) = \tilde{Y}(u,m)Y(u,m) = 0, \\ &\tilde{X}(u,m+1)Y(u,m-1) = \tilde{Y}(u,m-1)X(u,m+1) = 1 \end{aligned} \tag{3.3}$$

as well as the closure relation,

$$Y(u,m-1)\tilde{X}(u,m+1) + X(u,m+1)\tilde{Y}(u,m-1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

Let us introduce the function $\gamma(u,m) = \alpha q^{-m} u - \beta q^{m} u^{-1}$ and denote $\gamma(1,m) = \gamma_m$. The covariant vectors then satisfy the following intertwining relations with the $R$-matrix (2.1),

$$R_{12}(u/v) X_1(u,m+1) \otimes X_2(v,m) = b(qu/v) X_1(u,m) \otimes X_2(v,m+1),$$

$$R_{12}(u/v) Y_1(u,m) \otimes Y_2(v,m+1) = b(qu/v) Y_1(u,m+1) \otimes Y_2(v,m),$$

$$
\begin{aligned}
R_{12}(u/v)X_1(u,m+1)\otimes Y_2(v,m) &= \frac{b(u/v)\gamma_m}{\gamma_{m+1}}X_1(u,m+2)\otimes Y_2(v,m+1) \\
&\quad + \frac{\gamma(v/u,m+1)}{\gamma_{m+1}}Y_1(u,m)\otimes X_2(v,m+1), \\
R_{12}(u/v)Y_1(u,m)\otimes X_2(v,m+1) &= \frac{b(u/v)\gamma_{m+1}}{\gamma_m}Y_1(u,m-1)\otimes X_2(v,m) \\
&\quad + \frac{\gamma(u/v,m)}{\gamma_m}X_1(u,m+1)\otimes Y_2(v,m)
\end{aligned}
\tag{3.5}
$$

while the contravariant relations are given by,

$$
\begin{aligned}
&\tilde{X}_1(u,m+1)\otimes\tilde{X}_2(v,m)R_{12}(u/v) = b(qu/v)\tilde{X}_1(u,m)\otimes\tilde{X}_2(v,m+1), \\
&\tilde{Y}_1(u,m)\otimes\tilde{Y}_2(v,m+1)R_{12}(u/v) = b(qu/v)\tilde{Y}_1(u,m+1)\otimes\tilde{Y}_2(v,m), \\
&\tilde{X}_1(u,m+1)\otimes\tilde{Y}_2(v,m-2)R_{12}(u/v) = \frac{b(u/v)\gamma_{m+1}}{\gamma_m}\tilde{X}_1(u,m+2)\otimes\tilde{Y}_2(v,m-1) \\
&\qquad\qquad + \frac{\gamma(v/u,m)}{\gamma_m}\tilde{Y}_1(u,m-2)\otimes\tilde{X}_2(v,m+1), \\
&\tilde{Y}_1(u,m-1)\otimes\tilde{X}_2(v,m+2)R_{12}(u/v) \\
&\quad = \frac{b(u/v)\gamma_{m-1}}{\gamma_m}\tilde{Y}_1(u,m-2)\otimes\tilde{X}_2(v,m+1) \\
&\quad\quad + \frac{\gamma(u/v,m)}{\gamma_m}\tilde{X}_1(u,m+2)\otimes\tilde{Y}_2(v,m-1).
\end{aligned}
\tag{3.6}
$$

### 3.2. *Dynamical operators and transfer matrices*

Following [7], we introduce the dynamical operators

$$
\mathscr{C}(u,m) = \tilde{X}(u,m)K(u)X(u^{-1},m), \quad \mathscr{B}(u,m) = \tilde{Y}(u,m)K(u)Y(u^{-1},m), \tag{3.7}
$$

$$
\begin{aligned}
&\mathscr{A}(u,m) = \tilde{Y}(u,m-2)K(u)X(u^{-1},m), \\
&\hat{\mathscr{D}}(u,m) = \tilde{X}(u,m+2)K(u)Y(u^{-1},m),
\end{aligned}
\tag{3.8}
$$

and

$$
\hat{\mathscr{A}}(u,m) = \frac{\gamma_{m-1}}{\gamma_m}\mathscr{A}(u,m) - \frac{\gamma(u^2,m-1)}{b(qu^2)\gamma_m}\hat{\mathscr{D}}(u,m), \tag{3.9}
$$

$$
\mathscr{D}(u,m) = \frac{\gamma_{m+1}}{\gamma_m}\hat{\mathscr{D}}(u,m) - \frac{\gamma(u^{-2},m+1)}{b(qu^2)\gamma_m}\mathscr{A}(u,m), \tag{3.10}
$$

which satisfy the commutation relations given in Appendix A. The family of dynamical operators $\{\mathscr{A}(u,m), \mathscr{D}(u,m)\}$ is used to construct the Bethe vectors from the $\mathscr{B}(u,m)$ operator. In this case, which we will consider in more details, the transfer matrix can be decomposed in the following way,

$$
t(u) = t_d(u,m) + u^{-1}c(qu)\Big(\zeta_m\mathscr{B}(u,m) - \tilde{\zeta}_m\mathscr{C}(u,m) - \delta_m t_{ps}(u,m)\Big) \tag{3.11}
$$

where

$$\zeta_m = \frac{\kappa^2}{\gamma_m}\Big(\alpha q^{-m-1} + i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}\Big)\Big(\alpha q^{-m-1} + i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}\Big),$$

$$\tilde{\zeta}_m = \frac{\kappa^2}{\gamma_m}\Big(\beta q^{m-1} + i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}\Big)\Big(\beta q^{m-1} + i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}\Big), \tag{3.12}$$

$$\delta_m = \frac{\kappa^2}{\gamma_{m+1}}\Big(\alpha q^{-m-1} + i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}\Big)\Big(\beta q^{m+1} + i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}\Big). \tag{3.13}$$

The term $t_d(u,m)$, which is called the diagonal transfer matrix, is given by,

$$t_d(u,m) = \tilde{a}(u)\mathscr{A}(u,m) + \tilde{d}(u)\mathscr{D}(u,m), \quad \tilde{a}(u) = u^{-1}\phi(u)\tilde{k}^+(u),$$
$$\tilde{d}(u) = u^{-1}\tilde{k}^+(q^{-1}u^{-1}), \tag{3.14}$$

and corresponds to the only remaining contribution when there exist constraints between left and right boundary parameters [7]. The term $t_{ps}(u,m)$ is defined as,

$$t_{ps}(u,m) = \phi(q^{-1}u^{-1})\mathscr{A}(u,m) - \mathscr{D}(u,m). \tag{3.15}$$

Although it also involves the dynamical operators $\mathscr{A}(u,m)$ and $\mathscr{D}(u,m)$, this term disappears in the constrained case, however it has to be considered in the generic one.

**Remark 3.1.** In an analogous way, we can express the transfer matrix in terms of the operators $\{\hat{\mathscr{A}}(u,m), \hat{\mathscr{D}}(u,m)\}$. This is convenient for the construction of the Bethe vectors from the $\mathscr{C}(u,m)$ operator. We have,

$$t(u) = \hat{t}_d(u,m) + u^{-1}c(qu)\Big(\zeta_m\mathscr{B}(u,m) - \tilde{\zeta}_m\mathscr{C}(u,m) + \delta_{m-2}\hat{t}_{ps}(u,m)\Big) \tag{3.16}$$

where

$$\hat{t}_d(u,m) = \hat{a}(u)\hat{\mathscr{A}}(u,m) + \hat{d}(u)\hat{\mathscr{D}}(u,m), \quad \hat{a}(u) = u^{-1}\tilde{k}^+(qu),$$
$$\hat{d}(u) = u^{-1}\phi(u)\tilde{k}^+(u^{-1}) \tag{3.17}$$

and

$$\hat{t}_{ps}(u,m) = -\hat{\mathscr{A}}(u,m) + \phi(q^{-1}u^{-1})\hat{\mathscr{D}}(u,m). \tag{3.18}$$

In the following subsection, we determine the action of the dynamical transfer matrices on an ordered product of dynamical $\mathscr{B}$ (or $\mathscr{C}$) operators. This is a fundamental step in the execution of the ABA. In this subsection and the remaining of this paper, the set of $M$ variables $\{u_1, u_2, \ldots, u_M\}$ will be shortly notated by $\bar{u}$ with $\#\bar{u} = M$. Another important set is $\bar{u}_i = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_M\}$, where the element $u_i$ is removed. On the other hand, if an element $u$ is added to this last set we denote it by $\{u, \bar{u}_i\}$. All the needed functions are given in Appendix A. In addition, for products of functions regarding to the set $\bar{u}$ we use a shorthand notation; for example, for the products of the function $f$, we have

$$f(u,\bar{u}) = \prod_{i=1}^{M} f(u,u_i).$$

Similarly for the set $\bar{u}_i$ we denote

$$f(u_i,\bar{u}_i) = \prod_{j=1, j\neq i}^{M} f(u_i,u_j).$$

### 3.3. Multiple commutation relations and ABA framework

Let us introduce a string of dynamical $\mathscr{B}$ operators, with length $M$, namely

$$B(\bar{u}, m, M) = \mathscr{B}(u_1, m-2) \ldots \mathscr{B}(u_M, m-2M). \tag{3.19}$$

Using the dynamical commutation relations (A.28), (A.29), (A.30), we can show that the action of the diagonal dynamical operators $\{\mathscr{A}(u, m), \mathscr{D}(u, m)\}$ on this string is given by,

$$\begin{aligned}
&\mathscr{A}(u, m)B(\bar{u}, m, M) \\
&\quad = f(u, \bar{u})B(\bar{u}, m, M)\mathscr{A}(u, m-2M) \\
&\qquad + \sum_{i=1}^{M} g(u, u_i, m-2) f(u_i, \bar{u}_i)B(\{u, \bar{u}_i\}, m, M)\mathscr{A}(u_i, m-2M) \\
&\qquad + \sum_{i=1}^{M} w(u, u_i, m-2) h(u_i, \bar{u}_i)B(\{u, \bar{u}_i\}, m, M)\mathscr{D}(u_i, m-2M)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\mathscr{D}(u, m)B(\bar{u}, m, M) \\
&\quad = h(u, \bar{u})B(\bar{u}, m, M)\mathscr{D}(u, m-2M) \\
&\qquad + \sum_{i=1}^{M} k(u, u_i, m-2) h(u_i, \bar{u}_i)B(\{u, \bar{u}_i\}, m, M)\mathscr{D}(u_i, m-2M) \\
&\qquad + \sum_{i=1}^{M} n(u, u_i, m-2) f(u_i, \bar{u}_i)B(\{u, \bar{u}_i\}, m, M)\mathscr{A}(u_i, m-2M).
\end{aligned} \tag{3.21}$$

