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IMEX method convergence for a parabolic equation

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Abstract

Although implicit–explicit (IMEX) methods for approximating solutions to semilinear parabolic equations are relatively standard, most recent works examine the case of a fully discretized model. We show that by discretizing time only, one can obtain an elementary convergence result for an implicit–explicit method. This convergence result is strong enough to imply existence and uniqueness of solutions to a class of semilinear parabolic equations.

Keywords: IMEX; Semilinear parabolic equation; Classical solutions; Implicit–explicit method; Semigroups

1. Introduction

The use of implicit–explicit (IMEX) methods for approximating semilinear parabolic equations is well established [1]. Many of the recent works on these methods employ discretizations in both space and time. These fully discrete approximations can be computed directly by a computer. However, one can obtain a stronger condition for convergence of the approximation if only the time dimension is discretized [2]. We show how an even stronger condition for convergence is met by the Cauchy problem for

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x)u^i(x,t), \tag{1}$$

where $a_i \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and how convergence of this method provides an elementary proof of existence and uniqueness of solutions. Existence and uniqueness of solutions for (1)

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under reasonable initial conditions have been known for some time. For instance, [4] and [6] contain straightforward proofs using semigroup methods. The purpose of this paper is to show how a *more elementary* proof can be obtained from a sequence of explicitly computed discrete-time approximations.

The Cauchy problem for (1) arises in a variety of settings. Notably, some reaction–diffusion equations are of this form [3]. Another application is the special case

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - u^2(x,t) + a_0(x),$$

where a_0 is a nonzero function of x. This situation corresponds to a spatially-dependent logistic equation with a diffusion term, which can be thought of as a toy model of population growth with migration.

Following [2], the approximation to be used is

$$u_{n+1} = (I - h\Delta)^{-1}\left(u_n + h\sum_{i=0}^{\infty} a_i u_n^i\right), \tag{2}$$

which is obtained by inverting the linear portion of a discrete version of (1). For brevity, we shall call (2) *the* implicit–explicit method. (In the summary paper [1], this is called an SBDF method, to distinguish it from other implicit–explicit methods.) One can compute the operator $(I - h\Delta)^{-1}$ explicitly using Fourier transform methods, and obtain a proof of the numerical stability of the iteration as a whole.

2. A version of the fundamental inequality

In order to simplify the algebraic expressions, we make the following definitions.

Definition 1. Let

$$F\big(u(x,t)\big) = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x) u^i(x,t), \tag{3}$$

and

$$G\big(u(x,t)\big) = \sum_{i=0}^{\infty} a_i(x) u^i(x,t). \tag{4}$$

Definition 2. Define the analytic functions

$$g_1(z) = \sum_{i=0}^{\infty} \|a_i\|_1 z^i, \tag{5}$$

and

$$g_\infty(z) = \sum_{i=0}^{\infty} \|a_i\|_\infty z^i. \tag{6}$$

Since we do not discretize the spatial dimension, we can employ some of the theory of ordinary differential equations. We therefore first prove a variant of the fundamental inequality for (1) as is done in [5]. The fundamental inequality gives a sufficient condition for approximate solutions to converge. A slightly weaker version of Lemma 3 was obtained in Theorem 3.1 of [2], where the existence of solutions was required.

Lemma 3. *Suppose* $\{u_i\}_{i=1}^{\infty}$ *is a sequence of piecewise* C^1 *functions* $u_i : [0,T] \to C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*, such that*

(1) *there exist* $A, B > 0$ *so that for each* i *and* $t \in [0,T]$, $\|u_i(t)\|_1 \leqslant A$ *and* $\|u_i(t)\|_\infty \leqslant B$,
(2) *for each* i *and* $t \in [0,T]$*, the series* $g_1(\|u_i(t)\|_1)$ *and* $g_\infty(\|u_i(t)\|_\infty)$ *converge,*
(3) *for each* $t \in [0,T]$, $\|\frac{d}{dt}u_i(t) - F(u_i(t))\|_\infty < \epsilon_i$ *and* $\lim_{i\to\infty}\epsilon_i = 0$*, and*
(4) $u_1(0) = u_i(0)$ *for all* $i \geqslant 0$.

Then for each $t \in [0,T]$, $\{u_i(t)\}_{i=1}^{\infty}$ *is a Cauchy sequence in* $L^2(\mathbb{R}^n)$.

