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Persistently positive inverses of perturbed M-matrices

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Abstract

A well known property of an M-matrix A is that the inverse is element-wise non-negative, which we write as $A^{-1} \geqslant 0$. In this paper we consider perturbations of M-matrices and obtain bounds on the perturbations so that the non-negative inverse persists. The bounds are written in terms of decay estimates which characterize the decay (along rows) of the elements of the inverse matrix. We obtain results for diagonal and rank-1 perturbations of symmetric tridiagonal M-matrices and rank-1 perturbations of non-symmetric tridiagonal M-matrices.

1. Introduction

Ostrowski [1] introduced, with reference to the work of Minkowski [2,3], a rich class of matrices known as M-matrices in 1937. There are many characterizations of M-matrices. Bermann and Plemmons [4] give approximately 50 different but equivalent definitions. For our purposes we will say a matrix A is a nonsingular M-matrix if and only if A is nonsingular with $a_{ij} \leqslant 0$ for $i \neq j$, $a_{ii} > 0$ and $A^{-1} \geqslant 0$. A condition which is easy to check is that a matrix A is a nonsingular M-matrix if and only if $a_{ij} \leqslant 0$ for $i \neq j$, $a_{ii} > 0$ and A is generalized strictly diagonally dominant. A matrix A is said to be generalized (strictly) diagonally dominant if there exists a diagonal matrix D with positive entries so that AD is (strictly) diagonally dominant. It is clear that a sufficient, but not necessary, condition for A to be an M-matrix is that A is strictly diagonally dominant with non-positive off-diagonal entries.

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We consider the effect of perturbations on the inverse of M-matrices. Specifically, we perturb a tridiagonal M-matrix through the addition of sub- and super-diagonals. If these diagonal perturbations are non-positive, then a sufficient condition to ensure the inverse is non-negative is obtained by imposing the required diagonal dominance property. In this paper we explore perturbations in the form of *positive* sub- and super-diagonals and ask under what conditions is the inverse of the resulting matrix non-negative.

The remainder of the paper is as follows. In Section 2 we review known results about the inverses of symmetric tridiagonal M-matrices. Specifically, we review decay estimates which characterize the behavior of the elements of the inverse. In Section 3 we use these decay estimates and the Shermann–Morrison–Woodbury formula to obtain a bound on the size of diagonal perturbations for which the non-negative inverse persists. In Section 4 we extend the results to handle perturbations of non-symmetric tridiagonal M-matrices. Section 5 contains some numerical experiments to validate our results. We conclude in Section 6 with some comments and discussion of ongoing work.

2. Inverses of symmetric tridiagonal *M*-matrices

Consider a symmetric tridiagonal M-matrix

$$M = \begin{pmatrix} a_1 & -b_2 & & & \\ -b_2 & a_2 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & a_{n-1} & -b_n \\ & & & -b_n & a_n \end{pmatrix}.$$

In accordance with the definition of a M-matrix provided above, we assume the entries a_i, b_i are non-negative and $a_i > b_i + b_{i+1}$, i.e. the matrix is strictly diagonally dominant. In all that follows we set $b_1 = b_{n+1} = 0$.

Characterizations of the inverse of banded matrices have been considered by many authors, cf. [5–8]. Concus, Golub and Meurant [9–11] have used explicit Cholesky (LU) and UL factorizations of M to detail the inverses of symmetric tridiagonal matrices. We quote their result as Lemma 1.

Lemma 1. *The entries of the inverse of M are given explicitly as*

$$M_{ij}^{-1} = b_{i+1} \cdots b_j \frac{d_{j+1} \cdots d_n}{\delta_i \cdots \delta_n} \quad \text{for all } i, \text{ and } j > i \tag{1}$$

and

$$M_{ii}^{-1} = \frac{d_{i+1} \cdots d_n}{\delta_i \cdots \delta_n} \quad \text{for all } i.$$

The quantities d_i and δ_i are given by the recurrences

$$d_n = a_n, \quad d_i = a_i - \frac{b_{i+1}^2}{d_{i+1}}, \quad i = n-1, n-2, \ldots, 1$$

and

$$\delta_1 = a_1, \quad \delta_i = a_i - \frac{b_i^2}{\delta_{i-1}}, \quad i = 2, \ldots, n,$$

respectively.