Then, taking into account the following functional relations among coefficients of the commutation relations (see Appendix A for explicit formulas),

$$\begin{aligned}
&\tilde{a}(u)g(u, v, m) + \tilde{d}(u)n(u, v, m) \\
&\quad = \tilde{F}(u, v)\phi(q^{-1}v^{-1})\tilde{a}(v) + \chi_{m+2}u^{-1}c(qu)\phi(q^{-1}v^{-1}),
\end{aligned} \tag{3.22}$$

$$\tilde{a}(u)w(u, v, m) + \tilde{d}(u)k(u, v, m) = -\tilde{F}(u, v)\phi(v)\tilde{d}(v) - \chi_{m+2}u^{-1}c(qu), \tag{3.23}$$

$$\phi(q^{-1}u^{-1})g(u, v, m) - n(u, v, m) = \phi(q^{-1}v^{-1})\Big(G(u, v)b(v^2) + \rho_{m+2}\Big), \tag{3.24}$$

$$\phi(q^{-1}u^{-1})w(u, v, m) - k(u, v, m) = -\Big(G(u, v)b(q^{-2}v^{-2}) + \rho_{m+2}\Big), \tag{3.25}$$

where

$$\chi_m = i\tilde{\kappa}\kappa(q - q^{-1})\frac{\gamma(\tilde{\xi}/\xi, m)}{\gamma_{m-1}}, \qquad \rho_m = (q - q^{-1})\frac{q^{-m}\alpha + q^m\beta}{\gamma_{m-1}}, \tag{3.26}$$

we find that the action of the dynamical diagonal transfer matrix (3.14) on (3.19) is written as,

$$\begin{aligned}
&t_d(u, m)B(\bar{u}, m, M) \\
&\quad = B(\bar{u}, m, M)\Big(f(u, \bar{u})\tilde{a}(u)\mathscr{A}(u, m-2M) + h(u, \bar{u})\tilde{d}(u)\mathscr{D}(u, m-2M)\Big)
\end{aligned}$$

$$+\sum_{i=1}^{M}\tilde{F}(u,u_i)B(\{u,\bar{u}_i\},m,M)\big(\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\tilde{a}(u_i)\mathscr{A}(u_i,m-2M)$$
$$-\phi(u_i)h(u_i,\bar{u}_i)\tilde{d}(u_i)\mathscr{D}(u_i,m-2M)\big)$$
$$+\chi_m u^{-1}c(qu)\sum_{i=1}^{M}B(\{u,\bar{u}_i\},m,M)\big(\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\mathscr{A}(u_i,m-2M)$$
$$-h(u_i,\bar{u}_i)\mathscr{D}(u_i,m-2M)\big) \tag{3.27}$$

while for the action of (3.15) on (3.19) we have,

$$t_{ps}(u,m)B(\bar{u},m,M)$$
$$=B(\bar{u},m,M)\Big(\phi(q^{-1}u^{-1})f(u,\bar{u})\mathscr{A}(u,m-2M)-h(u,\bar{u})\mathscr{D}(u,m-2M)\Big)$$
$$+\sum_{i=1}^{M}G(u,u_i)B(\{u,\bar{u}_i\},m,M)\big(b(u_i^2)\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\mathscr{A}(u_i,m-2M)$$
$$-b(q^{-2}u_i^{-2})h(u_i,\bar{u}_i)\mathscr{D}(u_i,m-2M)\big)$$
$$+\rho_m\sum_{i=1}^{M}B(\{u,\bar{u}_i\},m,M)\big(\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\mathscr{A}(u_i,m-2M)$$
$$-h(u_i,\bar{u}_i)\mathscr{D}(u_i,m-2M)\big). \tag{3.28}$$

**Remark 3.2.** Similarly, we can introduce the string of dynamical $\mathscr{C}$ operators

$$C(\bar{u},m,M)=\mathscr{C}(u_1,m+2)\dots\mathscr{C}(u_M,m+2M), \tag{3.29}$$

and, from the dynamical commutation relations (A.31), (A.32), (A.33) and the functional relations,

$$\hat{d}(u)\hat{g}(u,v,m)+\hat{a}(u)\hat{n}(u,v,m)$$
$$=\tilde{F}(u,v)\phi(q^{-1}v^{-1})\hat{d}(v)+\hat{\chi}_{m-2}u^{-1}c(qu)\phi(q^{-1}v^{-1}), \tag{3.30}$$
$$\hat{a}(u)\hat{k}(u,v,m)+\hat{d}(u)\hat{w}(u,v,m)=-\tilde{F}(u,v)\phi(v)\hat{a}(v)-\hat{\chi}_{m-2}u^{-1}c(qu) \tag{3.31}$$
$$-\hat{k}(u,v,m)+\phi(q^{-1}u^{-1})\hat{w}(u,v,m)=-G(u,v)b(q^{-2}v^{-2})+\hat{\rho}_{m-2} \tag{3.32}$$
$$\phi(q^{-1}u^{-1})\hat{g}(u,v,m)-\hat{n}(u,v,m)=\phi(q^{-1}v^{-1})\big(G(u,v)b(v^2)-\hat{\rho}_{m-2}\big) \tag{3.33}$$

where

$$\hat{\chi}_m=\chi_m\frac{\gamma(m-1)}{\gamma(m+1)},\qquad \hat{\rho}_m=\rho_m\frac{\gamma(m-1)}{\gamma(m+1)}, \tag{3.34}$$

we are able to obtain,

$$\hat{t}_d(u,m)C(\bar{u},m,M)$$
$$=C(\bar{u},m,M)\Big(h(u,\bar{u})\hat{a}(u)\hat{\mathscr{A}}(u,m+2M)+f(u,\bar{u})\hat{d}(u)\hat{\mathscr{D}}(u,m+2M)\Big)$$
$$+\sum_{i=1}^{M}\tilde{F}(u,u_i)C(\{u,\bar{u}_i\},m,M)\big(-\phi(u_i)h(u_i,\bar{u}_i)\hat{a}(u_i)\hat{\mathscr{A}}(u_i,m+2M)$$

$$
\begin{aligned}
&+\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\hat{d}(u_i)\hat{\mathscr{D}}(u_i,m+2M)\big)\\
&+\hat{\chi}_m u^{-1}c(qu)\sum_{i=1}^{M}C(\{u,\bar{u}_i\},m,M)\big(-h(u_i,\bar{u}_i)\hat{\mathscr{A}}(u_i,m+2M)\\
&+\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\mathscr{D}(u_i,m+2M)\big)
\end{aligned}
\tag{3.35}
$$

and

$$
\begin{aligned}
&\hat{t}_{ps}(u,m)C(\bar{u},m,M)\\
&=C(\bar{u},m,M)\Big(-h(u,\bar{u})\hat{\mathscr{A}}(u,m+2M)+\phi(q^{-1}u^{-1})f(u,\bar{u})\hat{\mathscr{D}}(u,m+2M)\Big)\\
&\quad+\sum_{i=1}^{M}G(u,u_i)C(\{u,\bar{u}_i\},m,M)\big(-b(q^{-2}u_i^{-2})h(u_i,\bar{u}_i)\hat{\mathscr{A}}(u_i,m+2M)\\
&\quad+b(u_i^2)\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\hat{\mathscr{D}}(u_i,m+2M)\big)\\
&\quad-\hat{\rho}_m\sum_{i=1}^{M}C(\{u,\bar{u}_i\},m,M)\big(-h(u_i,\bar{u}_i)\hat{\mathscr{A}}(u_i,m+2M)\\
&\quad+\phi(q^{-1}u_i^{-1})f(u_i,\bar{u}_i)\hat{\mathscr{D}}(u_i,m+2M)\big).
\end{aligned}
\tag{3.36}
$$

## 4. Representation theory

We follow [7] and construct the highest weight vector by means of the covariant vector (3.1). For the choice of the gauge parameter,

$$
\alpha=\alpha_{hw}=iq^{m_0+N}\frac{\tau\mu}{\tilde{\tau}\tilde{\mu}},
\tag{4.1}
$$

one can show that the vector

$$
|\Omega_{m_0}^N\rangle=\otimes_{i=1}^{N}X_i(v_i,m_0+i)=X_1(v_1,m_0+1)\otimes\cdots\otimes X_N(v_N,m_0+N),
\tag{4.2}
$$

is the highest weight vector, such that the actions of the dynamical operators at the point $m=m_0$ are given by,

$$
\mathscr{A}(u,m_0)|\Omega_{m_0}^N\rangle=u\,\tilde{k}^-(u)\Lambda(u)|\Omega_{m_0}^N\rangle,
\tag{4.3}
$$

$$
\mathscr{D}(u,m_0)|\Omega_{m_0}^N\rangle=u\,\phi(q^{-1}u^{-1})\tilde{k}^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})|\Omega_{m_0}^N\rangle,
\tag{4.4}
$$

$$
\mathscr{C}(u,m_0)|\Omega_{m_0}^N\rangle=0,
\tag{4.5}
$$

where

$$
\Lambda(u)=\prod_{j=1}^{N}b(qu/v_j)b(quv_j).
\tag{4.6}
$$

**Remark 4.1.** The actions of the dynamical operators are obtained in [7] by introducing an additional family of local gauged operators. We refer to the work [7] for more details on this point.

