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New characterizations of classical orthogonal polynomials

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ABSTRACT

Classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel are characterized as the only orthogonal polynomials (up to a linear change of variable) such that
(i) (Bochner) they satisfy a second order differential equation of the form

$$\ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x);$$

and
(ii) (Hahn) their derivatives of any fixed order are also orthogonal.

Here, we give several new characterizations of classical orthogonal polynomials including extensions of the above two characterizations.

1. INTRODUCTION

In 1929, Bochner [2] classified all polynomial solutions of a second-order Sturm–Liouville type differential equation

$$L_2[y](x) = \ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda_n y(x), \tag{1.1}$$

where $\ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20} \not\equiv 0$ and $\ell_1(x) = \ell_{11}x + \ell_{10}$ are polynomials independent of n and

$$\lambda_n = n(n-1)\ell_{22} + n\ell_{11} \tag{1.2}$$

is the eigenvalue parameter depending on $n = 0, 1, \ldots$. He showed that up to a linear change of variable, the only polynomial systems that arise as eigenfunctions of the differential equation (1.1) are the

(a) Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ $(\alpha, \beta, \alpha+\beta+1 \notin \{-1,-2,\ldots\})$;
(b) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ $(\alpha \notin \{-1,-2,\ldots\})$;
(c) Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$;
(d) Bessel polynomials $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ $(\alpha \notin \{0,-1,-2,\ldots\}$ and $\beta \neq 0)$;
(e) $\{x^n\}_{n=0}^{\infty}$.

The orthogonality of the Jacobi polynomials for α and $\beta > -1$, the Laguerre polynomials for $\alpha > -1$, and the Hermite polynomials was known long before Bochner's work. They are all orthogonal relative to some positive-definite moment functionals. The orthogonality of Jacobi polynomials for α or $\beta < -1$, Laguerre polynomials for $\alpha < -1$, and Bessel polynomials was first observed by Krall [9]. They are all orthogonal relative to some regular moment functionals (see [10] and [16]). It is easy to see that the polynomial system $\{x^n\}_{n=0}^{\infty}$ in case (e) above cannot be orthogonal. Even though, Bochner [2] did not mention the orthogonality of the polynomial systems that he found, he thus implicitly classified all orthogonal polynomial solutions of the differential equation (1.1). In fact, Lesky [13] showed that Jacobi polynomials for α and $\beta > -1$, Laguerre polynomials for $\alpha > -1$, and Hermite polynomials are essentially the only orthogonal polynomials relative to some positive-definite moment functionals, that satisfy the differential equation (1.1). These four orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel are now known as classical orthogonal polynomials.

Besides Bochner's characterization, there are many properties common to all classical orthogonal polynomials (see Al-Salam [1] for an excellent survey of characterizations of various kinds of orthogonal polynomials including classical orthogonal polynomials). For example, the derivatives of any classical orthogonal polynomials are also orthogonal. Conversely, Hahn [4] showed that the only orthogonal polynomials, whose derivatives are also orthogonal, are the classical orthogonal polynomials. In fact, Hahn [4] considered only orthogonal polynomials relative to positive-definite moment functionals (this result appeared also in Sonine [17]) and Krall [9] extended the result to the general orthogonal polynomials (see also [6] and [18]). Later, Hahn [5] extended his result by showing that the only orthogonal polynomials, whose derivatives of any fixed order are also orthogonal, must be classical orthogonal polynomials (see also [7]).

In this work, we obtain several new characterizations of classical orthogonal polynomials. In particular, we extend Hahn's result by showing that the only orthogonal polynomials, whose derivatives of any fixed order are quasi-orthogonal, must be classical orthogonal polynomials. We also extend Bochner's result by characterizing classical orthogonal polynomials through differential equations of higher order.

2. PRELIMINARIES

All polynomials in this work are assumed to be real polynomials in the real variable x and we let $\mathcal{P}$ be the space of all these real polynomials. We denote the degree of a polynomial $\psi(x)$ by $\deg(\psi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$. Note that a PS forms a basis of $\mathcal{P}$. We call any linear functional σ on $\mathcal{P}$ a moment functional and denote its action on a polynomial $\psi(x)$ by

$$\langle \sigma, \psi \rangle.$$

In particular, we call

$$\{(\sigma)_n := \langle \sigma, x^n \rangle\}_{n=0}^{\infty}$$

the moments of σ. Any PS $\{P_n(x)\}_{n=0}^{\infty}$ determines a unique sequence of moment functionals $\{u_n\}_{n=0}^{\infty}$, called the dual sequence of $\{P_n(x)\}_{n=0}^{\infty}$ (Maroni [15]), by the conditions

$$(2.1) \qquad \langle u_n, P_m \rangle = \delta_{mn} \quad (m \text{ and } n \geq 0),$$

where δ_{mn} is the Kronecker delta function. In particular, we call u_0 the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Definition 2.1 (cf. [14]). We call a PS $\{P_n(x)\}_{n=0}^{\infty}$ a quasi-orthogonal polynomial system (QOPS) (respectively, an orthogonal polynomial system (OPS)) if there is a non-zero moment functional σ such that

$$(2.2) \qquad \langle \sigma, P_m P_n \rangle = K_n \delta_{mn} \quad (m \text{ and } n \geq 0),$$

where K_n are real (respectively, non-zero real) constants. In either case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a QOPS or an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

For example, $\{x^n\}_{n=0}^{\infty}$ is a QOPS (but not an OPS) relative to the Dirac moment functional δ, defined by

$$\langle \delta, x^n \rangle = \delta_{n0} \quad (n \geq 0).$$

Note that if $\{P_n(x)\}_{n=0}^{\infty}$ is a QOPS, then its orthogonalizing moment functional σ must be a non-zero constant multiple of the canonical moment functional u_0 of the PS $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n(x)\}_{n=0}^{\infty}$ is a QOPS relative to u_0.

