Maximum principles for fractional differential equations derived from Mittag–Leffler functions

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We present two new maximum principles for a linear fractional differential equation with initial or periodic boundary conditions. Some properties of the classical Mittag–Leffler functions are crucial in our arguments.

These comparison results allow us to study the corresponding nonlinear fractional differential equations and to obtain approximate solutions.

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1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, and goes back to time when Leibnitz and Newton invented differential calculus. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers. Fractional calculus techniques are widely used in rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas et al. [1], Kiryakova [2], Miller and Ross [3], Oldham and Spanier [4], Podlubny [5], and Samko et al. [6] and the references therein. Some recent contributions to the theory of fractional differential equations can be seen in [7–14].

The Mittag–Leffler functions appear naturally when solving linear fractional differential equations and they have been recently used to study different models; see [15–17]. However some of the properties of Mittag–Leffler functions seemed to be unknown to many researchers.

In this note we employ some basic properties of the Mittag–Leffler functions to present the analytic solution to a linear fractional differential equation with an initial condition or with a periodic boundary condition. Then we obtain some new maximum principles for the fractional order problem.

The maximum principles are important tools for studying differential equations and obtaining approximate solutions of the corresponding nonlinear problems.

Let \( 0 < \alpha < 1, \lambda \in \mathbb{R}, \sigma \in C[0, T], T > 0 \). We consider the following linear fractional differential equation:

\[
D^\alpha u(t) - \lambda u(t) = \sigma(t), \quad t \in J := (0, 1], 0 < \alpha < 1,
\]

where \( D^\alpha u \) is the Riemann–Liouville fractional derivative (see [5,6]). The appropriate initial condition for (1) is

\[
\lim_{t \to 0^+} t^{1-\alpha} u(t) = u_0 \in \mathbb{R}.
\]
We also consider the boundary condition of periodic type
\[
\lim_{t \to T} t^{1-\alpha} u(t) = u(T) .
\]  
(3)

This note is organized as follows. In Section 2 we recall some basic definitions and facts on fractional calculus, on linear fractional differential equations, and on Mittag–Leffler functions. Then we prove some comparison principles for the initial problems (1)–(2) and for the periodic boundary problems (1)–(3). We note that our results improve previous theorems and include the integer case \( \alpha = 1 \).

2. Preliminary results

In this section, we introduce some notation, definitions, and preliminarily facts which are used throughout this paper.

Let \( C[0, 1] \) be the Banach space of all continuous real functions defined on \([0, 1]\) with the norm \( \|f\| = \sup\{|f(t)| : t \in [0, 1]\} \). Define for \( t \in [0, 1] \), \( f(t) = t^r f(t) \). Let \( C_r[0, 1] \), \( r \geq 0 \) be the space of all functions \( f \) such that \( f_r \in C[0, 1] \) which turns out to be a Banach space when endowed with the norm \( \|f\|_r = \sup\{|f_r(t)| : t \in [0, 1]\} \).

By \( L^1[0, 1] \) we denote the space of all real functions defined on \([0, 1]\) which are integrable. Obviously \( C_r[0, 1] \subset L^1[0, 1] \) if \( r < 1 \).

The Riemann–Liouville fractional integral of order \( 0 < \alpha < 1 \) of a function \( u : (0, 1) \to \mathbb{R} \) is given by [5,6]
\[
I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) \, d\tau,
\]
provided the right side is pointwise defined on \((0, 1)\), and where \( \Gamma \) is the classical gamma function. For instance, \( I^\alpha u \) exists for all \( \alpha > 0 \), when \( u \in C([0, 1]) \cap L^1([0, 1]) \); note also that when \( u \in C([0, 1]) \), \( I^\alpha u \in C([0, 1]) \) and moreover \( I^\alpha u(0) = 0 \).

Let \( 0 < \alpha < 1 \), if \( u \in C_r[0, 1] \) with \( \beta < \alpha \), then \( I^\alpha u \in C_r[0, 1] \), with \( I^\alpha u(0) = 0 \).

The Riemann–Liouville fractional derivative of order \( 0 < \alpha < 1 \) of a continuous function \( u : (0, 1) \to \mathbb{R} \) is given by [5,6]
\[
D^\alpha u(t) = \frac{d}{dt} I^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} u(\tau) \, d\tau = \frac{d}{dt} I^{1-\alpha} u(t) .
\]

We have \( D^\alpha I^\alpha u = u \) for all \( u \in C([0, 1]) \cap L^1([0, 1]) \). If we assume \( u \in C([0, 1]) \cap L^1([0, 1]) \), then the fractional differential equation \( D^\alpha u = 0 \) has solutions \( u(t) = c t^{\alpha-1} \), \( c \in \mathbb{R} \), as solutions. Hence, for \( u \in C([0, 1]) \cap L^1([0, 1]) \) with a fractional derivative of order \( 0 < \alpha < 1 \) that belongs to \( C([0, 1]) \cap L^1([0, 1]) \), we have \( D^\alpha u(t) = u(t) + ct^{\alpha-1} \) for some \( c \in \mathbb{R} \).