Using Lemma 1 the authors are able to quantify the rate of decay of the entries of M^{-1}. We quote this result as Lemma 2.

Lemma 2. *An element of M^{-1}, $M^{-1}_{i,j+1}$, may be bounded in terms of M^{-1}_{ij} as*

$$\hat{\rho}_j M^{-1}_{ij} < M^{-1}_{i,j+1} < \rho_j M^{-1}_{ij},$$

where

$$\rho_j = \frac{b_{j+1}}{a_{j+1} - b_{j+2}} \quad \text{and} \quad \hat{\rho}_j = \frac{b_{j+1}}{a_{j+1}}.$$

If M is strictly diagonally dominant M-matrix, we have $0 < \hat{\rho}_j < \rho_j < 1$ for each j. It is clear that this may be extended to compare any off-diagonal entry of M^{-1} to the diagonal entry in that row,

$$\hat{\rho}^{|i-j|} M^{-1}_{ii} < M^{-1}_{ij} < \rho^{|i-j|} M^{-1}_{ii}, \tag{2}$$

where we have defined

$$\rho = \max_{1 \leqslant j \leqslant n-1} \frac{b_{j+1}}{a_{j+1} - b_{j+2}} \quad \text{and} \quad \hat{\rho} = \min_{1 \leqslant j \leqslant n-1} \frac{b_{j+1}}{a_{j+1}}. \tag{3}$$

To compare M^{-1}_{ij} to an entry closer to the diagonal (along a row) we have

$$\hat{\rho}^{|k-j|} M^{-1}_{ik} \leqslant M^{-1}_{ij} \leqslant \rho^{|k-j|} M^{-1}_{ik} \quad \text{for } |j-i| > |k-i|.$$

And to bound M^{-1}_{ij} in terms of an entry further from the diagonal (along a row) we have

$$\frac{1}{\rho^{|k-j|}} M^{-1}_{ik} \leqslant M^{-1}_{ij} \leqslant \frac{1}{\hat{\rho}^{|k-j|}} M^{-1}_{ik} \quad \text{for } |j-i| < |k-i|.$$

Due to symmetry the estimates also work column wise. Care must be taken that the entries being compared are on the same side of the diagonal.

In addition, we require a bound on the diagonal entries of M^{-1}. Ostrowski [12] obtained the following bound on M^{-1}_{ii} for strictly row diagonally dominant matrices:

$$M^{-1}_{ii} \leqslant \frac{1}{|M_{ii}|(1-\xi_i)}, \quad \text{where } \xi_i = \frac{1}{M_{ii}} \sum_{j \neq i} |M_{ij}|.$$

This bounds are typically quite pessimistic. The decay estimates above give much better bounds which are easy to compute.

To this end, we let $C = M^{-1}$ and equate diagonal entries of the matrix identity $CM = I$ to give

$$-b_i c_{i,i-1} + a_i c_{ii} - b_{i+1} c_{i,i+1} = 1.$$

Using the decay estimates above we have

$$a_i c_{ii} = 1 + b_i c_{i,i-1} + b_{i+1} c_{i,i-1} \leqslant 1 + \rho(b_i + b_{i+1}) c_{ii},$$

which is satisfied if

$$c_{ii} \leqslant \frac{1}{a_i - \rho(b_i + b_{i+1})} \equiv \mu_i.$$

We introduce

$$\mu = \max_{1 \leqslant i \leqslant n} \mu_i, \tag{4}$$

as an upper bound on the diagonal entries of M^{-1}.

In the case of the tridiag$\{-1, 4, -1\}$ matrix this gives $M^{-1}_{ii} \leqslant 3/10$ as compared to Ostrowski's bound of $1/2$. The actual bound, found numerically, is approximately 0.2887.

3. Positivity subject to a perturbation

In this section we investigate diagonal perturbations of symmetric tridiagonal M-matrices. Specifically we wish to find the maximum allowable perturbation for which the non-negative inverse persists. As we will see, a bound on the perturbation may be expressed in terms of the decay estimates from the previous section.