We say that a moment functional σ is regular or quasi-definite (respectively, positive-definite) if its moments $\{(\sigma)_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$(2.3) \qquad \Delta_n(\sigma) := \det[(\sigma)_{i+j}]_{i,j=0}^{n} \neq 0 \quad (\text{respectively, } \Delta_n(\sigma) > 0)$$

for every $n \geq 0$. It is well known (see Chapter 1 in Chihara [3]) that a moment functional σ is regular if and only if there is an OPS relative to σ.

For a moment functional σ and a polynomial $\phi(x)$, we let σ', the derivative of σ, and $\phi\sigma$, the multiplication of σ by $\phi(x)$ to be the moment functionals defined by

$$(2.4) \qquad \langle \sigma', \psi \rangle = -\langle \sigma, \psi' \rangle \quad (\psi \in \mathcal{P})$$

and

(2.5) $\langle \phi\sigma, \psi \rangle = \langle \sigma, \phi\psi \rangle \quad (\psi \in \mathcal{P})$.

Then we have the following Leibniz rule:

(2.6) $(\phi\sigma)' = \phi'\sigma + \phi\sigma'$

and $\sigma' = 0$ if and only if $\sigma = 0$.

The proof of the following lemma can be found in [14; Proposition 2.2].

Lemma 2.1. *Let σ be a regular moment functional and $\{P_n(x)\}_{n=0}^{\infty}$ an OPS relative to σ. Then we have*

(i) *for any polynomial $\phi(x)$, $\phi(x)\sigma = 0$ if and only if $\phi(x) \equiv 0$;*

(ii) *for any moment functional τ and any integer $k \geq 0$, $\langle \tau, P_n \rangle = 0$ for $n > k$ if and only if $\tau = \psi(x)\sigma$ for some polynomial $\psi(x)$ of degree $\leq k$.*

Lemma 2.2 (Maroni [15]). *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS and $\{u_n\}_{n=0}^{\infty}$ the dual sequence of $\{P_n(x)\}_{n=0}^{\infty}$. Then for any moment functional τ and any integer $k \geq 0$, the following two statements are equivalent.*

(i) *$\langle \tau, P_k \rangle \neq 0$ and $\langle \tau, P_n \rangle = 0$ for $n > k$.*

(ii) *There exist real constants $\{e_j\}_{j=0}^{k}$ such that $e_k \neq 0$ and*

(2.7) $$\tau = \sum_{j=0}^{k} e_j u_j.$$

Proof. See Lemma 1.1 in [15]. □

Lemma 2.3 (Maroni [15]). *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ the dual sequences of the PS's $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x) = (1/(n+1))P'_{n+1}(x)\}_{n=0}^{\infty}$, respectively. Then, we have*

(2.8) $v'_n = -(n+1)u_{n+1} \quad (n \geq 0)$.

Proof. Since $\langle v'_n, P_m \rangle = -\langle v_n, P'_m \rangle = -m\langle v_n, Q_{m-1} \rangle = -m\delta_{n,m-1}$ for n and $m \geq 0$ ($Q_{-1}(x) \equiv 0$), we have (2.8) by Lemma 2.2. □

Proposition 2.4 (Maroni [15]). *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS and $\{u_n\}_{n=0}^{\infty}$ the dual sequence of $\{P_n(x)\}_{n=0}^{\infty}$. Then the following two statements are equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is an OPS.*

(ii) *For each $n \geq 0$, there is a non-zero real constant C_n such that*

(2.9) $u_n = C_n P_n(x) u_0$.

Proof. See Proposition 1.1 in [15]. □

We call an OPS $\{P_n(x)\}_{n=0}^{\infty}$ a classical OPS if for each $n \geq 0$, $P_n(x)$ satisfies the second order differential equation (1.1). As mentioned in the introduction, there are essentially only four distinct classical OPS's of Jacobi, Laguerre, Hermite, and Bessel polynomials.

Proposition 2.5. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then, the following statements are all equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS.*

(ii) *$\{P_n'(x)\}_{n=1}^{\infty}$ is an OPS.*

(iii) *$\{P_n'(x)\}_{n=1}^{\infty}$ is a QOPS.*

(iv) *There are polynomials $\ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20} \not\equiv 0$ and $\ell_1(x) = \ell_{11}x + \ell_{10}$ with $\ell_{11} \neq 0$ such that σ satisfies*

$$(2.10) \quad (\ell_2\sigma)' - \ell_1\sigma = 0.$$

Proof. See Theorem 2.1 in [11] and Theorem 3.1 in [14]. □

The equivalence of the statements (i) and (ii) in Proposition 2.5 was first proved by Hahn [4] and Krall [9] (see also [6], [17], and [18]).

Remark. As an immediate consequence of Proposition 2.5, we obtain: if $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS satisfying the differential equation (1.1), then $\{P_n'(x)\}_{n=1}^{\infty}$ is also a classical OPS satisfying the differential equation

$$\ell_2 y'' + (\ell_1 + \ell_2')y' = (\lambda_n - \ell_{11})y.$$

By induction, for any integer $r \geq 1$, $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ is also an (classical) OPS. Conversely, Hahn [5] proved that if both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ are OPS's relative to positive-definite moment functionals for some integer $r \geq 1$, then $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS (see also [7]).

Definition 2.2 (Maroni [14]). A moment functional σ is said to be semi-classical if

(i) σ is regular

and

(ii) there is a pair of polynomials $(\phi(x), \psi(x)) \neq (0,0)$ such that

$$(2.11) \quad (\phi\sigma)' + \psi\sigma = 0.$$

For any semi-classical moment functional σ, we call

$$(2.12) \quad s := \min\{\max(\deg(\phi) - 2, \deg(\psi) - 1)\}$$

the class number of σ, where the minimum is taken over all pairs of polynomials $(\phi, \psi) \neq (0,0)$ satisfying the equation (2.11). In this case, we call σ a semi-classical moment functional of class s and an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ is called a semi-classical OPS of class s.

We can restate the equivalence of the statements (i) and (iv) in Proposition 2.5 as: an OPS is a classical OPS if and only if it is a semi-classical OPS of class 0.