Proof. Let $i, j > 0$ be given. Let $\eta(t) = \|u_i(t) - u_j(t)\|_2^2 = \int (u_i(t) - u_j(t))^2\,dx$. Notice that the fourth condition in the hypothesis gives $\eta(0) = 0$,

$$\eta'(t) = 2\int \big(u_i'(t) - u_j'(t)\big)\big(u_i(t) - u_j(t)\big)\,dx.$$

But, $\|\frac{d}{dt}u_i(t) - F(u_i(t))\|_\infty < \epsilon_i$ is equivalent to the statement that for each $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$F\big(u_i(x,t)\big) - \epsilon_i < u_i'(x,t) < F\big(u_i(x,t)\big) + \epsilon_i,$$

giving

$$\begin{aligned}
\eta'(t) &\leqslant 2\int \big(F\big(u_i(t)\big) - F\big(u_j(t)\big)\big)\big(u_i(t) - u_j(t)\big)\,dx + 2(\epsilon_i + \epsilon_j)\int \big|u_i(t) - u_j(t)\big|\,dx \\
&\leqslant 2\int \big(\Delta u_i(t) + G\big(u_i(t)\big) - \Delta u_j(t) - G\big(u_j(t)\big)\big)\big(u_i(t) - u_j(t)\big)\,dx \\
&\quad + 2(\epsilon_i + \epsilon_j)\big\|u_i(t) - u_j(t)\big\|_1 \\
&\leqslant 2\int \big(\Delta\big(u_i(t) - u_j(t)\big)\big)\big(u_i(t) - u_j(t)\big)\,dx \\
&\quad + 2\int \big(G\big(u_i(t)\big) - G\big(u_j(t)\big)\big)\big(u_i(t) - u_j(t)\big)\,dx + 2(\epsilon_i + \epsilon_j)\big\|u_i(t) - u_j(t)\big\|_1 \\
&\leqslant -2\int \big\|\nabla\big(u_i(t) - u_j(t)\big)\big\|^2\,dx + 2\big\|G\big(u_i(t)\big) - G\big(u_j(t)\big)\big\|_2\big\|u_i(t) - u_j(t)\big\|_2 \\
&\quad + 2(\epsilon_i + \epsilon_j)\big\|u_i(t) - u_j(t)\big\|_1 \\
&\leqslant 2\big\|G\big(u_i(t)\big) - G\big(u_j(t)\big)\big\|_2\big\|u_i(t) - u_j(t)\big\|_2 + 2(\epsilon_i + \epsilon_j)\big\|u_i(t) - u_j(t)\big\|_1.
\end{aligned}$$

Now also

$$\begin{aligned}
&\left\|G\big(u_i(t)\big)-G\big(u_j(t)\big)\right\|_2\\
&\quad=\left\|\sum_{k=0}^{\infty}a_k\big(u_i^k(t)-u_j^k(t)\big)\right\|_2\\
&\quad\leqslant\sum_{k=0}^{\infty}\|a_k\|_\infty\left\|u_i^k(t)-u_j^k(t)\right\|_2\\
&\quad\leqslant\sum_{k=0}^{\infty}\|a_k\|_\infty\sqrt{\int\big(u_i^k(x,t)-u_j^k(x,t)\big)^2\,dx}\\
&\quad\leqslant\sum_{k=0}^{\infty}\|a_k\|_\infty\sqrt{\int\big(u_i(x,t)-u_j(x,t)\big)^2\left(\sum_{m=0}^{k-1}u_i^m(x,t)u_j^{k-m-1}(x,t)\right)^2dx}\\
&\quad\leqslant\sum_{k=0}^{\infty}\|a_k\|_\infty\left\|\sum_{m=0}^{k-1}u_i^m(t)u_j^{k-m-1}(t)\right\|_\infty\left\|u_i(t)-u_j(t)\right\|_2\\
&\quad\leqslant\left(\sum_{k=0}^{\infty}\|a_k\|_\infty kB^{k-1}\right)\left\|u_i(t)-u_j(t)\right\|_2\\
&\quad\leqslant g_\infty'(B)\left\|u_i(t)-u_j(t)\right\|_2,
\end{aligned}$$