Specifically, we consider the matrix B given by

$$B = \begin{pmatrix} a_1 & -b_2 & h & & & & \\ -b_2 & a_2 & -b_3 & h & & & \\ h & -b_3 & a_3 & -b_4 & h & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & h & -b_{n-3} & a_{n-2} & -b_{n-1} & h \\ & & & h & -b_{n-1} & a_{n-1} & -b_n \\ & & & & h & -b_n & a_n \end{pmatrix}, \tag{5}$$

where $a_i, b_i > 0$, $h \neq 0$, and the tridiagonal part of B is an M-matrix. For what values of h is the inverse of B non-negative?

If $h < 0$, then it will suffice to choose h so that B is strictly diagonally dominant (or generalized strictly diagonally dominant). If $h > 0$ then the entries of B no longer satisfy the sign pattern necessary to be an M-matrix. However, due to continuity, if h is chosen small enough then we would expect $B^{-1} \geqslant 0$. Our goal is to find a bound on h which will ensure a non-negative inverse.

3.1. A rank-1 perturbation

We begin by considering a simpler case. Suppose B is given as $B = M + E$, where M is the tridiagonal M-matrix with entries $\{-b_{i-1}, a_i, -b_i\}$ and $E = uv^{\mathrm{T}}$ is a rank one matrix. We choose u and v so that the $(1, 3)$ entry of B is h. The vectors $u = (h, 0, \ldots, 0)^{\mathrm{T}}$ and $v = (0, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$ give the correct matrix. An expression for B^{-1} follows from the Sherman–Morrison formula [13,14]

$$B^{-1} = (M + uv^{\mathrm{T}})^{-1} = M^{-1} - \frac{M^{-1}uv^{\mathrm{T}}M^{-1}}{1 + v^{\mathrm{T}}M^{-1}u}.$$

A quick calculation shows that $v^{\mathrm{T}}M^{-1}u = hM^{-1}_{31}$ and $uv^{\mathrm{T}}M^{-1}$ is a matrix whose first row is h times the third row of M^{-1} and the rest zeros.

So $B^{-1} \geqslant 0$ if

$$M^{-1} \cdot \frac{h}{1 + hM^{-1}_{31}} \cdot \begin{pmatrix} M^{-1}_{31} & M^{-1}_{32} & \cdots & M^{-1}_{3n} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \leqslant M^{-1}.$$

We will now use the decay estimates (2) to find an upper bound on h to ensure $B^{-1} \geqslant 0$. The $(1, 1)$ entry of the matrix product on the left of the above inequality is $sM_{11}^{-1}M_{31}^{-1}$ where $s = h/(1 + hM_{31}^{-1})$. So comparing the $(1, 1)$ entries we require

$$sM_{11}^{-1}M_{31}^{-1} \leqslant M_{11}^{-1}.$$

Using the decay estimates we have the following sequence of inequalities:

$$sM_{11}^{-1}M_{31}^{-1} \leqslant s\rho^2 M_{11}^{-1}M_{33}^{-1} \leqslant s\rho^2\mu M_{11}^{-1} \leqslant M_{11}^{-1}.$$

This indicates that $s \leqslant 1/\mu\rho^2$ is a sufficient requirement.

For the $(1, 2)$ entries we have to find s so that

$$sM_{11}^{-1}M_{32}^{-1} \leqslant M_{12}^{-1}.$$

In this case, the sequence of inequalities

$$sM_{11}^{-1}M_{32}^{-1} \leqslant s\rho M_{11}^{-1}M_{33}^{-1} \leqslant s\rho\mu M_{11}^{-1} \leqslant \hat{\rho} M_{11}^{-1} \leqslant M_{12}^{-1},$$

implies that $s \leqslant \hat{\rho}/\mu\rho$ is sufficient.

To compare the $(1, k)$ entries for $k \geqslant 3$ we note that $M_{3k}^{-1} > M_{1k}^{-1}$ since the M_{3k}^{-1} entries are closer to the diagonal. Taking advantage of symmetry we use the decay estimates column-wise to obtain

$$M_{3k}^{-1} \leqslant \frac{M_{1k}^{-1}}{\hat{\rho}^2}$$

and therefore

$$sM_{11}^{-1}M_{3k}^{-1} \leqslant s\frac{\mu}{\hat{\rho}^2}M_{1k}^{-1} \leqslant M_{1k}^{-1} \quad \text{if } s \leqslant \hat{\rho}^2/\mu.$$