Lemma 2.6 (Maroni [15]). *Let σ be a semi-classical moment functional satisfying*

$$
(2.13) \quad \begin{cases} (\phi_1\sigma)' + \psi_1\sigma = 0 & (s_1 := \max(t_1 - 2, p_1 - 1)) \\ (\phi_2\sigma)' + \psi_2\sigma = 0 & (s_2 := \max(t_2 - 2, p_2 - 1)), \end{cases}
$$

where $t_j = \deg(\phi_j)$ and $p_j = \deg(\psi_j)$, $j = 1, 2$. Let $\phi(x)$ be a common factor of $\phi_1(x)$ and $\phi_2(x)$ of the highest degree. Then, there is a polynomial $\psi(x)$ such that

$$
(\phi\sigma)' + \psi\sigma = 0,
$$

where $s := \max(\deg(\phi) - 2, \deg(\psi) - 1) = s_1 - t_1 + \deg(\phi) = s_2 - t_2 + \deg(\phi)$.

Proof. See Lemma 3.1 in [15]. □

Proposition 2.7. *Let σ be a semi-classical moment functional of class s. If (ϕ, ψ) and (ϕ_1, ψ_1) are pairs of non-zero polynomials satisfying the equation (2.11) and if $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$, then $\phi_1(x)$ is divisible by $\phi(x)$.*

Proof. Let $\alpha(x)$ be a common factor of $\phi(x)$ and $\phi_1(x)$ of the highest degree. Then by Lemma 2.6, there is a polynomial $\beta(x)$ such that

$$
(\alpha\sigma)' + \beta\sigma = 0
$$

and $s_0 := \max(\deg(\alpha) - 2, \deg(\beta) - 1) = s - \deg(\phi) + \deg(\alpha)$. Since $s_0 \geq s$, $\deg(\alpha) \geq \deg(\phi)$ so that $\alpha(x) = c\phi(x)$ for some non-zero constant c. Hence, $\phi(x)$ must divide $\phi_1(x)$. □

3. MAIN THEOREMS

We now consider a linear differential equation of order $N \geq 1$ of the form

$$
(3.1) \qquad L_N[y](x) = \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \mu_n y(x),
$$

where $\ell_i(x) = \sum_{j=0}^{i} \ell_{ij} x^j$ are polynomials of degree $\leq i$ (independent of n), $\ell_N(x) \not\equiv 0$, and

$$
(3.2) \qquad \mu_n = n\ell_{11} + n(n-1)\ell_{22} + \cdots + n(n-1)\cdots(n-N+1)\ell_{NN}.
$$

In 1938, Krall [8] found a necessary and sufficient condition for the differential equation (3.1) to have an OPS as solutions. Recently, Kwon, Littlejohn, and Yoo [12] found a new and simple proof of Krall's theorem as well as some other equivalent conditions.

Proposition 3.1. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then the following statements are all equivalent.*

(i) *For each $n \geq 0$, $P_n(x)$ satisfies the differential equation* (3.1).

(ii) *The moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ satisfy*

$$
(3.3) \qquad S_k(m) := \sum_{i=2k+1}^{N} \sum_{j=0}^{i} \binom{i-k-1}{k} \frac{(m-2k-1)!}{(i-2k-1)!} \ell_{i,i-j}(\sigma)_{m-j} = 0
$$

for $k = 0, 1, \ldots, [(N-1)/2]$ *and* $m = 2k+1, 2k+2, \ldots$, *where* $[x]$ *is the integer part of a real number* x.

(iii) *There are* $r := [(N+1)/2]$ *moment functionals* $\{\tau_i\}_{i=1}^r$ *such that* $\tau_r \neq 0$ *and*

$$\sum_{i=1}^{r} \langle \tau_i, P_m^{(i)} P_n^{(i)} \rangle = M_n \delta_{mn} \quad (m \text{ and } n \geq 0), \tag{3.4}$$

where M_n *are real constants.*

(iv) *There are* $r := [(N+1)/2]$ *moment functionals* $\{\tau_i\}_{i=1}^r$ *such that* $\tau_r \neq 0$ *and*

$$L_{2r}(\phi)\sigma = \sum_{i=1}^{r} (-1)^i [\phi^{(i)} \tau_i]^{(i)} \tag{3.5}$$

for every polynomial $\phi(x)$.

Moreover, in this case, $N = 2r$ *must be even and the moment functionals* σ *and* $\{\tau_i\}_{i=1}^r$ *are related by the equations*

$$\ell_k(x)\sigma = \sum_{[(k+1)/2]}^{\min(r,k)} (-1)^i \binom{i}{k-i} \tau_i^{(2i-k)}, \quad k = 1, 2, \ldots, 2r. \tag{3.6}$$

Proof. See Theorem 2.4, Theorem 3.1, and Theorem 3.2 in [12]. □

The equivalence of the statements (i) and (ii) in Proposition 3.1 was first proved by Krall [8].

Using the formal calculus on moment functionals introduced in Section 2, the r equations in (3.3) for the moments of σ can be expressed as (see Section 2 in [12])

$$\tilde{R}_k(\sigma) := \sum_{i=2k+1}^{N} (-1)^i \binom{i-k-1}{k} (\ell_i \sigma)^{(i-2k-1)} = 0, \quad k = 0, 1, \ldots, r-1. \tag{3.7}$$

In particular, for $k = r-1$, we have

$$\tilde{R}_{r-1}(\sigma) = r(\ell_{2r}\sigma)' - \ell_{2r-1}\sigma = 0$$

so that any OPS satisfying a differential equation of the form (3.1) is a semi-classical OPS.

Now, we are ready to give our main results.