**Remark 4.2.** For the other family of dynamical operators, $\{\hat{\mathscr{A}}(u,m), \hat{\mathscr{D}}(u,m)\}$, the gauge parameter is fixed by,

$$\beta = \beta_{lw} = i q^{-m_0 - N} \frac{\tau \tilde{\mu}}{\tilde{\tau} \mu} \tag{4.7}$$

such that

$$\begin{aligned} |\hat{\Omega}^N_{m_0}\rangle &= \otimes_{i=1}^N Y_i(v_i, m_0 + 2N - i) \\ &= Y_1(v_1, m_0 + 2N - 1) \otimes \cdots \otimes Y_N(v_N, m_0 + N), \end{aligned} \tag{4.8}$$

is a lowest weight vector satisfying,

$$\hat{\mathscr{A}}(u, m_0 + 2N)|\hat{\Omega}^N_m\rangle = u\,\phi(q^{-1}u^{-1})\tilde{k}^-(qu)\Lambda(q^{-1}u^{-1})|\hat{\Omega}^N_{m_0}\rangle \tag{4.9}$$

$$\hat{\mathscr{D}}(u, m_0 + 2N)|\hat{\Omega}^N_m\rangle = u\,\tilde{k}^-(u^{-1})\Lambda(u)|\hat{\Omega}^N_{m_0}\rangle \tag{4.10}$$

$$\mathscr{B}(u, m_0 + 2N)|\hat{\Omega}^N_{m_0}\rangle = 0. \tag{4.11}$$

**Remark 4.3.** If we fix at the same time both the parameters $\alpha = \alpha_{hw}$ and $\beta = \beta_{lw}$, the highest and lowest weights vectors are related by,

$$\mathscr{B}\big(u_1, m_0 + 2(N-1)\big)\ldots\mathscr{B}(u_N, m_0)|\Omega^N_{m_0}\rangle = Z^N(\bar{u}|\bar{v})|\hat{\Omega}^N_{m_0}\rangle \tag{4.12}$$

and

$$\mathscr{C}\big(u_1, m_0 + 2\big)\ldots\mathscr{C}(u_N, m_0 + 2N)|\hat{\Omega}^N_{m_0}\rangle = \hat{Z}^N(\bar{u}, \bar{v})|\Omega^N_{m_0}\rangle, \tag{4.13}$$

where $Z^N(\bar{u}|\bar{v})$ and $\hat{Z}^N(\bar{u}|\bar{v})$ are the partition function for the trigonometric solid-on-solid model with one reflecting end and domain wall boundary conditions given explicitly in [37,21]. In other words, for such choice of the gauge parameters the dynamical creation and annihilation operators are nilpotent, i.e.,

$$B(\bar{u}, m_0 + 2(N+1), N+1) = C(\bar{u}, m_0, N+1) = 0. \tag{4.14}$$

Moreover, this choice of gauge parameters maps the original problem into a solid-on-solid model with left general and right diagonal boundaries.

## 5. MABA for two general boundaries

We are now in position to implement the MABA for the Heisenberg XXZ spin chain on the segment with two general boundaries. In order to construct the Bethe vectors from the highest weight vector and the dynamical $\mathscr{B}$ operator, we have to fix one of the gauge parameters (4.1). We still have at our disposal one free gauge parameter, which we use to bring the dynamical transfer matrix into a left lower and right upper dynamical form, and that allows us to follow the method introduced in [4] to implement the MABA. This is the case for the choice of parameters,

$$\alpha = \alpha_{hw}, \qquad \beta = \beta_{tl}(M) = -i q^{1 - m_0 - 2M} \frac{\tilde{\xi}\tilde{\kappa}}{\xi \kappa}, \tag{5.1}$$

with $M$ some integer with values in $\{0, 1, \ldots, N\}$. In this case, the coefficient of the operator $\mathscr{C}(u, m_0 + 2M)$ in the dynamical transfer matrix (3.11) at the point $m_0 + 2M$ vanishes, i.e.,

$$
\begin{aligned}
&t(u, m_0 + 2M) \\
&\quad = t_d(u, m_0 + 2M) + u^{-1}c(qu)\Big(\zeta_{m_0+2M}\mathscr{B}(u, m_0 + 2M) - \delta_{m_0+2M}t_{ps}(u, m_0 + 2M)\Big).
\end{aligned} \tag{5.2}
$$

From the highest weight vector (4.2) and the string of dynamical creation operators (3.19), we can construct the vectors

$$
\Psi^M_{m_0}(\bar{u}) = B(\bar{u}, m_0 + 2M, M)|\Omega^N_{m_0}\rangle. \tag{5.3}
$$

Using the relation (3.27) as well as the representation theory results (4.3), (4.4) we obtain,

$$
\begin{aligned}
&t_d(u, m_0 + 2M)\Psi^M_{m_0}(\bar{u}) \\
&\quad = \Lambda^M_{gd}(u, \bar{u})\Psi^M_{m_0}(\bar{u}) \\
&\qquad + \sum_{i=1}^{M}\big(\tilde{F}(u, u_i)E^M_{gd}(u_i, \bar{u}_i) + \chi_{m_0+2M}\,u^{-1}c(qu)W^M(u_i, \bar{u}_i)\big)\Psi^M_{m_0}(\{u, \bar{u}_i\}),
\end{aligned} \tag{5.4}
$$

with[3]

$$
\begin{aligned}
\Lambda^M_{gd}(u, \bar{u}) = {}& \phi(u)\tilde{k}^+(u)\tilde{k}^-(u)\Lambda(u)f(u, \bar{u}) \\
& + \phi(q^{-1}u^{-1})\tilde{k}^+(q^{-1}u^{-1})\tilde{k}^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})h(u, \bar{u}),
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
E^M_{gd}(u_i, \bar{u}_i) = {}& \phi(q^{-1}u_i^{-1})\phi(u_i)\Big(\tilde{k}^+(u_i)\tilde{k}^-(u_i)\Lambda(u_i)f(u_i, \bar{u}_i) \\
& - \tilde{k}^+(q^{-1}u_i^{-1})\tilde{k}^-(q^{-1}u_i^{-1})\Lambda(q^{-1}u_i^{-1})h(u_i, \bar{u}_i)\Big),
\end{aligned} \tag{5.6}
$$

and

$$
\begin{aligned}
&W^M(u_i, \bar{u}_i) \\
&\quad = u_i\phi(q^{-1}u_i^{-1})\Big(\tilde{k}^-(u_i)\Lambda(u_i)f(u_i, \bar{u}_i) - \tilde{k}^-(q^{-1}u_i^{-1})\Lambda(q^{-1}u_i^{-1})h(u_i, \bar{u}_i)\Big).
\end{aligned} \tag{5.7}
$$

Analogously, the action of $t_{ps}(u, m_0 + 2M)$ on (5.3) is given by,

$$
\begin{aligned}
&t_{ps}(u, m_0 + 2M)\Psi^M_{m_0}(\bar{u}) \\
&\quad = \Lambda^M_{ps}(u, \bar{u})\Psi^M_{m_0}(\bar{u}) \\
&\qquad + \sum_{i=1}^{M}\big(G(u, u_i)b(qu_i^2)E^M_{ps}(u_i, \bar{u}_i) + \rho_{m_0+2M}W^M(u_i, \bar{u}_i)\big)\Psi^M_{m_0}(\{u, \bar{u}_i\})
\end{aligned} \tag{5.8}
$$

with

$$
\Lambda^M_{ps}(u, \bar{u}) = u\phi(q^{-1}u^{-1})\Big(\tilde{k}^-(u)\Lambda(u)f(u, \bar{u}) - \tilde{k}^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})h(u, \bar{u})\Big), \tag{5.9}
$$

$$
\begin{aligned}
E^M_{ps}(u_i, \bar{u}_i) = {}& u_i\phi(q^{-1}u_i^{-1})\Big(\phi(q^{-1}u_i^{-1})\tilde{k}^-(u_i)\Lambda(u_i)f(u_i, \bar{u}_i) \\
& + \phi(u_i)\tilde{k}^-(q^{-1}u_i^{-1})\Lambda(q^{-1}u_i^{-1})h(u_i, \bar{u}_i)\Big).
\end{aligned} \tag{5.10}
$$

[3] Here and in the following the Bethe equations can be recovered from the corresponding eigenvalues using the relation,

$$
E^M(u_i, \bar{u}_i) = \lim_{u \to u_i}\Big(b(u_i/u)\Lambda^M(u, \bar{u})\Big).
$$

Gathering Eqs. (5.4), (5.8), it follows that the full action of the dynamical transfer matrix (5.2) on (5.3) is given by,

$$\begin{aligned} & t(u, m_0+2M)\Psi^M_{m_0}(\bar{u}) \\ & \quad = \Lambda^M_{gd}(u,\bar{u})\Psi^M_{m_0}(\bar{u}) + \sum_{i=1}^{M} \tilde{F}(u,u_i)E^M_{gd}(u_i,\bar{u}_i)\Psi^M_{m_0}(\{u,\bar{u}_i\}) \\ & \qquad + u^{-1}c(qu)\Big(\zeta_{m_0+2M}\mathscr{B}(u,m_0+2M)\Psi^M_{m_0}(\bar{u}) - \delta_{m_0+2M}\Lambda^M_{ps}(u,\bar{u})\Psi^M_{m_0}(\bar{u}) \\ & \qquad + \sum_{i=1}^{M}\big(\bar{\chi}_{m_0+2M}W^M(u_i,\bar{u}_i) - \delta_{m_0+2M}G(u,u_i)b(qu_i^2)E^M_{ps}(u_i,\bar{u}_i)\big)\Psi^M_{m_0}(\{u,\bar{u}_i\})\Big) \end{aligned} \tag{5.11}$$

where we introduce,

$$\bar{\chi}_m = \chi_m - \delta_m \rho_m. \tag{5.12}$$

Up to this point, we have basically a usual algebraic Bethe ansatz analysis. The new feature, which motivates the adjective *modified* for the ABA, is the presence of the modified creation operator $\mathscr{B}(u, m_0+2M)$ in the off-shell Eq. (5.11). For $M = N$ and $\bar{u} = \{u_1, \dots, u_N\}$, we can show from small chain calculations[4] that the vector $\mathscr{B}(u, m_0+2N)\Psi^N_{m_0}(\bar{u})$ has an off-shell action given by,