which allows

$$\begin{aligned}
\eta'(t)&\leqslant 2g_\infty'(B)\left\|u_i(t)-u_j(t)\right\|_2^2+2(\epsilon_i+\epsilon_j)\left\|u_i(t)-u_j(t)\right\|_1\\
&\leqslant 2g_\infty'(B)\eta(t)+2(\epsilon_i+\epsilon_j)\left\|u_i(t)-u_j(t)\right\|_1,
\end{aligned}$$

$$\begin{aligned}
\eta'(t)-2g_\infty'(B)\eta(t)&\leqslant 2(\epsilon_i+\epsilon_j)\left\|u_i(t)-u_j(t)\right\|_1,\\
\frac{d}{dt}\big(\eta(t)e^{-2g_\infty'(B)t}\big)&\leqslant 2(\epsilon_i+\epsilon_j)e^{-2g_\infty'(B)t}\left\|u_i(t)-u_j(t)\right\|_1,
\end{aligned}$$

so (recall $\eta(0)=0$)

$$\begin{aligned}
\eta(t)&\leqslant\left[2(\epsilon_i+\epsilon_j)\int_0^t e^{-2g_\infty'(B)s}\left\|u_i(s)-u_j(s)\right\|_1ds\right]e^{2g_\infty'(B)t}\\
&\leqslant\left[2(\epsilon_i+\epsilon_j)\int_0^t\left\|u_i(s)-u_j(s)\right\|_1ds\right]e^{2g_\infty'(B)t}\\
&\leqslant 4(\epsilon_i+\epsilon_j)Ate^{2g_\infty'(B)t}.
\end{aligned}$$

Hence as $i,j\to\infty$, $\eta(t)\to 0$ for each t. Thus for each t, $\{u_i(t)\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. □

Remark 4. Since $C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ and L^2 is complete, Lemma 3 gives conditions for existence and uniqueness of a short-time solution to (1).

Lemma 5. *Suppose $\{u_i(t)\}_{i=1}^\infty$ is the sequence of functions defined in Lemma* 3, *and that $u(t) = \lim_{i\to\infty} u_i(t)$ in $L^2(\mathbb{R}^n)$. Then*

$$u'(t,x) = \lim_{i\to\infty} u_i'(t,x) \quad \textit{for almost every } x, \tag{7}$$

wherever the limit exists.

Proof. Notice that since each $u_i(t) \in L^\infty(\mathbb{R}^n)$ and $\|u_i(t)\|_\infty \leqslant B$, the dominated convergence theorem allows for each $x \in \mathbb{R}^n$

$$\begin{aligned}\int_0^t \lim_{i\to\infty} u_i'(\tau,x)\,d\tau &= \lim_{i\to\infty} \int_0^t u_i'(\tau,x)\,d\tau \\ &= \lim_{i\to\infty} \big(u_i(t,x) - u_i(0,x)\big) \\ &= u(t,x) - u(0,x) \quad \text{for almost every } x.\end{aligned}$$

Hence, by differentiating in t,

$$u'(t,x) = \lim_{i\to\infty} u_i'(t,x) \quad \text{for almost every } x. \qquad \square$$

3. The implicit–explicit approximation

In this section, we consider the case of a 1-dimensional spatial domain, that is, $x \in \mathbb{R}$. There is no obstruction to extending any of these results to higher dimensions, though it complicates the exposition unnecessarily.

As is usual, the first task is to define the function spaces to be used. Initial conditions will be drawn from a subspace of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, as suggested by Lemma 3, and the first four spatial derivatives will be prescribed, for use in Lemma 10.

Definition 6. Let

$$W = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap \big\{ f \in C^\infty(\mathbb{R}) \bigm| f \text{ has bounded partial derivatives up to fourth order} \big\}.$$

For the remainder of this paper, we consider the case where each of the coefficients $a_i \in W$. Then let $X = \{f \in W \mid g_1(\|f\|_1) < \infty \text{ and } g_\infty(\|f\|_\infty) < \infty\}$. We consider the case where the initial condition is drawn from X.

An approximate solution given by the implicit–explicit iteration will be the piecewise linear interpolation through the iterates computed by (2). A smoother approximation will prove to be unnecessary, as will be shown in Lemma 11.