Now consider the jth row, for $j \geqslant 2$. We want to show

$$sM_{j1}^{-1}M_{3k}^{-1} \leqslant M_{jk}^{-1} \quad \text{for all } k = 1, \dots, n.$$

If $k \leqslant j$ we compare M_{j1}^{-1} to M_{jk}^{-1}, which is closer to the diagonal along a row, and we simply compare M_{3k}^{-1} to the maximum diagonal entry. So we have for $k \leqslant j$,

$$sM_{j1}^{-1}M_{3k}^{-1} \leqslant s\mu\rho^{k-1}M_{jk}^{-1} \leqslant M_{jk}^{-1} \quad \text{if } s \leqslant \frac{1}{\mu\rho^{k-1}}.$$

If $k > j$ then we compare M_{3k}^{-1} to M_{jk}^{-1}, which is closer to the diagonal along a column, and we compare M_{j1}^{-1} to the maximum diagonal entry. So we have

$$sM_{j1}^{-1}M_{3k}^{-1} \leqslant s\mu\rho^{j-3}M_{jk}^{-1} \quad \text{if } s \leqslant \frac{1}{\mu\rho^{j-3}}.$$

All the bounds on s obtained above are necessary to guarantee the non-negativity of specific elements of the inverse of the perturbed M-matrix B. The following condition on h ensures that all the required restrictions on h above are satisfied and hence is sufficient to ensure a non-negative inverse of the perturbed M-matrix.

Theorem 1. *Assume M is a symmetric tridiagonal M-matrix which is diagonally dominant. Let $u = (h, 0, 0, \dots, 0, 0)^{\mathrm{T}}$ and $v = (0, 0, 1, 0, \dots, 0)^{\mathrm{T}}$ form the rank-1 matrix uv^{T}. The matrix $B = M + uv^{\mathrm{T}}$ has a non-negative inverse (element-wise) if*

$$h \leqslant \frac{\hat{\rho}^2}{\mu},$$

where $\hat{\rho}$ and μ are defined in (3) *and* (4) *respectively*.

As an illustration consider the 40×40 M-matrix where $b_i = 1$ for all i and $a_i = 4$ for all i. Our bound implies $h \leqslant 0.208333$ to ensure positivity of the inverse. The actual largest value of h for which the inverse will be positive, is 0.25. We compute the actual value by numerically requiring $(M + uv^{\mathrm{T}})^{-1} \geqslant 0$ element-wise.

3.2. Higher rank perturbations

To generalize the result of the previous section we consider a perturbation given by $E = UV$ where $U = hI$ and V is a matrix of zeros except for a second super-diagonal of ones. The generalized Sherman–Morrison–Woodbury formula says, if $I + VM^{-1}U$ is nonsingular then

$$(M + UV)^{-1} = M^{-1} - M^{-1}U(I + VM^{-1}U)^{-1}VM^{-1}.$$

Rows 1 through $n-2$ of VM^{-1} are just the last $n-2$ rows of M^{-1}. Rows $n-1$ and n of VM^{-1} consist entirely of zeros.

To ensure $B^{-1} = (M + UV)^{-1}$ is non-negative we require

$$M^{-1}U(I + VM^{-1}U)^{-1}VM^{-1} \leqslant M^{-1}.$$

Under a suitable assumption[1] on h it is possible to show using a Neumann expansion that

$$(I + VM^{-1}U)^{-1} = (I + hVM^{-1})^{-1} \leqslant I. \tag{6}$$

From this we may deduce

$$M^{-1}U(I + VM^{-1}U)^{-1}VM^{-1} \leqslant M^{-1}UVM^{-1}.$$

We therefore wish to find a bound on h so that

$$hM^{-1}\begin{pmatrix} M^{-1}_{31} & M^{-1}_{32} & \cdots & M^{-1}_{3n} \\ M^{-1}_{41} & M^{-1}_{42} & \cdots & M^{-1}_{4n} \\ \vdots & & \ddots & \vdots \\ M^{-1}_{n1} & M^{-1}_{n2} & \cdots & M^{-1}_{nn} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \leqslant M^{-1}. \tag{7}$$

Computing directly we find the (ij)th entry of the matrix on the left hand side of (7) is given by

$$\sum_{k=1}^{n-2} hM^{-1}_{ik}M^{-1}_{k+2,j}.$$

Hence, with our findings from Section 3, we arrive at the following result.