Theorem 3.2. *For an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *relative to* σ *and an integer* $r \geq 1$, *the following statements are all equivalent.*

(i) $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ *is a QOPS.*

(ii) *For each* $n \geq 0$, $P_n(x)$ *satisfies a differential equation of order* $N = 2r$ *of the form* (3.1) *with* $\ell_i(x) \equiv 0$, $i = 1, \ldots, r-1$, *that is,*

$$L_{2r}[y](x) = \sum_{i=r}^{2r} \ell_i(x) y^{(i)}(x) = \mu_n y(x). \tag{3.8}$$

(iii) *There is a moment functional* τ ($\neq 0$) *such that*

(3.9) $$L_{2r}[\phi]\sigma = (-1)^r[\phi^{(r)}\tau]^{(r)} \quad (\phi \in \mathcal{P})$$

for some differential operator $L_{2r}[\cdot]$ *of order* $2r$ *of the form* (3.8).
(iv) *There are* $r+1$ *polynomials* $\{a_k(x)\}_r^{2r}$ *with* $a_{2r}(x) \not\equiv 0$, $\deg(a_k) \le k$, $k = r, r+1, \ldots, 2r$, *and*

(3.10) $$(a_k\sigma)' = a_{k-1}\sigma, \quad k = r+1, \ldots, 2r.$$

(v) *There are moment functional* τ $(\neq 0)$ *and* $r+1$ *polynomials* $\{a_k(x)\}_r^{2r}$ *with* $a_{2r}(x) \not\equiv 0$, $\deg(a_k) \le k$, $k = r, r+1, \ldots, 2r$ *and*

(3.11) $$\tau^{(2r-k)} = a_k(x)\sigma, \quad k = r, r+1, \ldots, 2r.$$

Moreover, the moment functionals σ *and* τ *are related by the equations*

(3.12) $$\ell_k(x)\sigma = (-1)^r\binom{r}{k-r}\tau^{(2r-k)}, \quad k = r, r+1, \ldots, 2r.$$

Proof. (i) $\Rightarrow$ (ii): Assume that $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ is a QOPS relative to τ, that is, $\tau \neq 0$ and

$$\langle \tau, P_m^{(r)}P_n^{(r)}\rangle = 0, \quad m \neq n.$$

Then the condition (iii) in Proposition 3.1 holds with $\tau_r = \tau$, $\tau_{r-1} = \cdots = \tau_1 = 0$. Hence, by Proposition 3.1, each $P_n(x)$ satisfies a differential equation (3.1) with $N = 2r$, of which the coefficients $\{\ell_k(x)\}_1^{2r}$ satisfy the equations (3.6). Since $\tau_1 = \cdots = \tau_{r-1} = 0$, we have, from (3.6), $\ell_k(x)\sigma = 0$, $k = 1, \ldots, r-1$. Therefore, $\ell_k(x) \equiv 0$, $k = 1, \ldots, r-1$ by Lemma 2.1 (i).
(ii) $\Rightarrow$ (iii): Assume that each $P_n(x)$ satisfies the differential equation (3.8). Then, by Proposition 3.1, there are r moment functionals $\{\tau_i\}_{i=1}^r$ such that $\tau_r \neq 0$ and the equations (3.5) and (3.6) hold. Since $\ell_1(x) \equiv \cdots \equiv \ell_{r-1}(x) \equiv 0$, we have from (3.6)

(3.13) $$0 = \sum_{[(k+1)/2]}^{k} (-1)^i\binom{i}{k-i}\tau_i^{(2i-k)}, \quad k = 1, \ldots, r-1.$$

Now, it is easy to see from (3.13) that $\tau_k = 0$, $k = 1, 2, \ldots, r-1$ by induction on k. Then the equation (3.5) reduces to the equation (3.9) with $\tau = \tau_r$.
(iii) $\Rightarrow$ (iv): Let $\tau(\neq 0)$ be a moment functional satisfying the equation (3.9). Then the condition (iv) in Proposition 3.1 holds with $\tau_r = \tau$ and $\tau_{r-1} = \cdots = \tau_1 = 0$. Hence, by Proposition 3.1, the equations (3.6) hold. In particular, the equations (3.12) hold. If we set $a_k(x) = (-1)^r\binom{r}{k-r}^{-1}\ell_k(x)$, $k = r, r+1, \ldots, 2r$, then $a_k(x)\sigma = \tau^{(2r-k)}$, $k = r, r+1, \ldots, 2r$, from which (3.10) follows immediately.
(iv) $\Rightarrow$ (v): Assume that the condition (iv) holds. Let $\tau = a_{2r}(x)\sigma$. Then $\tau \neq 0$ since $a_{2r}(x) \not\equiv 0$ and $\tau^{(2r-k)} = a_k(x)\sigma$, $k = r, r+1, \ldots, 2r$.
(v) $\Rightarrow$ (i): Assume that the condition (v) holds. Then we have for $r \le m < n$

$$\begin{aligned}\langle\tau, P_m^{(r)}P_n^{(r)}\rangle &= \langle a_{2r}\sigma, P_m^{(r)}P_n^{(r)}\rangle = (-1)^r\langle (P_m^{(r)}a_{2r}\sigma)^{(r)}, P_n\rangle \\ &= (-1)^r\sum_{i=0}^{r}\binom{r}{i}\langle P_m^{(2r-i)}(a_{2r}\sigma)^{(i)}, P_n\rangle \\ &= (-1)^r\sum_{i=0}^{r}\binom{r}{i}\langle P_m^{(2r-i)}a_{2r-i}\sigma, P_n\rangle \\ &= (-1)^r\sum_{i=0}^{r}\binom{r}{i}\langle \sigma, P_m^{(2r-i)}a_{2r-i}P_n\rangle = 0\end{aligned}$$

since $\deg(P_m^{(2r-i)}a_{2r-i}) \le m < n$. Hence, $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ is a QOPS relative to τ. □

Theorem 3.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ and $r \ge 1$ an integer. Then any one of the equivalent statements in Theorem 3.2 is also equivalent to*

(vi) *$\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS.*

In order to prove Theorem 3.3, we first need the following lemmas.