Definition 7. Suppose f_0 and $h > 0$ are given. Put

$$f_{n+1} = (I - h\Delta)^{-1}\big(f_n + hG(f_n)\big). \tag{8}$$

The function

$$u(t) = \left(1 - \left(\frac{t}{h} - n(t)\right)\right) f_{n(t)} + \left(\frac{t}{h} - n(t)\right) f_{n(t)+1}, \tag{9}$$

where $n(t) = \lfloor \frac{t}{h} \rfloor$, is called the *implicit–explicit iteration of size h beginning at* f_0.

Calculation 8. We explicitly compute the operator $(I - h\Delta)^{-1}$ using Fourier transforms. Suppose

$$(I - h\Delta)u(x) = u(x) - h\Delta u(x) = f(x).$$

Taking the Fourier transform (with transformed variable ω) gives

$$\hat{u}(\omega) + h\omega^2 \hat{u}(\omega) = \hat{f}(\omega),$$
$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{1 + h\omega^2}.$$

The Fourier inversion theorem yields

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \int \frac{e^{i\omega x}}{1 + h\omega^2} \int f(y) e^{-i\omega y}\, dy\, d\omega \\ &= \int f(y) \left(\frac{1}{2\pi} \int \frac{e^{i\omega(x-y)}}{1 + h\omega^2}\, d\omega \right) dy. \end{aligned}$$

Using the method of residues, this can be simplified to give

$$u(x) = \big((I - h\Delta)^{-1} f\big)(x) = \frac{1}{2\sqrt{h}} \int f(y) e^{-|y-x|/\sqrt{h}}\, dy. \tag{10}$$

Calculation 9. Bounds on the L^1 and L^∞ operator norms of $(I - h\Delta)^{-1}$ are now computed. First, let $f \in L^\infty(\mathbb{R})$. Then

$$\begin{aligned} \big|\big((I - h\Delta)^{-1} f\big)(x)\big| &= \left| \frac{1}{2\sqrt{h}} \int f(y) e^{-|y-x|/\sqrt{h}}\, dy \right| \\ &\leqslant \|f\|_\infty \frac{1}{2\sqrt{h}} \int e^{-|y-x|/\sqrt{h}}\, dy \\ &\leqslant \|f\|_\infty \frac{1}{\sqrt{h}} \int_0^\infty e^{-s/\sqrt{h}}\, ds \\ &\leqslant \|f\|_\infty, \end{aligned}$$

so $\|(I - h\Delta)^{-1}\|_\infty \leqslant 1$.

Now, let $f \in L^1(\mathbb{R})$. So then

$$\begin{aligned}
\|(I - h\Delta)^{-1} f\|_1 &= \int_{-\infty}^{\infty} \left| \frac{1}{2\sqrt{h}} \int_{-\infty}^{\infty} f(y) e^{-|y-x|/\sqrt{h}}\, dy \right| dx \\
&\leqslant \frac{1}{2\sqrt{h}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)| e^{-|y-x|/\sqrt{h}}\, dy\, dx \\
&\leqslant \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} |f(y)| \int_{0}^{\infty} e^{-|y-x|/\sqrt{h}}\, dx\, dy \\
&\leqslant \int_{-\infty}^{\infty} |f(y)|\, dy = \|f\|_1,
\end{aligned}$$

which means $\|(I - h\Delta)^{-1}\|_1 \leqslant 1$.

The third condition of Lemma 3 is a control on the slope error of the approximation. A bound on this error may be established for the implicit–explicit iteration as follows.

Lemma 10. *Suppose* $f_0 \in X$, $h > 0$. *Put* $f(x,t) = f_0(x) + tD(x)$, *where*

$$D = \frac{(I - h\Delta)^{-1}(f_0 + hG(f_0)) - f_0}{h}.$$

Then for every $0 < t < h$,

$$\|f'(t) - F(f(t))\|_\infty = O(h). \tag{11}$$

Proof. Recall every function in X will have bounded partial derivatives up to fourth order from Definition 6.