Theorem 2. *Assume M is a symmetric tridiagonal M-matrix which is diagonally dominant. Let $U = hI$ and V be a matrix with ones in the first super-diagonal and zeros elsewhere. The matrix $M + UV$ has a non-negative inverse* (*element-wise*) *if*

[1] The bound on h obtained in Theorem 2 is sufficient to ensure (6) is valid.

$$h \leqslant \frac{\hat{\rho}^2}{(n-2)\mu},$$

where $\hat{\rho}$ *and* μ *are defined in* (3) *and* (4) *respectively*.

3.3. A symmetric perturbation

We now consider a symmetric rank 2 perturbation, that is we only perturb the (1, 3) and (3, 1) entries of M by a quantity h. Let V be the matrix of zeros except for ones in the (1, 3) and (3, 1) positions. Once again the Shermann–Morrison–Woodbury formula guarantees that $B^{-1} \geqslant 0$ if

$$hM^{-1}(I + hVM^{-1})^{-1}VM^{-1} \leqslant M^{-1}. \tag{8}$$

The structure of $I + hVM^{-1}$ again allows us to deduce

$$hM^{-1}(I + hVM^{-1})^{-1}VM^{-1} \leqslant hM^{-1}VM^{-1}.$$

In this case the (i, j) entry of $hM^{-1}VM^{-1}$ is given by

$$h(M_{i1}^{-1}M_{3j}^{-1} + M_{i3}^{-1}M_{1j}^{-1}).$$

Therefore, inequality (8) will be satisfied if

$$h(M_{i1}^{-1}M_{3j}^{-1} + M_{i3}^{-1}M_{1j}^{-1}) \leqslant M_{ij}^{-1}.$$

Applying the decay estimates as in the previous sections we deduce the following result.

Theorem 3. *Assume M is a symmetric tridiagonal M-matrix which is diagonally dominant. Let $U = hI$ and V be the matrix with ones in the* (1, 3) *and* (3, 1) *positions and zeros elsewhere. The matrix $B = M + UV$ has a non-negative inverse* (*element-wise*) *if*

$$h \leqslant \frac{\hat{\rho}^2}{2\mu},$$

where $\hat{\rho}$ *and* μ *are defined in* (3) *and* (4) *respectively*.

3.4. Extensions

We are now in a position to return to the matrix B of (5). Writing B as $M + E$ we see that E is a combination of the perturbations discussed in the previous two sections. It will come as no surprise that a bound on h to ensure $B^{-1} \geqslant 0$ may be derived in a similar way to obtain the following result.

The matrix B from Eq. (5) *will have a non-negative inverse if*

$$h \leqslant \frac{\hat{\rho}^2}{2(n-2)\mu}.$$

This is a sufficient but not necessary condition.

4. Non-symmetric case

Our development does not depend in any way on the symmetry of the matrix M. M is only required to be a tridiagonal M-matrix. In the non-symmetric case, the decay estimates of Nabben [15,16] and Peluso and Politi [17] assist in extending the result.

Let

$$M = \begin{pmatrix} a_1 & -c_1 & & & \\ -b_1 & a_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-2} & a_{n-1} & -c_{n-1} \\ & & & -b_{n-1} & a_n \end{pmatrix} \tag{9}$$

be a non-symmetric M-matrix with c_i, b_i non-negative and $a_i > 0$. We will assume A is strictly diagonally dominant, that is $a_1 > c_1$, $a_n > b_{n-1}$ and $a_i > b_{i-1} + c_i$ for $i = 2, \dots, n-1$.

We define the following quantities, for $i = 1, \dots, n$

$$\tau_i = \frac{c_i}{a_i - b_{i-1}}, \quad \omega_i = \frac{b_{i-1}}{a_i - c_i},$$
$$\delta_i = \frac{c_i}{a_i + b_{i-1}}, \quad \gamma_i = \frac{b_{i-1}}{a_i + c_i},$$

where we set $b_0 = c_n = 0$ for consistency.

The following result was given in [16].