$$\begin{aligned} & u^{-1}c(qu)\zeta_{m_0+2N}\mathscr{B}(u,m_0+2N)\Psi^N_{m_0}(\bar{u}) \\ & \quad = \Big(\Lambda^N_g(u,\bar{u}) + u^{-1}c(qu)\delta_{m_0+2N}\Lambda^N_{ps}(u,\bar{u})\Big)\Psi^N_{m_0}(\bar{u}) \\ & \qquad + \sum_{i=1}^{N}\Big(u^{-1}c(qu)\big(\delta_{m_0+2N}G(u,u_i)b(qu_i^2)E^N_{ps}(u_i,\bar{u}_i) - \bar{\chi}_{m_0+2N}W^N(u_i,\bar{u}_i)\big) \\ & \qquad + \tilde{F}(u,u_i)E^N_g(u_i,\bar{u}_i)\Big)\Psi^N_{m_0}(\{u,\bar{u}_i\}) \end{aligned} \tag{5.13}$$

with

$$\begin{aligned} \Lambda^N_g(u,\bar{u}) &= -\kappa\tilde{\kappa}\tau\tilde{\tau}\Big(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\xi\tilde{\mu}}{\tilde{\xi}\mu}q^{N+1} + \frac{\tilde{\xi}\mu}{\xi\tilde{\mu}}q^{-N-1}\Big) \\ & \qquad \times c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})G(u,\bar{u}), \end{aligned} \tag{5.14}$$

$$\begin{aligned} E^N_g(u_i,\bar{u}_i) &= \kappa\tilde{\kappa}\tau\tilde{\tau}\Big(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\xi\tilde{\mu}}{\tilde{\xi}\mu}q^{N+1} + \frac{\tilde{\xi}\mu}{\xi\tilde{\mu}}q^{-N-1}\Big) \\ & \qquad \times \frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})G(u_i,\bar{u}_i). \end{aligned} \tag{5.15}$$

Eq. (5.13), together with (5.11), allows one to reach the main result of this paper, the off-shell action of the transfer matrix with two general boundaries given by,

[4] The unknown $\Lambda^N_g(u,\bar{u})$ and $E^N_g(u_i,\bar{u}_i)$ are fixed from the direct resolution of the case $N = 1$ and $N = 2$ using symbolic calculation on Mathematica for generic boundary parameters and variables $\bar{u}$ and then verified for $N = 3$ using arbitrary numerical parameters and variables.

$$t(u)\Psi^N_{m_0}(\bar{u}) = \Lambda^N(u,\bar{u})\Psi^N_{m_0}(\bar{u}) + \sum_{i=1}^{N} \tilde{F}(u,u_i)E^N(u_i,\bar{u}_i)\Psi^N_{m_0}(\{u,\bar{u}_i\}) \tag{5.16}$$

where

$$\Lambda^N(u,\bar{u}) = \Lambda^N_{gd}(u,\bar{u}) + \Lambda^N_g(u,\bar{u}), \quad E^N(u_i,\bar{u}_i) = E^N_{gd}(u_i,\bar{u}_i) + E^N_g(u_i,\bar{u}_i), \tag{5.17}$$

with $\Lambda^N_{gd}(u,\bar{u})$, $E^N_g(u_i,\bar{u}_i)$, $\Lambda^N_g(u,\bar{u})$ and $E^N_{gd}(u_i,\bar{u}_i)$ given respectively by (5.5), (5.6), (5.14), (5.15) and with the Bethe vector,

$$\Psi^N_{m_0}(\bar{u}) = \mathscr{B}(u_1, m_0 + 2(N-1))\dots\mathscr{B}(u_N, m_0)|\Omega^N_{m_0}\rangle \tag{5.18}$$

for the gauge parameters (5.1) with $M = N$.

**Remark 5.1.** The eigenvalues (5.17) correspond, up to a change of notation, to the ones obtained in [9,24] and investigated at the thermodynamic limit in [27]. We remark that the new term in the eigenvalue expression, characteristic of the modified T-Q Baxter equation introduced in [8], takes its algebraic origin in the off-shell action of the creation operator (5.13).

**Remark 5.2.** The Bethe vector (5.18) is constructed from the dynamical $\mathscr{B}$ operator with gauge parameters (5.1) which correspond to bring the transfer matrix into a dynamical lower/upper triangular form. It is also possible to characterise this Bethe vector using another set of gauge parameters which bring the transfer matrix into a dynamical general/diagonal form, with

$$\alpha = \alpha_{hw}, \quad \beta = \beta_{lw}. \tag{5.19}$$

In this case both the highest and lowest weight vector do exist at the same fixed $m = m_0$ and are related, see Remark 4.3. Using the relations between the dynamical creation operator $\mathscr{B}(u,m)$ with dynamical parameters $\alpha$ and $\beta_{tl}$ and the dynamical operators $\{\mathscr{A}(u,p), \mathscr{B}(u,p), \mathscr{C}(u,p), \mathscr{D}(u,p)\}$ with dynamical parameters $\alpha$ and $\beta_{lw}$, given by

$$\begin{aligned}
\mathscr{B}(u,m)|_{\beta_{tl}} &= q^{m-p}\frac{(\alpha - q^{p+m}\beta_{tl})(\alpha - q^{2+p+m}\beta_{tl})}{(\alpha - q^{2+2m}\beta_{tl})(\alpha - q^{2p}\beta_{lw})}\mathscr{B}(u,p)|_{\beta_{lw}} \\
&\quad + q^{3+m}\frac{(\alpha - q^{p+m}\beta_{tl})(q^m\beta_{tl} - q^p\beta_{lw})}{(\alpha - q^{2+2m}\beta_{tl})(\alpha - q^{2+2p}\beta_{lw})}t_{ps}(u,p)|_{\beta_{lw}} \\
&\quad - q^{m+p}\frac{(q^p\beta_{lw} - q^m\beta_{tl})(q^p\beta_{lw} - q^{2+m}\beta_{tl})}{(\alpha - q^{2+2m}\beta_{tl})(\alpha - q^{2p}\beta_{lw})}\mathscr{C}(u,p)|_{\beta_{lw}}
\end{aligned}$$

which can be obtained using (A.24), we can project (5.18) in this new basis of dynamical operators with $\alpha = \alpha_{hw}$ and $\beta = \beta_{lw}$, namely

$$\Psi^N_{m_0}(\bar{u}) = \sum_{i=0}^{N} \sum_{\bar{u}\Rightarrow\{\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}}\}} W^N_{N-i}(\bar{u}_{\mathrm{I}}|\bar{u}_{\mathrm{II}})\bar{\Psi}^i_{m_0}(\bar{u}_{\mathrm{II}}) \tag{5.20}$$

with

$$\bar{\Psi}^i_{m_0}(\{u_{i+1},\dots,u_N\}) = \mathscr{B}(u_{i+1}, m_0 - 2(i+1))\dots\mathscr{B}(u_N, m_0)|\Omega^N_{m_0}\rangle, \tag{5.21}$$

and where the second sum corresponds to each splitting of the set $\bar{u}$ into subsets $\bar{u}_{\mathrm{I}}$ and $\bar{u}_{\mathrm{II}}$ with $\#\bar{u}_{\mathrm{II}} = i$ and where the elements in every subset are ordered in such a way that the sequence of their subscripts is strictly increasing. The explicit coefficients $W^N_{N-i}(\bar{u}_{\mathrm{I}}|\bar{u}_{\mathrm{II}})$ can be fixed by the expansion and will be considered elsewhere.

**Remark 5.3.** The MABA can be similarly performed using the $\mathscr{C}(u,m)$ operators and the lowest weight vector (4.8) to construct the Bethe Vector. In this case, the lowest weight vector fixes the gauge parameter,

$$\beta = \beta_{lw} = iq^{-m_0-N}\frac{\tau\tilde{\mu}}{\tilde{\tau}\mu}, \tag{5.22}$$

and, by means of the choice,

$$\alpha = \alpha_{tu}(\hat{M}) = -iq^{1+m_0+2(N-\hat{M})}\frac{\xi\tilde{\kappa}}{\tilde{\xi}\kappa} \tag{5.23}$$

the transfer matrix acquires a dynamical upper/lower structure at the point $m_0 + 2(N - \hat{M})$, namely,

$$\begin{aligned}
&t(u, m_0 + 2(N - \hat{M}))\\
&\quad= \hat{t}_d(u, m_0 + 2(N - \hat{M})) - u^{-1}c(qu)\Big(\tilde{\zeta}_{m_0+2(N-\hat{M})}\mathscr{C}(u, m_0 + 2(N - \hat{M}))\\
&\qquad+ \delta_{m_0+2(N-\hat{M})-2}\hat{t}_{ps}(u, m_0 + 2(N - \hat{M}))\Big).
\end{aligned} \tag{5.24}$$