Lemma 3.4. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a monic OPS relative to σ. For an integer $r \ge 1$, let $\{Q_n(x) := (1/(P(n+r-1, r-1)))P_{n+r-1}^{(r-1)}(x)\}_{n=0}^{\infty}$ and $\{R_n(x) := (1/(n+1))Q'_{n+1}(x)\}_{n=0}^{\infty}$. If $\{R_n(x)\}_{n=0}^{\infty}$ is a QOPS relative to τ, then $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the following recurrence relation*:

$$Q_{n+1}(x) = (x-\beta_n)Q_n(x) - \gamma_n Q_{n-1}(x) - \sum_{j=0}^{n-2}\delta_n^j Q_j(x), \quad n \ge 1, \tag{3.14}$$

where β_n, γ_n, and δ_n^j are real constants with $\delta_1^0 = \delta_1^{-1} = 0$ and $\delta_n^1 = 0$, $n \ge 1$.

Proof. Since $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation (see Chihara [3]):

$$P_{n+1}(x) = (x-b_n)P_n(x) - c_nP_{n-1}(x), \quad n \ge 1, \tag{3.15}$$

where b_n and c_n are real constants with $c_n \ne 0$, $n \ge 1$. Replacing n by $n+r-1$ in (3.15) and then differentiating $r-1$ and r times, we obtain for $n \ge 0$

$$\left\{\begin{aligned} P_{n+r}^{(r-1)}(x) = (x-b_{n+r-1})P_{n+r-1}^{(r-1)}(x) - c_{n+r-1}P_{n+r-2}^{(r-1)}(x) \\ + (r-1)P_{n+r-1}^{(r-2)}(x),\end{aligned}\right. \tag{3.16}$$

and

$$P_{n+r}^{(r)}(x) = (x-b_{n+r-1})P_{n+r-1}^{(r)}(x) - c_{n+r-1}P_{n+r-2}^{(r)}(x) + rP_{n+r-1}^{(r-1)}(x). \tag{3.17}$$

On the other hand, as a monic PS, $\{R_n(x)\}_{n=0}^{\infty}$ satisfy

$$R_{n+1}(x) = (x-\tilde{b}_n)R_n(x) - \tilde{c}_nR_{n-1}(x) - \sum_{j=0}^{n-2}\tilde{\delta}_n^jR_j(x), \quad n \ge 1, \tag{3.18}$$

where $\tilde{b}_n$, $\tilde{c}_n$, and $\tilde{\delta}_n^j$ are real constants with $\tilde{\delta}_1^0 = \tilde{\delta}_1^{-1} = 0$ and $R_{-1}(x) \equiv 0$. Applying τ to (3.18) and using the quasi orthogonality of $\{R_n(x)\}_{n=0}^{\infty}$ relative to τ, we obtain $\tilde{\delta}_n^0 = 0$, $n \geq 2$ so that (3.18) reduces to

$$(3.19) \quad R_{n+1}(x) = (x - \tilde{b}_n)R_n(x) - \tilde{c}_n R_{n-1}(x) - \sum_{j=1}^{n-2} \tilde{\delta}_n^j R_j(x), \quad n \geq 2 \ (\tilde{\delta}_2^1 = 0).$$

From (3.19) with n replaced by $n-1$ and (3.17), we obtain

$$(3.20) \quad \begin{cases} rP_{n+r-1}^{(r-1)}(x) = \left(\frac{r}{n}x + b_{n+r-1} - \tilde{b}_{n-1}\frac{n+r}{n}\right)P_{n+r-1}^{(r)}(x) \\ \qquad + \left(c_{n+r-1} - \tilde{c}_{n-1}\frac{(n+r)(n+r-1)}{n(n-1)}\right)P_{n+r-2}^{(r)}(x) \\ \qquad - \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^j \frac{P(n+r,r)}{P(j+r,r)} P_{j+r}^{(r)}(x), \qquad n \geq 3. \end{cases}$$

Integrating (3.20), we obtain for $n \geq 3$

$$(3.21) \quad \begin{cases} P_{n+r-1}^{(r-2)}(x) = \left(\frac{x}{n+1} + \frac{nb_{n+r-1}}{r(n+1)} - \frac{(n+r)\tilde{b}_{n-1}}{r(n+1)}\right)P_{n+r-1}^{(r-1)}(x) \\ \qquad + \left(\frac{nc_{n+r-1}}{r(n+1)} - \frac{(n+r)(n+r-1)\tilde{c}_{n-1}}{r(n-1)(n+1)}\right)P_{n+r-2}^{(r-1)}(x) \\ \qquad - \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^j \frac{n}{r(n+1)} \frac{P(n+r,r)}{P(j+r,r)} P_{j+r}^{(r-1)}(x) + d_n, \end{cases}$$

where d_n is an integration constant. Substituting (3.21) into (3.16) yields

$$\begin{aligned} P_{n+r}^{(r-1)}(x) = & \left(\frac{n+r}{n+1}x - \frac{(n+r)b_{n+r-1}}{r(n+1)} - \frac{(n+r)(r-1)\tilde{b}_{n-1}}{r(n+1)}\right)P_{n+r-1}^{(r-1)}(x) \\ & - \left(\frac{n+r}{(n+1)r} + \frac{(r-1)(n+r)(n+r-1)\tilde{c}_{n-1}}{r(n-1)(n+1)}\right)P_{n+r-2}^{(r-1)}(x) \\ & - \sum_{j=1}^{n-3} \tilde{\delta}_{n-1}^j \frac{(r-1)n}{r(n+1)} \frac{P(n+r,r)}{P(j+r,r)} P_{j+r}^{(r-1)}(x) + (r-1)d_n, \quad n \geq 3. \end{aligned}$$

This last equation can be rewritten into the equation (3.14) by the definition of $Q_n(x)$ for $n \geq 3$. The equation (3.14) for $n = 1, 2$ is trivial. □

Lemma 3.5. *Let $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{R_n(x)\}_{n=0}^{\infty}$ be the same as in Lemma 3.4. Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, and $\{w_n\}_{n=0}^{\infty}$ be the dual sequences of $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{R_n(x)\}_{n=0}^{\infty}$ respectively. If $\{R_n(x)\}_{n=0}^{\infty}$ is a QOPS, then*

(i) *there are $r+1$ polynomials $\{a_k(x)\}_r^{2r}$ with $a_{2r}(x) \not\equiv 0$, $\deg(a_k) \leq k$, $k = r, \dots, 2r$, and*

$$(3.22) \quad w_0^{(2r-k)} = a_k(x)u_0, \quad k = r, \dots, 2r$$

and

(ii) *there are r polynomials $\{h_k(x)\}_{r+1}^{2r}$ with $h_{2r}(x) \not\equiv 0$, $\deg(h_k) \leq k$, $k = r+1, \dots, 2r$, and*

$$(3.23)\quad v_0^{(2r-k)} = h_k(x)u_0, \quad k = r+1, \dots, 2r.$$

Moreover, we also have $\deg(a_r) = r$ *and* $\deg(h_{r+1}) = r-1$.