Lemma 3. *The elements of $C = M^{-1}$ satisfy*

$$\delta_j c_{i+1,j} \leqslant c_{ij} \leqslant \tau_j c_{i+1,j}, \quad i = 1, \dots, j-1$$

and

$$\gamma_j c_{i-1,j} \leqslant c_{ij} \leqslant \omega_j c_{i-1,j}, \quad i = j+1, \dots, n.$$

Peluso and Politi [17] point out that if M is an M-matrix the lower bounds δ_i and γ_i may be replaced by the tighter bounds

$$\delta_i = \frac{c_i}{a_i} \quad \text{and} \quad \gamma_i = \frac{b_{i-1}}{a_i}.$$

To obtain a bound on the perturbation for the non-symmetric case we proceed in a manner similar to Section 3.

For purposes of illustration we consider the inverse of $B = M + uv^{\mathrm{T}}$ where M is given in (9) and u and v are chosen so that $B_{1,3} = h$. In the symmetric case we were able to rely on estimates which quantified the decay both along rows and columns. For nonsymmetric matrices M which are row diagonally dominant Theorem 3 quantifies decay along columns only.

As in Section 3, B^{-1} will be non-negative if

$$sM^{-1}_{j1}M^{-1}_{3k} \leqslant M^{-1}_{jk} \quad \text{for } j, k = 1, \dots, n, \tag{10}$$

where $s = h/(1 + hM^{-1}_{31})$.

Using the decay estimates from Theorem 3 we find the following bounds on s which ensure that the first row of the matrix inequality (10) is satisfied:

B^{-1}_{1k}	$k=1$	$k=2$	$k\geqslant 3$
$\max s$	$\frac{1}{\omega^2\mu}$	$\frac{\delta}{\mu\omega}$	$\frac{\delta^2}{\mu}$

Here ω and δ are defined as

$$\omega=\max_{1\leqslant i\leqslant n}\frac{b_{i-1}}{a_i-c_i}\quad\text{and}\quad\delta=\min_{1\leqslant i\leqslant n}\frac{c_i}{a_i}. \tag{11}$$

As in the symmetric case no further restriction is imposed on s from rows 2 through n. Since M is diagonally dominant, all the quantities τ_i, ω_i, δ_i and γ_i are less than one in magnitude. This implies $\delta<\frac{1}{\omega}$ and we have the following result.

Theorem 4. *Assume A is a non-symmetric tridiagonal M-matrix which is row diagonally dominant. Let $u=(h,0,0,\ldots,0)^{\mathrm{T}}$ and $v=(0,0,1,0,\ldots,0)^{\mathrm{T}}$ form the rank-1 matrix uv^{T}. The matrix $B=M+uv^{\mathrm{T}}$ has a non-negative inverse (element-wise) if*

$$h\leqslant\frac{\delta^2}{\mu},$$

where δ is defined in (11) and μ is a bound on the diagonal entries of M^{-1}.

The other results of Section 3 (higher rank and symmetric perturbations) may be obtained by a similar analysis – although the details become quite tedious.

5. Numerical results

We present a number of simple experiments to illustrate the behavior of M matrices under perturbations.

An irreducible M-matrix has an inverse with all positive elements. One might be tempted to conclude, that the size of the smallest element of M^{-1} is an indicator of the extent to which M may be perturbed and still retain the M-matrix property.

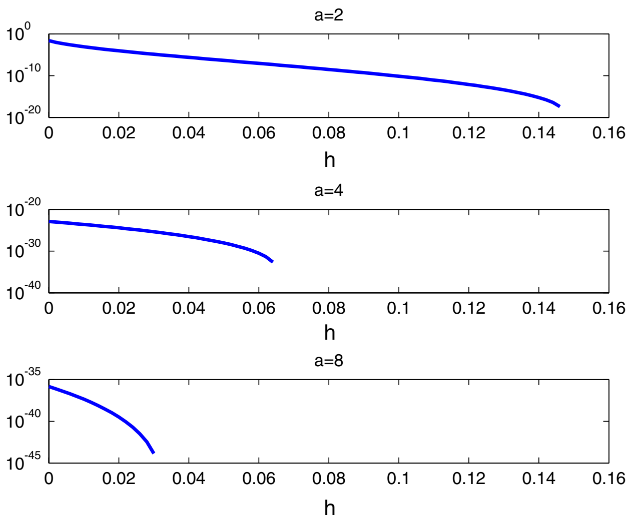

Fig. 1. Size of smallest element in $(M+hV)^{-1}$.