We introduce the vector,

$$\hat{\Psi}^{\hat{M}}_{m_0}(\bar{u}) = C(\bar{u}, m_0 + 2(N - \hat{M}), \hat{M})|\hat{\Omega}^N_{m_0}\rangle \tag{5.25}$$

such that at the point $m_0 + 2(N - \hat{M})$, we have the off-shell action,

$$\begin{aligned}
&t(u, m_0 - 2\hat{M} + 2N)\hat{\Psi}^{\hat{M}}_{m_0}(\bar{u})\\
&\quad= \left(\hat{\Lambda}^{\hat{M}}_d(u, \bar{u}) + u^{-1}c(qu)\delta_{m_0-2M+2N-2}\hat{\Lambda}^{\hat{M}}_{ps}(u, \bar{u})\right)\hat{\Psi}^{M}_{m_0}(\bar{u})\\
&\qquad+ \sum_{i=1}^{\hat{M}}\Big[\tilde{F}(u, u_i)\hat{E}^{\hat{M}}_d(u_i, \bar{u}) + u^{-1}c(qu)\left(\hat{\tilde{\chi}}_{m_0-2\hat{M}+2N}\hat{W}^{\hat{M}}(u_i, \bar{u}_i)\right.\\
&\qquad+ \left.\delta_{m_0-2\hat{M}+2N-2}G(u, u_i)b(q^{-1}u_i^{-2})\hat{E}^{\hat{M}}_{ps}(u_i, \bar{u}_i)\right)\Big]\hat{\Psi}^{\hat{M}}_{m_0}(\{u, \bar{u}_i\})\\
&\qquad- u^{-1}c(qu)\tilde{\zeta}_{m_0-2\hat{M}+2N}\mathscr{C}(u, m_0 - 2\hat{M} + 2N)\hat{\Psi}^{\hat{M}}_{m_0}(\bar{u})
\end{aligned} \tag{5.26}$$

where we define

$$\hat{\tilde{\chi}}_m = \hat{\chi}_m - \delta_{m-2}\hat{\rho}_m \tag{5.27}$$

and with the functions,

$$\begin{aligned}
\hat{\Lambda}^{\hat{M}}_{gd}(u, \bar{u}) &= \phi(u)\tilde{k}^+(u^{-1})\tilde{k}^-(u^{-1})\Lambda(u)f(u, \bar{u})\\
&\quad+ \phi(q^{-1}u^{-1})\tilde{k}^+(qu)\tilde{k}^-(qu)\Lambda(q^{-1}u^{-1})h(u, \bar{u}),
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
\hat{E}^{\hat{M}}_{gd}(u_i, \bar{u}_i) &= \phi(q^{-1}u_i^{-1})\phi(u_i)\Big(\tilde{k}^+(u_i^{-1})\tilde{k}^-(u_i^{-1})\Lambda(u_i)f(u_i, \bar{u}_i)\\
&\quad- \tilde{k}^+(qu_i)\tilde{k}^-(qu_i)\Lambda(q^{-1}u_i^{-1})h(u_i, \bar{u}_i)\Big),
\end{aligned} \tag{5.29}$$

$$\hat{\Lambda}_{ps}^{\hat{M}}(u,\bar{u}) = u\phi(q^{-1}u^{-1})\Big(\tilde{k}^{-}(u^{-1})\Lambda(u)f(u,\bar{u}) - \tilde{k}^{-}(qu)\Lambda(q^{-1}u^{-1})h(u,\bar{u})\Big), \tag{5.30}$$

$$\begin{aligned}\hat{E}_{ps}^{\hat{M}}(u_i,\bar{u}_i) = u_i\phi(q^{-1}u_i^{-1})\Big(&\phi(q^{-1}u_i^{-1})\tilde{k}^{-}(u_i^{-1})\Lambda(u_i)f(u_i,\bar{u}_i)\\ &+\phi(u_i)\tilde{k}^{-}(qu_i)\Lambda(q^{-1}u_i^{-1})h(u_i,\bar{u}_i)\Big),\end{aligned} \tag{5.31}$$

$$\hat{W}^{\hat{M}}(u_i,\bar{u}_i) = u\phi(q^{-1}u_i^{-1})\Big(\tilde{k}^{-}(u_i^{-1})\Lambda(u_i)f(u_i,\bar{u}_i) - \tilde{k}^{-}(qu_i)\Lambda(q^{-1}u_i^{-1})h(u_i,\bar{u}_i)\Big). \tag{5.32}$$

For $\hat{M} = N$, we can conjecture that

$$\begin{aligned}&-u^{-1}c(qu)\tilde{\zeta}_{m_0}\mathscr{C}(u,m_0)\hat{\Psi}_{m_0}^{N}(\bar{u})\\ &\quad=\Big(\hat{\Lambda}_{g}^{N}(u,\bar{u}) - u^{-1}c(qu)\delta_{m_0-2}\hat{\Lambda}_{ps}^{N}(u,\bar{u})\Big)\hat{\Psi}_{m_0}^{N}(\bar{u})\\ &\qquad+\sum_{i=1}^{N}\Big(-u^{-1}c(qu)\big(\delta_{m_0-2}G(u,u_i)b(q^{-1}u_i^{-2})\hat{E}_{ps}^{N}(u_i,\bar{u}_i) + \hat{\tilde{\chi}}_{m_0}\hat{W}^{N}(u_i,\bar{u}_i)\big)\\ &\qquad+\tilde{F}(u,u_i)\hat{E}_{g}^{N}(u_i,\bar{u}_i)\Big)\hat{\Psi}_{m_0}^{N}(\{u,\bar{u}_i\}),\end{aligned} \tag{5.33}$$

with

$$\begin{aligned}\hat{\Lambda}_{g}^{N}(u,\bar{u}) = -\kappa\tilde{\kappa}\tau\tilde{\tau}&\Big(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\tilde{\xi}\tilde{\mu}}{\xi\mu}q^{-N-1} + \frac{\xi\mu}{\tilde{\xi}\tilde{\mu}}q^{N+1}\Big)\\ &\times c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})G(u,\bar{u}),\end{aligned} \tag{5.34}$$

$$\begin{aligned}\hat{E}_{g}^{N}(u_i,\bar{u}_i) = \kappa\tilde{\kappa}\tau\tilde{\tau}&\Big(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\tilde{\xi}\tilde{\mu}}{\xi\mu}q^{-N-1} + \frac{\xi\mu}{\tilde{\xi}\tilde{\mu}}q^{N+1}\Big)\\ &\times\frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})G(u_i,\bar{u}_i),\end{aligned} \tag{5.35}$$

which gives the final result for the dynamical $\mathscr{C}$ operator, namely,

$$t(u)\hat{\Psi}_{m_0}^{N}(\bar{u}) = \hat{\Lambda}^{N}(u,\bar{u})\hat{\Psi}_{m_0}^{N}(\bar{u}) + \sum_{i=1}^{N}\tilde{F}(u,u_i)\hat{E}^{N}(u_i,\bar{u}_i)\hat{\Psi}_{m_0}^{N}(\{u,\bar{u}_i\}) \tag{5.36}$$

$$\hat{\Lambda}^{N}(u,\bar{u}) = \hat{\Lambda}_{gd}^{N}(u,\bar{u}) + \hat{\Lambda}_{g}^{N}(u,\bar{u}), \quad \hat{E}^{N}(u_i,\bar{u}_i) = \hat{E}_{gd}^{N}(u_i,\bar{u}_i) + \hat{E}_{g}^{N}(u_i,\bar{u}_i), \tag{5.37}$$

where the Bethe vector is given by

$$\hat{\Psi}_{m_0}^{N}(\bar{u}) = \mathscr{C}(u_1,m_0+2)\ldots\mathscr{C}(u_N,m_0+2N)|\hat{\Omega}_{m_0}^{N}\rangle \tag{5.38}$$

with gauge parameters (5.22) and (5.23) with $\hat{M} = N$. Due to the symmetry of the parametrisation (2.16), (2.17), the resulting on-shell eigenvalues are the same of those ones (5.17) which are obtained from the dynamical $\mathscr{B}$ operator in the off-shell case.

## 6. Some limiting cases

We consider, from the general result of the previous section, two special limits of the XXZ spin chain on the segment. Firstly, we consider the limit where the right boundary is right upper triangular and, next, cases where there exist constraints between left and right boundary couplings.

### 6.1. Limit to the right triangular boundary

Here we consider the limit $\tilde{\tau} \to 0$ from the previous results. In this limit, the $K^-$-matrix (2.3) becomes upper triangular and the off-shell action for the transfer matrix (5.16) is given by,

$$t(u)\Phi^N_{m_0}(\bar{u}) = \Lambda^N_{up}(u,\bar{u})\Phi^N_{m_0}(\bar{u}) + \sum_{i=1}^{N} \tilde{F}(u,u_i)\mathrm{E}^N_{up}(u_i,\bar{u}_i)\Phi^N_{m_0}(\{u,\bar{u}_i\}) \tag{6.1}$$

with

$$\Lambda^N_{up}(u,\bar{u}) = \Lambda^N_d(u,\bar{u}) + \Lambda^N_{gup}(u,\bar{u}), \quad \mathrm{E}^N_{up}(u_i,\bar{u}_i) = \mathrm{E}^N_d(u_i,\bar{u}_i) + \mathrm{E}^N_{gup}(u_i,\bar{u}_i) \tag{6.2}$$

where

$$\begin{aligned}\Lambda^N_d(u,\bar{u}) &= \phi(u)\tilde{k}^+(u)\big(uk^-(u)\big)\Lambda(u)f(u,\bar{u})\\ &\quad + \phi(q^{-1}u^{-1})\tilde{k}^+(q^{-1}u^{-1})\big(q^{-1}u^{-1}k^-(q^{-1}u^{-1})\big)\Lambda(q^{-1}u^{-1})h(u,\bar{u}),\end{aligned} \tag{6.3}$$

$$\begin{aligned}\mathrm{E}^N_d(u_i,\bar{u}_i) &= \phi(q^{-1}u_i^{-1})\phi(u_i)\Big(\phi(u_i)\tilde{k}^+(u_i)\tilde{k}^+(u)\big(u_ik^-(u_i)\big)\Lambda(u_i)f(u_i,\bar{u}_i)\\ &\quad - \tilde{k}^+(q^{-1}u_i^{-1})\big(q^{-1}u_i^{-1}k^-(q^{-1}u_i^{-1})\big)\Lambda(q^{-1}u_i^{-1})h(u_i,\bar{u}_i)\Big),\end{aligned} \tag{6.4}$$

and

$$\Lambda^N_{gup}(u,\bar{u}) = i\kappa\tilde{\kappa}\left(\nu_- q^{-N-1}\frac{\tilde{\xi}}{\xi} + i\frac{\kappa}{\tilde{\kappa}}\tau^2\right)c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})G(u,\bar{u}), \tag{6.5}$$

$$\begin{aligned}&\mathrm{E}^N_{gup}(u_i,\bar{u}_i)\\ &\quad = -i\kappa\tilde{\kappa}\left(\nu_- q^{-N-1}\frac{\tilde{\xi}}{\xi} + i\frac{\kappa}{\tilde{\kappa}}\tau^2\right)\frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})G(u_i,\bar{u}_i).\end{aligned} \tag{6.6}$$