$$\begin{aligned}
\|f'(t) - F(f(t))\|_\infty &= \|D - (\Delta(f_0 + tD) + G(f_0 + tD))\|_\infty \\
&= \left\| D - \left(\Delta(f_0 + tD) + \sum_{i=0}^{\infty} a_i (f_0 + tD)^i \right) \right\|_\infty \\
&\leqslant \left\| D - \Delta f_0 - t\Delta D - \sum_{i=0}^{\infty} a_i \left(\sum_{j=0}^{i} \binom{i}{j} f_0^j (tD)^{i-j} \right) \right\|_\infty \\
&\leqslant \left\| D - \Delta f_0 - t\Delta D - \sum_{i=0}^{\infty} a_i f_0^i \right\|_\infty + O(h) \\
&\leqslant \left\| \frac{(I - h\Delta)^{-1} - I}{h} f_0 - \Delta f_0 + ((I - h\Delta)^{-1} - I) G(f_0) \right\|_\infty + O(h).
\end{aligned}$$

Now, using the fact that $(I - h\Delta)^{-1} - I = (I - h\Delta)^{-1}(h\Delta)$,

$$\begin{aligned}\big\| f'(t) - F\big(f(t)\big)\big\|_\infty &\leqslant \big\|(I - h\Delta)^{-1}\Delta f_0 - \Delta f_0 + (I - h\Delta)^{-1}(h\Delta)G(f_0)\big\|_\infty + O(h) \\ &\leqslant \big\|(I - h\Delta)^{-1}(h\Delta)\big(\Delta f_0 + G(f_0)\big)\big\|_\infty + O(h) \\ &\leqslant h\big\|(I - h\Delta)^{-1}\big(\Delta F(f_0)\big)\big\|_\infty + O(h) \\ &\leqslant h\big\|(I - h\Delta)^{-1}\big\|_\infty\big\|\big(\Delta F(f_0)\big)\big\|_\infty + O(h) = O(h). \qquad \square\end{aligned}$$

Lemma 11. *Suppose* $0 < h_i \to 0$. *Let* u_i *be the implicit–explicit iteration of size* h_i *beginning at* $f_0 \in X$ *on* $t \in [0, T]$. *Then provided there exist* $A, B > 0$ *such that for each* i *and* $t \in [0, T]$, $\|u_i(t)\|_1 \leqslant A$ *and* $\|u_i(t)\|_\infty \leqslant B$, *then the sequence* $\{u_i(t)\}_{i=1}^\infty$ *converges pointwise to a function in* t. *The limit function is piecewise differentiable in* t.

Proof. Let u_i be the implicit–explicit iteration of size h_i. By Lemma 10, the slope error is bounded:

$$\big\|u_i'(t) - F\big(u_i(t)\big)\big\|_\infty = O(h_i) = \epsilon_i.$$

Notice that $\epsilon_i \to 0$. Then, since $X \subset C^2(\mathbb{R}^n)$, Lemma 3 applies, giving a pointwise limit function $u(t)$. Finally, since the slope error uniformly vanishes, Lemma 5 implies that the solution is piecewise differentiable. $\square$

4. "*A priori* estimates" for the approximate solutions

Now we demonstrate that the implicit–explicit method converges for all initial conditions in X. Specifically, for each $f_0 \in X$, there exist $A, B > 0$ such that for each i and $t \in [0, T]$, $\|u_i(t)\|_1 \leqslant A$ and $\|u_i(t)\|_\infty \leqslant B$, given sufficiently small T. We begin by recalling that from Calculation 9, the L^∞-norm of $(I - h\Delta)^{-1}$ is less than one. This means that for the implicit–explicit iteration,

$$\begin{aligned}\|f_{n+1}\|_\infty &\leqslant \big\|f_n + hG(f_n)\big\|_\infty \\ &\leqslant \|f_n\|_\infty + h\left\|\sum_{i=0}^\infty a_i f_n^i\right\|_\infty \\ &\leqslant \|f_n\|_\infty + h\sum_{i=0}^\infty \|a_i\|_\infty \big\|f_n^i\big\|_\infty \\ &\leqslant \|f_n\|_\infty + h\sum_{i=0}^\infty \|a_i\|_\infty \|f_n\|_\infty^i \\ &\leqslant \|f_n\|_\infty + hg_\infty\big(\|f_n\|_\infty\big).\end{aligned}$$

Hence the norm of each step of the implicit–explicit iteration will be controlled by the behavior of the recursion

$$f_{n+1} = f_n + hg_\infty(f_n), \tag{12}$$

for $f_n, h, a > 0$. Since we are only concerned with short-time existence and uniqueness, we look specifically at $h = T/N$ and $0 \leqslant n \leqslant N$, for fixed $T > 0$ and $N \in \mathbb{N}$.