Proof. Assume that $\{R_n(x)\}_{n=0}^{\infty}$ is a QOPS. Then w_0 is an orthogonalizing moment functional of $\{R_n(x)\}_{n=0}^{\infty}$. Hence we have (i) from the equivalence of the statements (i) and (v) in Theorem 3.2. By Lemma 3.4, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the recurrence relation (3.14). Applying v_1 to (3.14), we obtain $\langle xv_1, Q_n\rangle = 0$, $n \geq 3$ so that by Lemma 2.2

$$xv_1 = e_0v_0 + e_1v_1 + e_2v_2,$$

where $e_j = \langle xv_1, Q_j\rangle$, $j = 0, 1, 2$. Since $e_0 = \langle xv_1, Q_0\rangle = \langle v_1, x\rangle = \langle v_1, Q_1\rangle = 1$, we have by Lemma 2.3

$$(3.24)\quad v_0 = (-x + e_1)w_0' + \frac{e_2}{2}w_1'.$$

On the other hand, applying w_0 to (3.19), we obtain $\langle xw_0, R_n\rangle = 0$, $n \geq 2$ so that by Lemma 2.2

$$xw_0 = c_0w_0 + c_1w_1,$$

where $c_j = \langle xw_0, R_j\rangle$, $j = 0, 1$. If $c_1 = 0$, then $(x - c_0)w_0 = (x - c_0)a_{2r}(x)u_0 = 0$ by (3.22). This is a contradiction since u_0 is regular and $a_{2r}(x) \not\equiv 0$. Hence, $c_1 \neq 0$ and

$$(3.25)\quad w_1 = \frac{x - c_0}{c_1}w_0.$$

Substituting (3.25) into (3.24), we obtain

$$(3.26)\quad v_0 = \pi_0(x)w_0 + \pi_1(x)w_1(x),$$

where $\pi_j(x)$ is a polynomial of degree $\leq j$, $j = 0, 1$. Differentiating (3.26) successively, we obtain (3.23) from (3.22) with $h_k(x) = [\pi_0(x) + (2r-k)\pi_1'(x)]a_k(x) + \pi_1(x)a_{k-1}(x)$.

Finally we have

$$\langle w_0^{(r)}, P_n\rangle = (-1)^r\langle w_0, P_n^{(r)}\rangle = \begin{cases} 0 & \text{if } 0 \leq n < r \\ (-1)^r\langle w_0, P(n,r)R_{n-r}\rangle & \text{if } n \geq r \end{cases}$$
$$= (-1)^r r!\,\delta_{rn}$$

so that $a_r(x)u_0 = w_0^{(r)} = (-1)^r r!\,u_r = (-1)^r r!\,C_rP_r(x)u_0$ by Proposition 2.4. Hence $\deg(a_r) = r$.

Similarly we have $h_{r+1}(x)u_0 = v_0^{(r-1)} = (-1)^{r-1}(r-1)!\,C_{r-1}P_{r-1}(x)u_0$ so that $\deg(h_{r+1}) = r-1$. □

Proof of Theorem 3.3. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS. Then, by the remark following Proposition 2.5, $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ is also a classical OPS for any integer $r \geq 1$. Hence, the statement (i) in Theorem 3.2 holds. Conversely, we assume that the statement (ii) in Theorem 3.2 holds. If $r = 1$, $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS by the definition. Hence we assume $r \geq 2$. Then, by induction, it suffices to show that $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a differential equation of the form (3.8) of order $2(r-1)$. We may assume $\{P_n(x)\}_{n=0}^{\infty}$ is a monic PS and let $\{Q_n(x)\}_{n=0}^{\infty}$, $\{R_n(x)\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, and $\{w_n\}_{n=0}^{\infty}$ be the same as in Lemma 3.5. By the statement (i) in Theorem 3.2, $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ and so $\{R_n\}_{n=0}^{\infty}$ is a QOPS. Then, by Lemma 3.5, we have polynomials $\{a_k(x)\}_r^{2r}$ and $\{h_k(x)\}_{r+1}^{2r}$ satisfying (3.22) and (3.23). Hence, the moment functional u_0 satisfies

$$(3.27) \quad (a_k u_0)' = a_{k-1} u_0, \quad k = r+1, \ldots, 2r$$

and

$$(3.28) \quad (h_k u_0)' = h_{k-1} u_0, \quad k = r+2, \ldots, 2r.$$

Now, let s (≥ 0) be the class number of the semi-classical moment functional u_0 and $(\alpha(x), \beta(x)) \neq (0,0)$ a pair of polynomials satisfying

$$(\alpha u_0)' - \beta u_0 = 0$$

and

$$s = \max(\deg(\alpha) - 2, \deg(\beta) - 1).$$

Then we have from Proposition 2.7

$$(3.29) \quad a_k(x) = \tilde{a}_k(x)\alpha(x), \quad k = r+1, \ldots, 2r$$

and

$$(3.30) \quad h_k(x) = \tilde{h}_k(x)\alpha(x), \quad k = r+2, \ldots, 2r,$$

where $\tilde{a}_k(x)$ and $\tilde{h}_k(x)$ are polynomials. Hence we have from (3.27), (3.28), (3.29), and (3.30)

$$(3.31) \quad \tilde{a}_k'\alpha + \tilde{a}_k\beta = a_{k-1}, \quad k = r+1, \ldots, 2r;$$

$$(3.32) \quad \tilde{h}_k'\alpha + \tilde{h}_k\beta = h_{k-1}, \quad k = r+2, \ldots, 2r.$$

At this point, we divide the proof into two cases: $s = \deg(\alpha) - 2 \geq \deg(\beta) - 1$ and $s = \deg(\beta) - 1 > \deg(\alpha) - 2$.

