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Some Isoperimetric Inequalities for Membrane Frequencies and Torsional Rigidity*

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I. INTRODUCTION

Let Λ denote the fundamental frequency of a two-dimensional membrane G fixed on its boundary. Let A be the area of G, and L its perimeter. Makai [5, 6] has recently shown that if G is simply or doubly connected, the dimensionless quantity $\Lambda^2 A^2 L^{-2}$ is at most 3. Pólya [7] has improved this result to

$$\Lambda^2 \leqslant (\tfrac{1}{2}\pi)^2 L^2 A^{-2}. \tag{1.1}$$

The constant $(\frac{1}{2}\pi)^2$ is optimal, since equality is attained in the limiting case of an infinite rectangular strip. To obtain these results Makai and Pólya insert in the minimum principle for Λ^2 functions which depend only on the distance from the boundary.

In this paper we apply a similar method to a two-dimensional membrane G fixed on its exterior bounding curve C_0. The membrane is permitted to have interior bounding curves C_i (holes) along which it is free. We shall show that among all such membranes with given area A and given perimeter L of C_0 the highest fundamental frequency is attained when G is annular.

This fact gives the upper bound

$$\Lambda \leqslant 2\pi L^{-1} \mu \tag{1.2}$$

where μ is the lowest root of the transcendental equation

$$J_0(\mu) Y_1(\mu\Psi) = Y_0(\mu) J_1(\mu\Psi) \tag{1.3}$$

with

$$\Psi^2 = 1 - 4\pi A L^{-2}. \tag{1.4}$$

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The classical isoperimetric inequality [1, p. 83] shows that the expression on the right of (1.4) is always nonnegative, and vanishes if and only if G is a circle. The solution of (1.3) is graphed in Jahnke and Emde [3, pp. 207–208]. If G is simply-connected the inequality (1.2) is an improvement of (1.1).

The same method yields an isoperimetric inequality for membranes G which are elastically supported on C_0 and free along any inner boundaries C_i. The annular membrane has the largest fundamental frequency among all such membranes of given area, perimeter of C_0, and elastic constant.

In a similar manner we find a lower bound for the torsional rigidity of a simply connected domain. Again we obtain an improvement of the inequalities of Makai [5,6] and Pólya [7].

The inequalities of Makai and Pólya for the fundamental frequency and torsional rigidity hold for doubly connected (ring-shaped) as well as simply connected domains G.

Our bound (1.2) for the fundamental frequency applies when only the outer boundary C_0 of G is fixed. However, we may obtain a bound for a membrane G which is fixed along C_0 and along one or more inner boundaries C_i. To do this, we replace G by a membrane $\bar{G}$ which occupies the same domain and whose boundaries are fixed wherever those of G are fixed, as well as along straight-line paths connecting the fixed boundary components. Then the fundamental frequency $\bar{\Lambda}$ of $\bar{G}$ is greater than Λ. Moreover, $\bar{G}$ is fixed along a single curve $\bar{C}_0$ consisting of the fixed boundary components of G together with the connecting paths, covered twice. The perimeter $\bar{L}$ of $\bar{C}_0$ exceeds the total length L of the fixed boundary components of G by twice the total length of connecting lines. The area of $\bar{G}$ is again A.

Thus, we obtain the bound (1.2) with L replaced by $\bar{L}$ in (1.2) and (1.4). Whether or not this bound is better than (1.1) when G is ring-shaped depends upon the location of the hole.

Similar remarks apply to the torsional rigidity of multiply connected domains.

II. The Fundamental Frequency

Let G be a plane domain lying inside a simple closed bounding curve C_0, and possibly having interior holes bounded by smooth curves C_i. Let Λ^2 be the lowest eigenvalue of the membrane problem:

$$\begin{aligned} &\Delta u + \Lambda^2 u = 0 && \text{in } G \\ &u = 0 && \text{on } C_0 \\ &\partial u/\partial n = 0 && \text{on } C_i. \end{aligned} \tag{2.1}$$

It is well known [1, pp. 345–346; 9, p. 87] that

$$\Lambda^2 \leqslant \frac{\iint_G |\operatorname{grad} v|^2 \, dx\, dy}{\iint_G v^2 \, dx\, dy} \tag{2.2}$$

where v is any piecewise continuously differentiable function vanishing on C_0.