Remark 12. The recursion defined by (12) is an Euler solver for

$$\frac{dy}{dt} = g_\infty(y), \quad \text{with } y(0) = f_0. \tag{13}$$

This equation is separable, and g_∞ is analytic near f_0, so there exists a unique solution for the initial value problem (13) for sufficiently short time. Also, whenever $y(t) > 0$

$$\frac{d^2y}{dt^2} = g'_\infty\big(y(t)\big) > 0,$$

the function $y(t)$ is concave up. As a result, the exact solution to (13) provides an upper bound for the recursion (12). More precisely, we have the following result.

Lemma 13. *Suppose $y(0) = f_0 > 0$ in (13). Let $T > 0$ be given so that y is continuous on $[0, T]$, and let $N \in \mathbb{N}$. Then for each $0 \leqslant n \leqslant N$, $f_n \leqslant y(T)$, where f_n satisfies (12) with $h = T/N$.*

Proof. Since the right-hand side of (13) is strictly positive, the maximum of y is attained at T on any interval $[0, T]$ where y is continuous. Furthermore, since $y(0) > 0$, it follows from Remark 12 that y is concave up on all of $[0, T]$. Therefore, y is a convex function on $[0, T]$. Hence Euler's method, (12), will always underestimate the true value of y. Another way of stating this is that

$$f_n \leqslant y(nh) \leqslant y(T). \qquad \square$$

Using Lemma 13, the growth of iterates to (12) may be controlled independently of the step size. This provides a uniform bound on the sequence of implicit–explicit approximations.

Lemma 14. *Suppose $0 < h_i = T/i$ for $i \in \mathbb{N}$. Let u_i be the implicit–explicit iteration of size h_i beginning at $f_0 \in X$ on $t \in [0, T]$. Then there exists a $B > 0$ such that for each i and $t \in [0, T]$, we have $\|u_i(t)\|_\infty \leqslant B$ for sufficiently small $T > 0$.*

Proof. Suppose f_{in} is the nth step of the implicit–explicit iteration of size h_i. If we let $y(0) = \|f_0\|_\infty$, Lemma 13 implies that for any i and any $0 \leqslant n \leqslant i$

$$\|f_{in}\|_\infty \leqslant y(T)$$

for sufficiently small T. Hence by (9) and the triangle inequality, $\|u_i(t)\|_\infty \leqslant B$ for all i and $t \in [0, T]$. $\square$

With the bound on the suprema of the approximations, we can obtain a bound on the 1-norms.

Lemma 15. *Suppose $0 < h_i = T/i$ for $i \in \mathbb{N}$. Let u_i be the implicit–explicit iteration of size h_i beginning at $f_0 \in X$ on $t \in [0, T]$. Then there exists an $A > 0$ such that for each i and $t \in [0, T]$, we have $\|u_i(t)\|_1 \leqslant A$ for sufficiently small $T > 0$.*

Proof. First, notice that Lemma 14 implies that there is a $B > 0$ such that for each i and $t \in [0, T]$, we have $\|u_i(t)\|_\infty \leqslant A$ for sufficiently small $T > 0$. Again suppose f_{in} is the nth step of the implicit–explicit iteration of size h_i. Then we compute

$$\begin{aligned}
\|f_{i,n+1}\|_1 &\leqslant \|f_{in}\|_1 + h_i \big\|G(f_{in})\big\|_1 \\
&\leqslant \|f_{in}\|_1 + h_i \sum_{k=0}^{\infty} \big\|a_k f_{in}^k\big\|_1 \\
&\leqslant \|f_{in}\|_1 + h_i \sum_{k=0}^{\infty} \int \big|a_k f_{in}^k\big|\,dx \\
&\leqslant \|f_{in}\|_1 + h_i \sum_{k=1}^{\infty} \|f_{in}\|_\infty^{k-1} \|a_k\|_\infty \|f_{in}\|_1 + h_i \|a_0\|_1 \\
&\leqslant \|f_{in}\|_1 \left(1 + h_i \sum_{k=1}^{\infty} \|a_k\|_\infty B^{k-1}\right) + h_i \|a_0\|_1 \\
&\leqslant \|f_{in}\|_1 \left(1 + \frac{h_i}{B} g_\infty(B) - \frac{h_i}{B} \|a_0\|_\infty\right) + h_i \|a_0\|_1 \\
&\leqslant \|f_{in}\|_1 (1 + h_i C) + h_i \|a_0\|_1.
\end{aligned}$$