The Bethe vector is given by

$$\Phi^N_{m_0}(\bar{u}) = \mathscr{B}(u_1, m_0 - 2(N-1))\dots\mathscr{B}(u_N, m_0)|\Omega\rangle \tag{6.7}$$

and with

$$\beta = \beta_{tl} = -iq^{1-m_0-2N}\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}. \tag{6.8}$$

Let us remark that the parameter $\alpha_{hw}$ does not appear in the off-shell action (6.1). Indeed, this fact is expected since the general off-shell formula (5.16) depends explicitly on $\alpha_{hw}$ only through the highest weight vector (4.2). The dynamical $\mathscr{B}$ operator is independent of $\alpha_{hw}$ and the transfer matrix, invariant by any gauge transformation, is considered to be in the form (2.10) for the limit. Thus, to obtain the limit, we only need to consider the leading term of the dynamical highest weight vector (4.2), namely

$$\lim_{\tilde{\tau}\to 0}|\Omega^N_{m_0}\rangle \sim \tilde{\tau}^{-N}|\Omega\rangle \tag{6.9}$$

which gives the highest weight vector of the diagonal case [33], given in Appendix B. Moreover, we use the limit of the $\tilde{k}^-(u)$ function,

$$\lim_{\tilde{\tau}\to 0}\left(\tilde{k}^-(u)\right) = uk^-(u) \tag{6.10}$$

as well as the limit of the constant coefficient in the new term of the eigenvalue (5.34),

$$\lim_{\tilde{\tau}\to 0}\Big(-\kappa\tilde{\kappa}\tau\tilde{\tau}\Big(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}}+\frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau}+\frac{\xi\tilde{\mu}}{\tilde{\xi}\mu}q^{N+1}+\frac{\tilde{\xi}\mu}{\xi\tilde{\mu}}q^{-N-1}\Big)\Big)$$
$$=i\kappa\tilde{\kappa}\Big(\nu_- q^{-N-1}\frac{\tilde{\xi}}{\xi}+i\frac{\kappa}{\tilde{\kappa}}\tau^2\Big). \tag{6.11}$$

**Remark 6.1.** This result can also be obtained directly from the MABA. It is important to consider this fact since, rather than a simple limit, it allows one to emphasise some characteristics which are hidden in the generic case. Firstly, we put the transfer matrix into a modified diagonal form, i.e. with only $\mathscr{A}$ and $\mathscr{D}$ operators [4], at the point $m_0+2M$

$$t(u)=t_d(u,m_0+2M)=\tilde{a}(u)\mathscr{A}(u,m_0+2M)+\tilde{d}(u)\mathscr{D}(u,m_0+2M), \tag{6.12}$$

where we fix the gauge parameters to be

$$\alpha=\alpha_d(M)=-i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}q^{m_0+2M+1},\quad \beta=\beta_d(M)=-i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}q^{-m_0-2M+1}, \tag{6.13}$$

where $M$ is an integer in $\{0,1,\dots,M\}$. From the explicit relation between the initial operators (2.8) and the dynamical one (3.7) given in (A.24) and the action of the initial operators on the highest weight vector (B.2) we can find,

$$\mathscr{A}(u,m_0)|\Omega\rangle=u^2k^+(u)\Lambda(u)|\Omega\rangle+\mathscr{B}(u,m_0-2)|\Omega\rangle \tag{6.14}$$
$$\mathscr{D}(u,m_0)|\Omega\rangle=\phi(q^{-1}u^{-1})k^+(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})|\Omega\rangle-\phi(u)\mathscr{B}(u,m_0-2)|\Omega\rangle. \tag{6.15}$$

Such actions are called modified, due to the presence of an off-diagonal term, and were already pointed out in [5,4]. They are a new feature which can appear in the context of the MABA. Then, using the string of $\mathscr{B}$ operators and the highest weight vector (B.1), we introduce the vector

$$\Phi^M_{m_0}(\bar{u})=B(\bar{u},m_0+2M,M)|\Omega\rangle, \tag{6.16}$$

and compute, from the commutation relations (3.20), (3.21) as well as from the modified off-diagonal action (6.14), (6.15), the off-shell action of the modified transfer matrix (6.12), given by

$$t(u)\Phi^M_{m_0}(\bar{u})=\Lambda^M_d(u,\bar{u})\Phi^M_{m_0}(\bar{u})+\sum_{i=1}^{M}\tilde{F}(u,u_i)\mathrm{E}^M_d(u_i,\bar{u}_i)\Phi^M_{m_0}(u,\bar{u}_i)$$
$$+\kappa^2\gamma_{m_0-1}q^{-1}u^{-1}c(qu)\mathscr{B}(u,m_0+2M-2)\Phi^M_{m_0-2}(\bar{u}) \tag{6.17}$$

with $\Lambda^M_d(u,\bar{u})$ and $\mathrm{E}^M_d(u_i,\bar{u}_i)$ are given by (6.3), (6.4) with $N\to M$ and where we use the following identities to simplify the term $\mathscr{B}(u,m-2)\Phi^M_{m-2}(\bar{u})$,

$$\sum_{i=1}^{M}\Big(g(u,u_i,m)f(u_i,\bar{u}_i)-w(u,u_i,m)h(u_i,\bar{u}_i)\phi(u_i)\Big)=\frac{\gamma_{m-2M+1}}{\gamma_{m+1}}-f(u,\bar{u}), \tag{6.18}$$

$$\sum_{i=1}^{M}\Big(n(u,u_i,m)f(u_i,\bar{u}_i)-k(u,u_i,m)h(u_i,\bar{u}_i)\phi(u_i)\Big)$$
$$=-\phi(u)\Big(\frac{\gamma_{m-2M+1}}{\gamma_{m+1}}-h(u,\bar{u})\Big). \tag{6.19}$$

For $M = N$, the action of the dynamical creation operator has an off-shell structure that we conjecture to be of the form

$$\kappa^2 \gamma_{m_0-1} q^{-1} u^{-1} c(qu) \mathscr{B}(u, m_0 - 2 + 2N) \Phi^N_{m_0-2}(\bar{u})$$
$$= \Lambda^N_{gup}(u, \bar{u}) \Phi^N_{m_0}(\bar{u}) + \sum_{i=1}^{N} \tilde{F}(u, u_i) \mathrm{E}^N_{gup}(u_i, \bar{u}_i) \Phi^N_{m_0}(\{u, \bar{u}_i\}). \tag{6.20}$$

In particular we note that $\beta_d(N) = \beta_{tl}$ which corresponds to the result we obtained by limit from the general boundary case.

**Remark 6.2.** We can also obtain by limit the cases with two upper triangular boundaries and with lower and upper triangular boundaries considered in [4].[5] For the former, we consider the limit $\kappa \to 0$ and $\tilde{\tau} \to 0$ of the $\tilde{k}^-(u)$ function (6.10), of the $\tilde{k}^+(u)$ function

$$\lim_{\kappa\to 0} \left( \tilde{k}^+(u) \right) = u^{-1} k^+(u), \tag{6.21}$$

and of the parameters

$$\lim_{\kappa\to 0, \tilde{\tau}\to 0} \left( \delta_{m_0+2M} \right) = \lim_{\kappa\to 0, \tilde{\tau}\to 0} \left( \zeta_{m_0+2M} \right) = 0, \tag{6.22}$$

$$\lim_{\kappa\to 0, \tilde{\tau}\to 0} \left( \tilde{\zeta}_{m_0+2M} \right) = \lim_{\kappa\to 0, \tilde{\tau}\to 0} \left( \chi_{m_0+2M} \right) = 0, \tag{6.23}$$

$$\lim_{\tilde{\kappa}\to 0} \left( \beta_{tl}(M) \right) \to q^{1-m_0-2M} \frac{\tilde{\kappa}}{\epsilon_-}, \tag{6.24}$$

on the off-shell action (5.11). For $m_0 = -2M$, this allows to recover the off-shell action given by Eq. (4.12) in [4]. The form of the vectors $Y$ and $\tilde{Y}$ behave, up to a factor that changes $\tilde{F}$ to $F$, as

$$Y_{up}(u, m) = \begin{pmatrix} \frac{\kappa}{\epsilon_-} q^{m+1} u^{-1}, \\ 1 \end{pmatrix}, \quad \tilde{Y}_{up}(u, m) = \begin{pmatrix} -\frac{\kappa}{\epsilon_-} q^{m+1} u^{-1}, & 1 \end{pmatrix}. \tag{6.25}$$

To recover the notation of [4] we have to choose the vectors $X$ and $\tilde{X}$ to be

$$X_{up}(u, m) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{X}_{up}(u, m) = (0, \quad 1). \tag{6.26}$$

This is allowed from the invariance of the transfer matrix by any gauge transformation. For the latter, we took the limit $\tilde{\kappa} \to 0$ of the coefficient (6.11), of the function $\tilde{k}^+(u)$

$$\lim_{\tilde{\kappa}\to 0} \left( \tilde{k}^+(u) \right) = u^{-1} k^+(u), \tag{6.27}$$

and of the gauge parameter

$$\lim_{\tilde{\kappa}\to 0} \left( \beta_{tl}(M) \right) = 0 \tag{6.28}$$

in the off-shell action (6.1) in order to recover Eq. (5.9) of [4]. The form of the vectors $Y$ and $\tilde{Y}$ behave, up to a factor that change $\tilde{F}$ to $F$, as

$$Y_{lo}(u, m) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{Y}_{lo}(u, m) = (0, \quad 1). \tag{6.29}$$

[5] At this point, and only in this remark, we remove the square on the off-diagonal boundary parameters to fit with [4].

To recover the notation of [4] we have to choose the vectors $X$ and $\tilde{X}$ to be

$$X_{lo}(u,m) = \begin{pmatrix} 1 \\ \frac{\kappa}{\epsilon_-} q^{m-1} u \end{pmatrix}, \quad \tilde{X}_{lo}(u,m) = \begin{pmatrix} -\frac{\kappa}{\epsilon_-} q^{m-1} u, & 1 \end{pmatrix}. \tag{6.30}$$

While the expressions (3.3) for scalar products and (3.4) for the closure relation are the same, the intertwining relations with the $R$-matrix (3.5), (3.6) are simpler. In fact, in these cases all the functions $\gamma$ are set to the unity. For this reason, the commutation relations among the dynamical operators will depend on the dynamical integer $m$ only through the operators, see [4].