We define C_δ to be the curve consisting of points inside C_0 at distance δ from C_0. It was shown by Sz.-Nagy [11] that the length $\bar{l}(\delta)$ of C_δ is well defined for almost all values of δ, and that $\bar{l}(\delta) + 2\pi\delta$ is non-increasing in δ. Thus if $l(\delta)$ is the length of the portion of C_δ which lies in G,

$$l(\delta) \leqslant \bar{l}(\delta) \leqslant L - 2\pi\delta \tag{2.3}$$

where $L = \bar{l}(0)$ is the length of C_0.

Let $a(\delta)$ be the area of the portion of G lying between C_0 and C_δ. Then

$$a(\delta) = \int_0^\delta l(\delta)\, d\delta. \tag{2.4}$$

Integrating (2.3) gives

$$a(\delta) \leqslant L\delta - \pi\delta^2. \tag{2.5}$$

Inserting (2.3) in this inequality yields

$$\left(\frac{da}{d\delta}\right)^2 = l^2 \leqslant L^2 - 4\pi a(\delta). \tag{2.6}$$

We define a function $r(\delta)$ by

$$4\pi^2 r^2 = L^2 - 4\pi a(\delta). \tag{2.7}$$

If we interpret this equation as a mapping of the portion of C_δ in G onto the circle of radius $r(\delta)$, we find that C_0 is mapped into a circle of equal perimeter and that the portion of G between C_0 and C_δ goes into an annulus of equal area $a(\delta)$. We differentiate (2.7) and use (2.6) and the fact that

$$|\operatorname{grad} \delta| = 1 \tag{2.8}$$

almost everywhere to show that

$$|\operatorname{grad} r|^2 \leqslant 1 \tag{2.9}$$

almost everywhere in G.

We now let the function v in (2.2) depend only on r. In view of (2.9),

$$|\operatorname{grad} v|^2 \leqslant \left(\frac{dv}{dr}\right)^2. \tag{2.10}$$

Since the mapping (2.7) is area-preserving, (2.2) becomes

$$\Lambda^2 \leqslant \frac{\int_{r_1}^{r_2} \left(\frac{dv}{dr}\right)^2 r\,dr}{\int_{r_1}^{r_2} v^2\, r\,dr}, \tag{2.11}$$

where

$$\begin{aligned} r_1 &= (L^2 - 4\pi A)^{1/2}/2\pi \equiv L\Psi/2\pi, \\ r_2 &= L/2\pi, \end{aligned} \tag{2.12}$$

and v is any differentiable function of r satisfying

$$v(r_2) = 0. \tag{2.13}$$

The right-hand side of (2.11) is the Rayleigh quotient for the annular membrane $\mathring{G}$ whose area is A and whose outer boundary has perimeter L. Its minimum under the condition (2.13) is the lowest eigenvalue for the membrane $\mathring{G}$ fixed on the outer boundary and free along the inner boundary. Thus we have established that $\mathring{G}$ has the highest fundamental frequency among all membranes G with given A and L.

The minimum value of the expression on the right of (2.11) is attained for

$$v = J_0(2\pi L^{-1}\mu r)Y_0(\mu) - Y_0(2\pi L^{-1}\mu r)J_0(\mu) \tag{2.14}$$

where μ is determined in such a way that $v'(r_1) = 0$. It is the lowest root of the Eq. (1.3) (cf. [3, pp. 207—208]), and therefore depends upon the dimensionless quantity Ψ defined by (1.4). Substituting (2.14) in (2.11) leads to the bound

$$\Lambda \leqslant 2\pi L^{-1}\mu. \tag{2.15}$$

If G has no holes C_i, a lower bound for Λ^2 in terms of the area A is given by the isoperimetric inequality of Faber [2] and Krahn [4].

$$\Lambda^2 \geqslant \pi j^2 A^{-1}. \tag{2.16}$$

Here $j(\approx 2.4048)$ is the first zero of the Bessel function J_0. Equality in (2.16) is attained when G is a circle.