This, however, is not the case. To illustrate, let

$$M(x) = \begin{pmatrix} \sqrt{2}+x & -1 & 0 \\ -1 & \sqrt{2}+x & -1 \\ 0 & -1 & \sqrt{2}+x \end{pmatrix},$$

clearly $M(0)$ is singular, the following is the explicit expression for the inverse. From this it is easy to see that for positive x the matrix is an M-matrix, whereas for negative x it is not.

$$M(x)^{-1} = \begin{bmatrix} \frac{1+2\sqrt{2}x+x^2}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{\sqrt{2}+x}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{1}{x\left(4+3\sqrt{2}x+x^2\right)} \\ \frac{\sqrt{2}+x}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{\left(\sqrt{2}+x\right)^2}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{\sqrt{2}+x}{x\left(4+3\sqrt{2}x+x^2\right)} \\ \frac{1}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{\sqrt{2}+x}{x\left(4+3\sqrt{2}x+x^2\right)} & \frac{1+2\sqrt{2}x+x^2}{x\left(4+3\sqrt{2}x+x^2\right)} \end{bmatrix}.$$

As $x \to 0+$, the elements of the inverse become arbitrarily large, whereas the M-matrix property of $M(x)$ becomes more susceptible to perturbations.

In our experiments we choose a matrix dimension of $N = 40$, and we work with the matrix

$$M = \begin{pmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{pmatrix}.$$

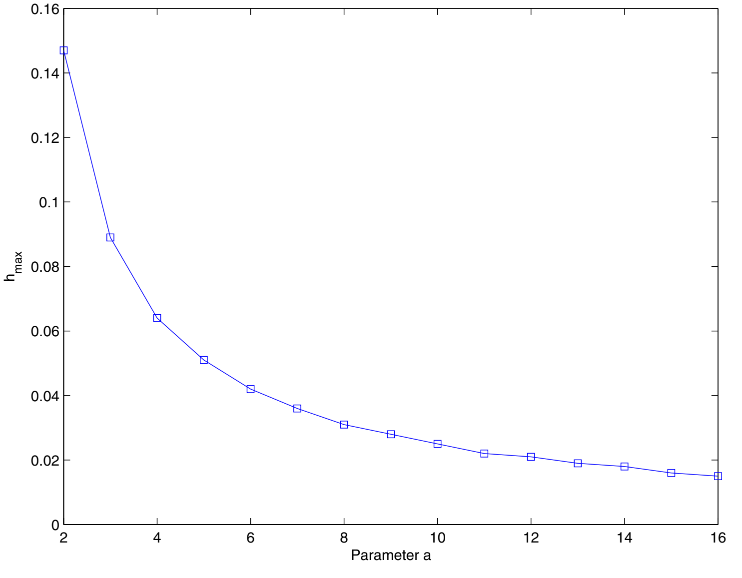

Fig. 2. Largest perturbation $h_{\max}$ as a function of a ($N = 40$).

We consider the perturbed matrix $M + hV$ where V is the matrix with all ones along the second sub- and super-diagonals and h is a scalar. The first plot, Fig. 1, shows the size of the smallest element in $(M + hV)^{-1}$ as a function of h for three values of a, namely $a = 2, 4, 8$. No line is plotted for values of h for which $M + hV$ has negative elements. It is evident that for larger a one can only perturb with smaller values of h.

Fig. 2 shows the largest allowable perturbation h as a function of the parameter a for $N = 40$. As a is increased the largest allowable perturbation in the first super- and sub-diagonals decrease. This is expected since for fixed N the upper bound on h decays like $\hat{\rho}^2/\mu$ which for this matrix scales like $1/a$.