For \( \lambda = 0 \), Eq. (1) is \( D^\alpha u(t) = \sigma(t) \), \( t \in (0, T] \), and its general solution is \( u(t) = ct^{\alpha-1} + I^\alpha \sigma(t) \), \( t \in (0, T] \), with \( c \) an arbitrary constant. For \( \alpha > 0 \), \( \beta > 0 \), the Mittag–Leffler function is defined as
\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.
\]
In particular, for \( \beta = 1 \) and for \( \alpha = \beta \), we have \( E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \), \( z \in \mathbb{C} \), and \( E_{\alpha, \alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \), \( z \in \mathbb{C} \), respectively. These functions are entire functions of order \( \rho = \frac{1}{\alpha} \) and of type 1.

For \( 0 < \alpha < 1 \) and \( x \in \mathbb{R}, x > 0 \), it is clear that \( E_{\alpha, \alpha}(x) \geq \frac{x^\alpha}{\Gamma(1+\alpha)} > 0 \). For \( x < 0 \), showing that \( E_{\alpha, \alpha}(x) > 0 \) is not so trivial, but using the fact that \( E_{\alpha}(-x), x \in (0, \infty) \) is completely monotonic [18,19] we have that \( E_{\alpha, \alpha}(x) > 0 \) for every \( x \in \mathbb{R} \).

3. The initial problem

The linear fractional equation (1) with the initial condition (2) has a unique solution given by [1]
\[
u(t) = u_0 \Gamma(\alpha) t^{\alpha-1} E_{\alpha, \alpha}(\lambda t^\alpha) + \int_0^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda t - s^\alpha) \sigma(s) \, ds.
\]
(4)

For \( \alpha = 1 \) this representation is still valid since \( \Gamma(1) = 1 \) and \( E_{1,1}(z) = e^z \). We thus recover the usual solution for the first-order ordinary differential equation \( u'(t) - \lambda u(t) = \sigma(t) \) with the initial condition \( u(0) = u_0 \):
\[
u(t) = u_0 e^{\lambda t} + \int_0^t e^{\lambda(t-s)} \sigma(s) \, ds.
\]
(5)

We have the following maximum principle:

**Theorem 3.1.** Let \( \alpha \in (0, 1), \lambda \in \mathbb{R}, \) and \( u \in C_{1-\alpha}[0, T] \) be such that
\[
D^\alpha u(t) - \lambda u(t) \geq 0, \quad \lim_{t \to 0^+} t^{1-\alpha} u(t) \geq 0 .
\]

Then \( u(t) \geq 0 \) for \( t \in (0, T] \).
Proof. Using the integral representation (4) and the fact that $E_{\alpha,\alpha}(\lambda) > 0$, $\lambda \in \mathbb{R}$, it suffices to take $\sigma(t) = D^{\alpha}u(t) - \lambda u(t) \geq 0$ and the initial condition $u_0 = \lim_{t \to 0^+} t^{1-\alpha} u(t)$. □

This result improves Lemma 2.1 in [14] where the author required $\lambda < \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}$.

For $\alpha = 1$, the maximum principle (see, for example, [20]) is still valid using the representation (5).

4. The periodic problem

We now consider the linear fractional equation (1) with the boundary condition (3). For every initial condition $u_0$ the solution of Eq. (1) is given by (4), and thus

$$u(T) = u_0 \Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}(\lambda T^\alpha) + \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) \sigma(s) \, ds.$$ 

Equation $u(T) = u_0$ has a unique solution if and only if

$$\Gamma(\alpha) T^{\alpha-1} E_{\alpha,\alpha}(\lambda T^\alpha) \neq 1.$$ 

For example for $T = 1$ this condition is satisfied and the solution of the problem (1)--(3) is given by (see [21])

$$u(t) = \int_0^1 G_{\alpha,\alpha}(t, s) \sigma(s) \, ds,$$

where

$$G_{\alpha,\alpha}(t, s)$$

is defined by

$$\frac{\Gamma(\alpha) E_{\alpha,\alpha}(\lambda t^\alpha) E_{\alpha,\alpha}(\lambda (1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} + (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha)$$

for $0 \leq s \leq t \leq 1,$

and by

$$\frac{\Gamma(\alpha) E_{\alpha,\alpha}(\lambda t^\alpha) E_{\alpha,\alpha}(\lambda (1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)}$$

for $0 \leq t < s \leq 1.$

Theorem 4.1. Let $T = 1$, $\alpha \in (0, 1)$, $\lambda \in \mathbb{R}$, and $E_{\alpha,\alpha}(\lambda) < \frac{1}{\Gamma(\alpha)}$. Suppose $u \in C_{1-\alpha}[0, T]$ is such that

$$D^{\alpha} u(t) - \lambda u(t) \geq 0, \quad \lim_{t \to 0^+} t^{1-\alpha} u(t) = u(T).$$

Then $u(t) \geq 0$ for $t \in (0, T]$.

Proof. We have $G_{\alpha,\alpha}(t, s) > 0$ for every $(t, s) \in [0, 1] \times [0, 1]$ since $1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda) > 0$. Using the integral representation (6) we obtain the conclusion. □

For $\alpha = 1$ the problem is $u'(t) - \lambda u(t) = \sigma(t)$, $t \in [0, 1]$, $u(0) = u(1)$ and it has a unique solution [20] given by (6). It coincides with the usual representation of the solution; see [20].

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