If in (2.11) we choose

$$v = J_0(j\,[\pi A^{-1}(r^2 - r_1{}^2)]^{1/2}), \tag{2.17}$$

which satisfies (2.13), we obtain the upper bound

$$\begin{aligned} \Lambda^2 &\leqslant \pi^2 j^2 A^{-1}[1 + (J_1{}^{-2}(j) - 1)\Psi^2 (1 - \Psi^2)^{-1}] \\ &\leqslant \pi j^2 A^{-1}\,[1 + 2.712\Psi^2(1 - \Psi^2)^{-1}]. \end{aligned} \tag{2.18}$$

Here Ψ^2 is the dimensionless quantity defined by (1.4). Again equality is attained when G is a circle.

The inequalities (2.16) and (2.18) show that if G is simply connected and nearly circular in the sense that Ψ is small, the fundamental frequency Λ is near that of the circle of equal area.

Since the function (2.14) yields the best upper bound for Λ^2, the inequality (2.15) is in general sharper than (2.18).

III. The Elastically Supported Membrane

We consider the lowest eigenvalue $\Lambda^2(k)$ of the problem

$$\begin{aligned} &\Delta u + \Lambda^2 u = 0 && \text{in } G, \\ &\partial u/\partial n + ku = 0 && \text{on } C_0, \\ &\partial u/\partial n = 0 && \text{on } C_i. \end{aligned} \tag{3.1}$$

The elastic constant k is positive. For any piecewise continuously differentiable function v we have the inequality (cf. [1, pp. 345–346]).

$$\Lambda^2(k) \leqslant \frac{\displaystyle\iint_G |\text{grad}\, v|^2\, dx\, dy + k \oint_{C_0} v^2\, ds}{\displaystyle\iint_G v^2\, dx\, dy}. \tag{3.2}$$

We introduce the new variable r as in Section II and let v be a function of r only. This gives the upper bound

$$\Lambda^2(k) \leqslant \frac{\int_{r_1}^{r_2} \left(\frac{dv}{dr}\right)^2 r\,dr + 2\pi k r_2\, v^2(r_2)}{\int_{r_1}^{r_2} v^2\, r\,dr} \tag{3.3}$$

where r_1 and r_2 are given by (2.12). The right hand side of (3.3) is the Rayleigh quotient for the annular membrane $\overset{\circ}{G}$ of area A elastically supported (with elastic constant k) on the outer boundary of perimeter L, and free on the inner boundary. The minimum of the Rayleigh quotient is the lowest eigenvalue of this membrane. Thus we have shown that $\overset{\circ}{G}$ gives the highest fundamental frequency among all membranes G of given A, L, and k. This fact leads to the upper bound

$$\Lambda(k) \leqslant 2\pi L^{-1} \mu \tag{3.4}$$

where μ is the lowest root of the equation

$$Y_1(\mu\Psi)\,[kLJ_0(\mu) - 2\pi\mu J_1(\mu)] = J_1(\mu\Psi)\,[kLY_0(\mu) - 2\pi\mu Y_1(\mu)]. \tag{3.5}$$

If k in problem (3.1) is a nonnegative function of arc length rather than a constant, the inequality (3.4) still holds with kL in (3.5) replaced by $\oint_{C_0} k\,ds$.

IV. Torsional Rigidity

Let G be a simply connected domain of area A bounded by the closed curve C_0 of perimeter L. The torsional rigidity P of G is defined by [9, p. 87].

$$P = \max \frac{\left[2\iint_G v\,dx\,dy\right]^2}{\iint_G |\operatorname{grad} v|^2\,dx\,dy} \tag{4.1}$$

among sufficiently regular functions v which vanish on C_0.

We define the variable r as in section 2 and let

$$v = \frac{1}{2}(r_2{}^2 - r^2) + r_1{}^2 \log \frac{r}{r_2}. \tag{4.2}$$

Using the results of Section II leads immediately to the bound

$$P \geqslant \frac{A^2}{2\pi} [1 - 2\Psi^2(1 - \Psi^2)^{-1} - 4\Psi^4(1 - \Psi^2)^{-2} \log \Psi] \qquad (4.3)$$

where Ψ is given by (1.4). An upper bound for P in terms of A is given by the isoperimetric inequality

$$P \leqslant A^2/2\pi \qquad (4.4)$$

which was conjectured by St. Venant [10] and proved by Pólya [8]. Again we see that if G is nearly circular in the sense that Ψ is small, its torsional rigidity is close to that of the circle of equal area.

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