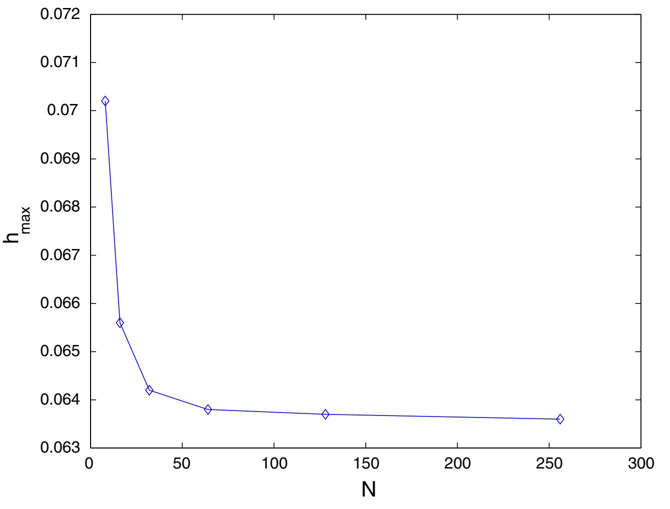

Fig. 3. Actual maximum perturbation, h as a function of N ($a = 4$).

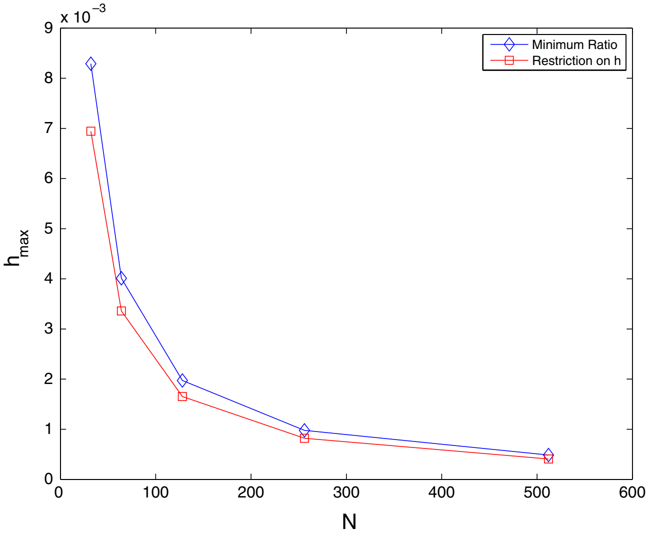

Fig. 4. Comparison of the largest perturbation $h_{\max}$ as a function of N ($a = 4$).

The actual largest allowable h for which the inverse is non-negative is shown in Fig. 3 as a function of N for $a = 4$. This bound was found numerically by searching for the maximum value of h which ensures the inverse of the perturbed matrix is non-negative. This result was confirmed by two approaches: the use of higher-precision numerical routines (up to 128 digits) and by adding an appropriately scaled random matrix to the perturbed matrix and re-running the experiment.

Fig. 4 illustrates the largest allowable h as a function of N for $a = 4$ computed using our analysis in two ways. The upper curve (with diamond markers) represents the maximum allowable perturbation as computed numerically by enforcing $hM^{-1}VM^{-1} \leqslant M^{-1}$ element-wise. The lower curve (with square markers) is the upper bound of h allowed by the result of Section 3.4. This figure demonstrates that our bound on h obtained from $hM^{-1}VM^{-1} \leqslant M^{-1}$ is quite good. A comparison with Fig. 3, however, demonstrates that although we have theoretically found a sufficient condition to ensure a non-negative inverse the requirement is far from necessary. The true bound (Fig. 3) shows an initial decay in the maximum perturbation as the size of the matrix (and the number of perturbed entries) is increased, however, there is no continued scaling with n as our theoretical bounds indicate.

6. Conclusions

We have established sufficient upper bounds on constant super- and sub-diagonal perturbations of symmetric tridiagonal M-matrices which ensure the non-negative inverse persists. A similar bound is presented for a rank-1 perturbation of a general non-symmetric tridiagonal M-matrix. These bounds depend on the decay rate of the entries of the inverse.

There are several possible improvements which are the subject of ongoing research. If the tridiagonal M-matrix is non-constant then replacing the decay estimates ρ_j and $\hat{\rho}_j$ by their maximum and minimum values (respectively) will certainly effect the tightness of the bounds. The decay estimates may also be improved in an iterative fashion [16] which will yield tighter estimates. As shown in the previous section, numerically there is a n independent bound on the maximum allowable positive perturbation of an M-matrix. This bound is not reflected in our current analysis and is the focus of current investigation. Generalizations of these results by considering more general perturbations and extensions involving perturbations of general banded matrices are also underway.

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