Case I: $s = \deg(\alpha) - 2 \geq \deg(\beta) - 1$. Counting degrees on both sides of the equation (3.31), we have

$$\deg(a_{k-1}) + 1 \leq \deg(a_k), \quad k = r+1, \ldots, 2r$$

since $\deg(\alpha) \geq \deg(\beta) + 1$. Hence we have

$$\deg(a_k) = k, \quad k = r, \ldots, 2r$$

since $\deg(a_r) = r$ and $\deg(a_k) \leq k$, $k = r, \ldots, 2r$. Similarly, counting degrees on both sides of the equation (3.32), we have

$$\deg(h_{k-1}) + 1 \leq \deg(h_k), \quad k = r+2, \ldots, 2r.$$

We now claim that

$$(3.33) \quad \deg(h_{k-1}) + 1 = \deg(h_k), \quad k = r+2, \ldots, 2r.$$

If not, let j be the first integer $\geq r+2$ such that $\deg(h_{j-1}) + 1 < \deg(h_j)$. Then,

$\deg(h_k) = k - 2, k = r + 1, \ldots, j - 1$ and $j - 2 < \deg(h_j) \leq j$ since $\deg(h_{r+1}) = r - 1$. Since $r + 2 \leq j \leq 2r$, $\deg(h_j) = m = \deg(a_m)$ for some $m = r + 1, \ldots, 2r$. Let A ($\neq 0$) and B ($\neq 0$) be the leading coefficients of $a_m(x)$ and $h_j(x)$ respectively. Multiplying the equation (3.31) for $k = m$ by B and the equation (3.32) for $k = j$ by A and subtracting these two equations, we obtain

(3.34) $$(B\tilde{a}'_m - A\tilde{h}'_j)\alpha + (B\tilde{a}_m - A\tilde{h}_j)\beta = Ba_{m-1} - Ah_{j-1}.$$

We then have $\deg(Ba_{m-1} - Ah_{j-1}) = m - 1$ since $\deg(a_{m-1}) = m - 1 > j - 3 = \deg(h_{j-1})$. However, the degree of the left hand side of the equation (3.34) is at most $m - 2$ since $\deg(Ba_m - Ah_j) \leq m - 1$ and $\deg(\beta) \leq \deg(\alpha) - 1$. It is a contradiction so that we have (3.33).
Since $\deg(h_{r+1}) = r - 1$, we have from (3.33)

(3.35) $$\deg(h_k) = k - 2, \quad k = r + 1, \ldots, 2r.$$

If we set $g_k(x) = h_{k+2}(x)$, $k = r - 1, \ldots, 2(r - 1)$, then $\{g_k\}_{r-1}^{2(r-1)}$ satisfy the condition (v) in Theorem 3.2 with r replaced by $r - 1$ and σ replaced by u_0 by (3.28) and (3.35). Hence, by Theorem 3.2, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a differential equation of the form (3.8) of order $2(r - 1)$ since $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to u_0.

Case II: $s = \deg(\beta) - 1 > \deg(\alpha) - 2$. Counting degrees on both sides of the equation (3.31), we have

$$\deg(a_k) = \deg(a_{k-1}) + \deg(\alpha) - \deg(\beta), \quad k = r + 1, \ldots, 2r$$

so that

(3.36) $$\begin{cases} \deg(a_k) = \deg(a_r) + (k - r)(\deg(\alpha) - \deg(\beta)) \\ \qquad\qquad = r + (k - r)(\deg(\alpha) - \deg(\beta)), \qquad k = r + 1, \ldots, 2r. \end{cases}$$

In particular, we have for $k = 2r$ in (3.36)

$$\deg(a_{2r}) = r(\deg(\alpha) - s).$$

Since $\deg(a_{2r}) \geq \deg(\alpha) \geq 0$, $s \leq \deg(\alpha) < s + 2$ so that $\deg(\alpha)$ is either s or $s + 1$. If $\deg(\alpha) = s$, then $s = 0$ and so $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS. If $\deg(\alpha) = s + 1$, then we have by counting degrees on both sides of the equation (3.32)

$$\deg(h_k) = \deg(h_{k-1}), \quad k = r + 2, \ldots, 2r$$

so that

(3.37) $$\deg(h_k) = r - 1, \quad k = r + 1, \ldots, 2r.$$

If we set $g_k(x) = h_{k+2}(x)$, $k = r - 1, \ldots, 2(r - 1)$, then $\{g_k(x)\}_{r-1}^{2(r-1)}$ satisfy the condition (v) in Theorem 3.2 with r replaced by $r - 1$ and σ replaced by u_0 by (3.28) and (3.37). Hence, by Theorem 3.2, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a differential equation of the form (3.8) of order $2(r - 1)$ since $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to u_0. □

Finally, for any classical OPS $\{P_n(x)\}_{n=0}^{\infty}$ we show the way of constructing the differential equation of the form (3.8), which has $\{P_n(x)\}_{n=0}^{\infty}$ as polynomial solutions. First, we need the following lemma.

Lemma 3.6. *If the differential equation* (1.1) *has an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *of polynomial solutions, then* $\lambda_n \neq 0$, $n \geq 1$ *and* $\lambda_m \neq \lambda_n$ *for* $0 \leq m < n$.