### 6.2. Limit to the cases with constraints between right and left boundary

Let us recover previous results in the literature. From the off-shell equations (5.11) and (5.26), we note that special cases can be obtained by setting,

$$\zeta_{m_0+2M} = \delta_{m_0+2M} = \chi_{m_0+2M} = 0 \tag{6.31}$$

and

$$\tilde{\zeta}_{m_0-2\hat{M}+2N} = \delta_{m_0-2\hat{M}+2N-2} = \hat{\chi}_{m_0-2\hat{M}+2N} = 0. \tag{6.32}$$

In fact, if relations (6.31) and (6.32) are satisfied, all off-diagonal contributions to the off-shell actions (5.11) and (5.26) disappear, remaining only the usual off-shell diagonal terms, allowing the usual Bethe ansatz execution. The constraints between the boundary parameters which arise from (6.31) and (6.32) can be written as, respectively,

$$-\frac{\tilde{\kappa}\tilde{\mu}\tilde{\tau}\xi}{\kappa\mu\tau\tilde{\xi}} q^{1+2M-N} = 1 \tag{6.33}$$

and

$$-\frac{\kappa\xi\tau\tilde{\mu}}{\tilde{\kappa}\tilde{\tau}\tilde{\xi}\mu} q^{N-1-2\hat{M}} = 1. \tag{6.34}$$

If we require the relation between $M$ and $\hat{M}$ to be,

$$M + \hat{M} = N - 1 \tag{6.35}$$

and use the parametrisation,

$$\begin{gathered} \tau = e^{\frac{\theta_-}{2}} \sqrt{\frac{\kappa_-}{2}}, \quad \tilde{\tau} = e^{-\frac{\theta_-}{2}} \sqrt{\frac{\kappa_-}{2}}, \quad \kappa = i e^{-\frac{\theta_+}{2}} \sqrt{\frac{\kappa_+}{2}}, \quad \tilde{\kappa} = i e^{\frac{\theta_+}{2}} \sqrt{\frac{\kappa_+}{2}}, \\ \mu = -i e^{-\alpha_-}, \quad \tilde{\mu} = -e^{\beta_-}, \quad \xi = i e^{\alpha_+}, \quad \tilde{\xi} = -e^{-\beta_+}, \end{gathered} \tag{6.36}$$

we find that the constraints (6.33) and (6.34) reduce to the ones obtained in [29], namely

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k, \qquad k = N - 2M - 1. \tag{6.37}$$

We observe that, in this case, the gauge transformation used to construct the Bethe vector from the dynamical $\mathscr{B}$ operator is not the same that for the one used to build the Bethe vector from the dynamical $\mathscr{C}$ operator. On the order hand, if $M$ and $\hat{M}$ are related by

$$M + \hat{M} = N, \tag{6.38}$$

with the parametrisation

$$\tau = e^{-\frac{\tau_0}{2}}\sqrt{\frac{1}{2}}, \quad \tilde{\tau} = ie^{\frac{\tau_0}{2}}\sqrt{\frac{1}{2}}, \quad \kappa = e^{\frac{\bar{\tau}}{2}}\sqrt{\frac{1}{2}}, \quad \tilde{\kappa} = ie^{-\frac{\bar{\tau}}{2}}\sqrt{\frac{1}{2}},$$
$$\mu = -ie^{-\delta}, \quad \tilde{\mu} = -e^{-\zeta}, \quad \xi = -ie^{-\bar{\delta}}, \quad \tilde{\xi} = -e^{-\bar{\zeta}}, \tag{6.39}$$

we recover the constraints in [21] and we further observe, in this case, that the gauge parameters $\alpha$ and $\beta$ are the same for both the $\mathscr{B}$ and $\mathscr{C}$ cases. These values of $\alpha$ and $\beta$ correspond to the ones required to have highest and lowest weight vector (see Remark 4.3).

## 7. Conclusion

In this work, we have constructed the Bethe vector of the Heisenberg XXZ spin chain on the segment with generic boundary parameters by means of the MABA approach. The off-shell action of the transfer matrix on the Bethe vector is also given. The construction follows from the conjecture of a specific off-shell structure for the action of the so-called modified creation operator on the Bethe vector.

From the general result, we obtain by limit the case of the Heisenberg XXZ spin chain on the segment with left generic and right upper triangular boundary parameters, considered in [30] from another approach. Also, we recover the limiting case where the left and right boundary parameters are related by constraints [29,21].

The proof of the conjecture of the off-shell action of the modified creation operator on the Bethe vector, which is the key feature of the MABA, remains to be done. It could be handled in different ways, following for instance [13], whose proof is based on the analytical properties of the modified creation operator and in the fact that such operator is invertible. Other possibilities would be to consider the *off-diagonal Bethe ansatz* method presented in [10] or by means of an explicit link with the SOV approach [20].

Our results can be used to consider the scalar product of the Bethe vector. This is a crucial step to consider the correlation functions of the Heisenberg XXZ spin chain on the segment with generic boundary parameters, within the algebraic Bethe ansatz framework. This will extend the known results for the Heisenberg XXZ spin chain on the circle [23], on the segment with diagonal boundary [25] or with general constrained boundary [37] conditions. A key step will be to find a simple realisation of the scalar product between an off-shell and an on-shell Bethe vector, if possible, in terms of a unique determinant.

Finally, it is interesting to extend the MABA to other models without $U(1)$ symmetry, in particular for finite models with higher spin and higher rank symmetry algebra. In addition, it should be interesting to consider models with an infinite dimensional Hilbert space, such as spin chains with defect or the Bose gas on the segment or on the half-line.

**Note added.** After this work was completed, we became aware of the paper [11], where the Bethe vector, in a different parametrisation, is obtained by another method.

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## Appendix A. Functions and commutation relations

We use the following functions along the text,

$$b(u)=\frac{u-u^{-1}}{q-q^{-1}},\quad k^-(u)=\nu_-u+\nu_+u^{-1},$$
$$k^+(u)=\epsilon_+u+\epsilon_-u^{-1},\quad c(u)=u^2-u^{-2},\tag{A.1}$$
$$\tilde{k}^-(u)=i\tilde{\tau}\tau(\mu u+\mu^{-1}u^{-1})(\tilde{\mu}^{-1}u+\tilde{\mu}u^{-1}),$$
$$\tilde{k}^+(u)=i\tilde{\kappa}\kappa(\tilde{\xi}u+\tilde{\xi}^{-1}u^{-1})(\xi^{-1}u+\xi u^{-1}),\tag{A.2}$$
$$\phi(u)=\frac{b(q^2u^2)}{b(qu^2)},\quad G(u,v)=\frac{1}{b(u/v)b(quv)},$$
$$F(u,v)=G(u,v)\frac{b(q^2u^2)}{\phi(v)},\quad \tilde{F}(u,v)=(v/u)F(u,v),\tag{A.3}$$
$$f(u,v)=\frac{b(qv/u)b(uv)}{b(v/u)b(quv)},\quad g(u,v)=\frac{\phi(q^{-1}v^{-1})}{b(u/v)},\quad w(u,v)=-\frac{1}{b(quv)},\tag{A.4}$$
$$h(u,v)=\frac{b(q^2uv)b(qu/v)}{b(quv)b(u/v)},\quad k(u,v)=\frac{\phi(u)}{b(v/u)},\quad n(u,v)=\frac{\phi(u)\phi(q^{-1}v^{-1})}{b(quv)},\tag{A.5}$$
$$s(u,v)=\frac{\phi(q^{-1}u^{-1})}{b(v/u)b(qv^2)},\quad x(u,v)=\frac{\phi(q^{-1}u^{-1})b(qu/v)}{b(u/v)b(quv)},\tag{A.6}$$
$$y(u,v)=-\frac{1}{b(qv^2)b(quv)},\quad r(u,v)=\frac{\phi(q^{-1}u^{-1})}{b(v/u)},\quad q(u,v)=\frac{b(uv)}{b(u/v)b(quv)}.\tag{A.7}$$

Direct calculation gives the following relations,

$$g(u,v)\phi(u)k^{\pm}(u)+n(u,v)k^{\pm}(q^{-1}u^{-1})=F(u,v)\phi(q^{-1}v^{-1})\phi(v)k^{\pm}(v),\tag{A.8}$$
$$k(u,v)k^{\pm}(q^{-1}u^{-1})+w(u,v)\phi(u)k^{\pm}(u)=-F(u,v)\phi(v)k^{\pm}(q^{-1}v^{-1}).\tag{A.9}$$