This recurrence leads to

$$\begin{aligned}
\|f_{in}\|_1 &\leqslant \|f_0\|_1 (1 + h_i C)^n + h_i \|a_0\|_1 \sum_{m=0}^{n-1} (1 + h_i C)^m \\
&\leqslant \|f_0\|_1 (1 + h_i C)^n + h_i \|a_0\|_1 \frac{(1 + h_i C)^n - 1}{h_i C} \\
&\leqslant \left(\|f_0\|_1 + \frac{1}{C} \|a_0\|_1\right) (1 + h_i C)^n - \frac{1}{C} \|a_0\|_1 \\
&\leqslant \left(\|f_0\|_1 + \frac{1}{C} \|a_0\|_1\right) \left(1 + \frac{CT}{i}\right)^n - \frac{1}{C} \|a_0\|_1 \\
&\leqslant \left(\|f_0\|_1 + \frac{1}{C} \|a_0\|_1\right) \left(1 + \frac{CT}{i}\right)^i - \frac{1}{C} \|a_0\|_1 \\
&\leqslant \left(\|f_0\|_1 + \frac{1}{C} \|a_0\|_1\right) e^{CT} - \frac{1}{C} \|a_0\|_1 = A.
\end{aligned}$$

Once again, by referring to (9) and using the triangle inequality, it follows that $\|u_i(t)\|_1 \leqslant B$ for all i and $t \in [0, T]$. □

Theorem 16. *Suppose $0 < h_i = T/i$ for $i \in \mathbb{N}$. Let u_i be the implicit–explicit iteration of size h_i beginning at $f_0 \in X$ on $t \in [0, T]$. Then, for sufficiently small $T > 0$, the sequence $\{u_i(t)\}_{i=1}^{\infty}$ converges pointwise to a function in t. The limit function is piecewise differentiable in t.*

Proof. This compiles the results of Lemmas 11, 14, and 15. □

Remark 17. These proofs can be generalized further to handle all equations of the form

$$\frac{\partial u(t)}{\partial t} = L\big(u(t)\big) + G(u),$$

where G is as in (4). If the operator L satisfies

- $L : L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a sectorial linear operator [4],
- $\|(I - hL)^{-1}\|_1 \leqslant 1$ and $\|(I - hL)^{-1}\|_\infty \leqslant 1$,

then the implicit–explicit iteration

$$f_{n+1} = (I - hL)^{-1}\big(f_n + hG(f_n)\big)$$

converges for whenever $f \in X$.

Remark 18. Additionally, the techniques can be easily extended to handle the initial boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + \sum_{i=0}^{\infty} a_i(x) u^i(x,t), \quad \text{for } x \in K \subset \mathbb{R}^n,\ t > 0,$$

with $u(x,t) = v(x,t)$ a given Lipschitz function along $\partial K \times [0,\infty)$, for K compact with smooth boundary. In this case, a boundary term appears in the estimate for $\eta'(t)$ in Lemma 3, which depends on the Lipschitz constant of v. Additionally, in Definition 7, one defines f_{n+1} to be the unique solution to the linear elliptic boundary value problem

$$(I - h\Delta) f_{n+1} = f_n + hG(f_n)$$

with $f_{n+1}(x) = v(x, nh)$ for $x \in \partial K$.

5. Conclusions

The convergence proof for the implicit–explicit method presented here has a number of advantages. First of all, like all implicit–explicit methods, each approximation to the solution is computed explicitly. As a result, a fully discretized version (as is standard in the literature) is easy to program on a computer. Theorem 16 therefore assures the convergence of these fully discrete methods.

However, since the implicit–explicit method presented here is discretized only in time, the convergence proof actually shows the existence of a semigroup of solutions. As a result, the convergence proof forms a bridge between the functional-analytic viewpoint of differential equations, namely that of semigroups, and the numerical methods used to approximate solutions. While the existence and uniqueness of solutions for (1) has been known via semigroup methods, the proof provided here gives a more elementary explanation of how this occurs. In particular, it approximates the semigroup action directly.

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