Proof. Assume that (1.1) has an OPS $\{P_n(x)\}_{n=0}^{\infty}$ of polynomial solutions and let σ be the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ and σ satisfies the equation (2.10). Suppose $\lambda_n = 0$ for some $n \geq 1$. Then we have by (2.10)

$$\begin{aligned} 0 = \lambda_n P_n \sigma &= (\ell_2 P_n'' + \ell_1 P_n')\sigma = (\ell_2 P_n' \sigma)' - P_n'(\ell_2 \sigma)' + P_n'(\ell_1 \sigma) \\ &= (\ell_2 P_n' \sigma)'. \end{aligned}$$

Hence, $\ell_2 P_n' \sigma = 0$ so that $\ell_2 P_n' \equiv 0$ by Lemma 2.1 (i). Since $\ell_2(x) \not\equiv 0$, $P_n(x) \equiv 0$ which implies $n = 0$ contradicting the fact that $n \geq 1$. Hence $\lambda_n \neq 0$, $n \geq 1$, which implies $\lambda_m \neq \lambda_n$ for $m \neq n$ since

$$\begin{aligned} (n+m)(\lambda_n - \lambda_m) &= (n-m)(n+m)(\ell_{22}(n+m-1) + \ell_{11}) \\ &= (n-m)\lambda_{n+m}. \qquad \square \end{aligned}$$

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a classical OPS satisfying the equation (1.1). Consider the following simultaneous equations:

$$(3.38) \qquad \begin{cases} \lambda_1 c_1 + \lambda_1^2 c_2 + \cdots + \lambda_1^{r-1} c_{r-1} = \lambda_1^r \\ \lambda_2 c_1 + \lambda_2^2 c_2 + \cdots + \lambda_2^{r-1} c_{r-1} = \lambda_2^r \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_{r-1} c_1 + \lambda_{r-1}^2 c_2 + \cdots + \lambda_{r-1}^{r-1} c_{r-1} = \lambda_{r-1}^r. \end{cases}$$

Since $\det[\lambda_i^j]_{i,j=1}^{r-1} \neq 0$ by Lemma 3.6, the simultaneous equations (3.38) have a unique solution $\{c_j\}_{j=1}^{r-1}$. Then $\{P_n(x)\}_{n=0}^{\infty}$ also satisfies the equation

$$(3.39) \qquad M_{2r}[y](x) = \sum_{i=1}^{2r} m_i(x) y^{(i)}(x) = L_2^r[y](x) - \sum_{j=1}^{r-1} c_j L_2^j[y](x) = \mu_n y(x),$$

where $\mu_n = \lambda_n^r - (c_1 \lambda_n + \cdots + c_{r-1}\lambda_n^{r-1})$, $n \geq 0$, and $m_i(x)$, $i = 1, 2, \ldots, 2r$, are polynomials of degree $\leq i$. Since $M_{2r}[P_k](x) = \sum_{i=1}^{k} m_i(x) P_k^{(i)}(x) = \mu_k P_k(x) = 0$ for $k = 1, 2, \ldots, r-1$, we have by induction $m_1(x) \equiv m_2(x) \equiv \cdots \equiv m_{r-1}(x) \equiv 0$. Thus the equation (3.39) is of the form (3.8).

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REFERENCES

1. Al-Salam, W.A. – Characterization theorems for orthogonal polynomial, in: Orthogonal Polynomials: Theory and Practice. NATO ASI Series vol. **294**, Kluwer Acad., Dordrecht, Netherlands, 1–24 (1990).
2. Bochner, S. – Über Sturm–Liouvillesche Polynomsysteme. Math. Z. **29**, 730–736 (1929).
3. Chihara, T.S. – An Introduction to Orthogonal Polynomials. Gordon and Breach, N.Y. (1977).
4. Hahn, W. – Über die Jacobischen Polynome und zwei verwandte Polynomklassen. Math. Z. **39**, 634–638 (1935).
5. Hahn, W. – Über höhere Ableitungen von Orthogonalpolynomen. Ibid. **43**, 101 (1937).
6. Krall, H.L. – On derivatives of orthogonal polynomials. Bull. Amer. Math. Soc. **42**, 423–428 (1936).
7. Krall, H.L. – On higher derivatives of orthogonal polynomials. Bull. Amer. Math. Soc. **42**, 867–870 (1936).
8. Krall, H.L. – Certain differential equations for Tchebychev polynomials. Duke Math. J. **4**, 705–719 (1938).
9. Krall, H.L. – On derivatives of orthogonal polynomials II. Bull. Amer. Math. Soc. **47**, 261–264 (1941).
10. Krall, H.L. and O. Frink – A new class of orthogonal polynomials: The Bessel polynomials. Trans. Amer. Math. Soc. **65**, 100–115 (1949).
11. Kwon, K.H., J.K. Lee and B.H. Yoo – Characterizations of classical orthogonal polynomials. Results in Mathematics **24**, 119–128 (1993).
12. Kwon, K.H., L.L. Littlejohn and B.H. Yoo – Characterizations of orthogonal polynomials satisfying differential equations. SIAM J. Math. Anal. **25**, 976–990 (1994).
13. Lesky, P. – Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm–Liouvillesche Differentialgleichungen. Arch. Rat. Mech. Anal. **10**, 341–352 (1962).
14. Maroni, P. – Prolégomènes a l'étude des polynomes orthonaux semi-classiques. Ann. Mat. Pura Appl. **149**, 165–184 (1987).
15. Maroni, P. – Variations around classical orthogonal polynomials. Connected problems. J. Comp. Appl. Math. **48**, 133–155 (1993).
16. Morton, R.D. and A.M. Krall – Distributional weight functions for orthogonal polynomials. SIAM J. Math. Anal. **9** (4), 604–626 (1978).
17. Sonine, N.J. – Über die angenäherte Berechnung der bestimmten Integrale und über die dabei vorkommenden ganzen Funktionen. Warsaw Univ. Izv. **18** (1887), (in Russian), Summary in Jbuch Fortschritte Math. **19**, 282 (1889).
18. Webster, M.S. – Orthogonal polynomials with orthogonal derivatives. Bull. Amer. Math. Soc. **44**, 880–888 (1938).