From the reflection algebra (2.5), one can extract the commutations relations between the operators $\mathscr{A}$, $\mathscr{D}$, $\mathscr{C}$ and $\mathscr{B}$. To order monomials of such operators in the basis span by operator valued series

$$\mathscr{M}_{bdac}(\bar{u},\bar{v},\bar{w},\bar{x})=\mathscr{B}(\bar{u})\mathscr{D}(\bar{v})\mathscr{A}(\bar{w})\mathscr{C}(\bar{x})\tag{A.10}$$

one need the following commutation relations

$$\mathscr{A}(u)\mathscr{B}(v)=f(u,v)\mathscr{B}(v)\mathscr{A}(u)+g(u,v)\mathscr{B}(u)\mathscr{A}(v)+w(u,v)\mathscr{B}(u)\mathscr{D}(v),\tag{A.11}$$
$$\mathscr{C}(v)\mathscr{A}(u)=f(u,v)\mathscr{A}(u)\mathscr{C}(v)+g(u,v)\mathscr{A}(v)\mathscr{C}(u)+w(u,v)\mathscr{D}(v)\mathscr{C}(u),\tag{A.12}$$
$$\mathscr{D}(u)\mathscr{B}(v)=h(u,v)\mathscr{B}(v)\mathscr{D}(u)+k(u,v)\mathscr{B}(u)\mathscr{D}(v)+n(u,v)\mathscr{B}(u)\mathscr{A}(v),\tag{A.13}$$
$$\mathscr{C}(v)\mathscr{D}(u)=h(u,v)\mathscr{D}(u)\mathscr{C}(v)+k(u,v)\mathscr{D}(v)\mathscr{C}(u)+n(u,v)\mathscr{A}(v)\mathscr{C}(u),\tag{A.14}$$
$$\mathscr{C}(u)\mathscr{B}(v)=\mathscr{B}(v)\mathscr{C}(u)+s(u,v)\mathscr{A}(u)\mathscr{A}(v)+x(u,v)\mathscr{A}(v)\mathscr{A}(u)+y(u,v)\mathscr{D}(u)\mathscr{A}(v)$$
$$+r(u,v)\mathscr{A}(u)\mathscr{D}(v)+q(u,v)\mathscr{A}(v)\mathscr{D}(u)+w(u,v)\mathscr{D}(u)\mathscr{D}(v),\tag{A.15}$$
$$\mathscr{A}(u)\mathscr{D}(v)=\mathscr{D}(v)\mathscr{A}(u)+k(v,u)\big(\mathscr{B}(u)\mathscr{C}(v)-\mathscr{B}(v)\mathscr{C}(u)\big)\tag{A.16}$$

and

$$\mathscr{A}(u)\mathscr{A}(v) = \mathscr{A}(v)\mathscr{A}(u) + w(u,v)\big(\mathscr{B}(u)\mathscr{C}(v) - \mathscr{B}(v)\mathscr{C}(u)\big), \tag{A.17}$$

$$\mathscr{D}(u)\mathscr{D}(v) = \mathscr{D}(v)\mathscr{D}(u) - \phi(u)\phi(v)w(u,v)\big(\mathscr{B}(u)\mathscr{C}(v) - \mathscr{B}(v)\mathscr{C}(u)\big), \tag{A.18}$$

$$\mathscr{B}(u)\mathscr{B}(v) = \mathscr{B}(v)\mathscr{B}(u), \tag{A.19}$$

$$\mathscr{C}(u)\mathscr{C}(v) = \mathscr{C}(v)\mathscr{C}(u). \tag{A.20}$$

Let us remark that this set of relations is complete, i.e. they are isomorphic to the reflection equation. We can also define another ordering,

$$\hat{\mathscr{M}}_{cadb}(\bar{u},\bar{v},\bar{w},\bar{x}) = \mathscr{C}(\bar{u})\hat{\mathscr{A}}(\bar{v})\hat{\mathscr{D}}(\bar{w})\mathscr{B}(\bar{x}). \tag{A.21}$$

Here we give only the two most relevant relations for the ABA,

$$\hat{\mathscr{A}}(u)\mathscr{C}(v) = h(u,v)\mathscr{C}(v)\hat{\mathscr{A}}(u) + k(u,v)\mathscr{C}(u)\hat{\mathscr{A}}(v) + n(u,v)\mathscr{C}(u)\hat{\mathscr{D}}(v), \tag{A.22}$$

$$\hat{\mathscr{D}}(u)\mathscr{C}(v) = f(u,v)\mathscr{C}(v)\hat{\mathscr{D}}(u) + g(u,v)\mathscr{C}(u)\hat{\mathscr{D}}(v) + w(u,v)\mathscr{C}(u)\hat{\mathscr{A}}(v). \tag{A.23}$$

The dynamical operators (A.24) can be expressed in terms of the initial operators (2.7),

$$\begin{aligned}\mathscr{B}(u,m) &\\ &= \frac{qu}{\gamma_{m+1}}\Big(\mathscr{B}(u) + q^m\beta\big(qu\phi(q^{-1}u^{-1})\mathscr{A}(u) - u^{-1}\mathscr{D}(u)\big) - (q^m\beta)^2\mathscr{C}(u)\Big)\end{aligned} \tag{A.24}$$

$$\begin{aligned}\mathscr{A}(u,m) = \frac{qu}{\gamma_{m-1}}&\Big(\mathscr{B}(u) + q^{m-2}\beta\Big(\big(uq^{2-2m}\frac{\alpha}{\beta} - \frac{u^{-1}}{b(qu^2)}\big)\mathscr{A}(u) - u^{-1}\mathscr{D}(u)\Big)\\ &- q^{-2}\alpha\beta\mathscr{C}(u)\Big)\end{aligned} \tag{A.25}$$

$$\begin{aligned}\mathscr{D}(u,m) = \frac{qu}{\gamma_{m-1}}&\Big(q^{m-1}\beta\Big((q^{-2m}u^{-1}\frac{\alpha}{\beta} + \frac{u}{b(qu^2)})\mathscr{D}(u) - u\phi(u)\phi(q^{-1}u^{-1})\mathscr{A}(u)\Big)\\ &- \phi(u)(\mathscr{B}(u) - q^{-2}\alpha\beta\mathscr{C}(u))\Big)\end{aligned} \tag{A.26}$$

$$\mathscr{C}(u,m) = \frac{qu}{\gamma_{m-1}}\Big(q^{-2m}\alpha^2\mathscr{C}(u) - q^{-m}\alpha\big(qu\phi(q^{-1}u^{-1})\mathscr{A}(u) - u^{-1}\mathscr{D}(u)\big) - \mathscr{B}(u)\Big) \tag{A.27}$$

It allows to find the dynamical commutation relations,

$$\mathscr{B}(u,m+2)\mathscr{B}(v,m) = \mathscr{B}(v,m+2)\mathscr{B}(u,m), \tag{A.28}$$

$$\begin{aligned}\mathscr{A}(u,m+2)\mathscr{B}(v,m) &= f(u,v)\mathscr{B}(v,m)\mathscr{A}(u,m)\\ &+ g(u,v,m)\mathscr{B}(u,m)\mathscr{A}(v,m) + w(u,v,m)\mathscr{B}(u,m)\mathscr{D}(v,m),\end{aligned} \tag{A.29}$$

$$\begin{aligned}\mathscr{D}(u,m+2)\mathscr{B}(v,m) &= h(u,v)\mathscr{B}(v,m)\mathscr{D}(u,m)\\ &+ k(u,v,m)\mathscr{B}(u,m)\mathscr{D}(v,m) + n(u,v,m)\mathscr{B}(u,m)\mathscr{A}(v,m),\end{aligned} \tag{A.30}$$

$$\mathscr{C}(u,m-2)\mathscr{C}(v,m) = \mathscr{C}(v,m-2)\mathscr{C}(u,m), \tag{A.31}$$

$$\begin{aligned}\hat{\mathscr{A}}(u,m-2)\mathscr{C}(v,m) &= h(u,v)\mathscr{C}(v,m)\hat{\mathscr{A}}(u,m)\\ &+ \hat{k}(u,v,m)\mathscr{C}(u,m)\hat{\mathscr{A}}(v,m) + \hat{n}(u,v,m)\mathscr{C}(u,m)\hat{\mathscr{D}}(v,m),\end{aligned} \tag{A.32}$$

$$\begin{aligned}\hat{\mathscr{D}}(u,m-2)\mathscr{C}(v,m) &= f(u,v)\mathscr{C}(v,m)\hat{\mathscr{D}}(u,m)\\ &+ \hat{g}(u,v,m)\mathscr{C}(u,m)\hat{\mathscr{D}}(v,m) + \hat{w}(u,v,m)\mathscr{C}(u,m)\hat{\mathscr{A}}(v,m).\end{aligned} \tag{A.33}$$

The first term of the commutation relations (A.29), (A.30) and (A.32), (A.33) corresponds to the wanted terms for the ABA. They have the same form of the commutation relations for the diagonal case (A.11), (A.13) and (A.22), (A.23). For the unwanted term we have a new form that depends of $m$ and of the gauge parameters $\alpha$ and $\beta$, namely

$$g(u,v,m)=\frac{\gamma(u/v,m+1)}{\gamma_{m+1}}g(u,v),\quad w(u,v,m)=\frac{\gamma(uv,m)}{\gamma_{m+1}}w(u,v), \tag{A.34}$$

$$k(u,v,m)=\frac{\gamma(v/u,m+1)}{\gamma_{m+1}}k(u,v),\quad n(u,v,m)=\frac{\gamma(1/(uv),m+2)}{\gamma_{m+1}}n(u,v), \tag{A.35}$$

$$\hat{g}(u,v,m)=\frac{\gamma(v/u,m-1)}{\gamma_{m-1}}g(u,v),\quad \hat{w}(u,v,m)=\frac{\gamma(1/(uv),m)}{\gamma_{m-1}}w(u,v), \tag{A.36}$$

$$\hat{k}(u,v,m)=\frac{\gamma(u/v,m-1)}{\gamma_{m-1}}k(u,v),\quad \hat{n}(u,v,m)=\frac{\gamma(uv,m-2)}{\gamma_{m-1}}n(u,v). \tag{A.37}$$

## Appendix B. Representation theory for triangular boundary case

For finite dimensional representation of the quantum one-row monodromy matrix, we always have a highest weight representation [35]. For the fundamental representation we used here, the quantum one-row monodromy matrix is given by the product $R_{a1}(u/v_1)\dots R_{aN}(u/v_N)$, or by its reflected inverse, and the highest weight vector is

$$|\Omega\rangle=\otimes_{k=1}^{N}\begin{pmatrix}1\\0\end{pmatrix}_k . \tag{B.1}$$

When the right boundary is upper triangular, i.e., $\tilde{\tau}^2=0$, the vector (B.1) turns out to be a reference state also for the double-row operators (2.7). This can be easily seen by expressing the operators $\{\mathscr{A}(u),\mathscr{B}(u),\mathscr{C}(u),\mathscr{D}(u)\}$ in terms of the entries of the quantum one-row monodromy matrix, see e.g. [4] for explicit formulas. Then, from their actions on the highest weight vector (B.1) we find,

$$\mathscr{A}(u)|\Omega\rangle=k^-(u)\Lambda(u)|\Omega\rangle, \tag{B.2}$$

$$\mathscr{D}(u)|\Omega\rangle=\phi(q^{-1}u^{-1})k^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})|\Omega\rangle, \tag{B.3}$$

$$\mathscr{C}(u)|\Omega\rangle=0. \tag{B.4}$$

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