Global Existence of Solutions to the Derivative 2D Ginzburg–Landau Equation

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In this paper we study a complex derivative Ginzburg–Landau equation with two spatial variables (2D). We obtain sufficient conditions for the existence and uniqueness of global solutions for the initial boundary value problem of the derivative 2D Ginzburg–Landau equation and improve the known results. © 2000 Academic Press

Key Words: derivative 2D Ginzburg–Landau equation; initial boundary value problem; a priori estimate; global existence; regularity of the solution.

1. INTRODUCTION

There has been extensive work on the Ginzburg–Landau equations (GLE) of type

$$u_t = \gamma u + (1 + iv)\Delta u - (1 + i\mu)|u|^2u,$$

(1.1)

which describes various pattern formations and the onset of instabilities in nonequilibrium fluid dynamical systems, as well as in the theory of phase transitions and superconductivity. Here $i = \sqrt{-1}$, $\gamma$, $v$, and $\mu$ are given real numbers. $\gamma > 0$ is the instability parameter and $v$ the dispersive

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parameter of either sign. Doering et al. [6], Ghidaglia and Héron [15], Promislov [25], etc., studied the finite dimensional global attractor and related dynamical issues for the 1D or 2D (i.e., one or two spatial dimensional) GLE with cubic nonlinearity ($\sigma = 1$). Levermore and Oliver [20] studied 1D GLE as a model problem. In [3] Bu obtained the global existence of the Cauchy problem of the 2D GLE with $\sigma = 1$ and $\sigma = 2$, $\nu \mu > 0$, or $|\mu| \leq \frac{\sqrt{2}}{4}$. Bartuccelli et al. [1] investigated “strong” and “weak” turbulence of (1.1) in the case of cubic nonlinearity in 2D. Doering et al. [7] showed the global existence and regularity of solutions of (1.1) in all cases of spatial dimensions and parameters. Ginibre and Velo [16] studied the solutions in local spaces, that is, the spaces of functions satisfying only the local regularity condition but with no restrictions on their behavior at infinity. Mielke [21] discussed the solution of (1.1) in weighted $L^p$ space and derived some new bounds and investigated some properties of attractors.

A 1D generalized (derivative) Ginzburg–Landau equation has the form (see Doelman [4, 5, 11])

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x - \alpha_5 |u|^2 u, \quad (1.2)$$

where $\alpha_0 > 0$, $\alpha_j = \alpha_j + ib_j$, $j = 1,\ldots,5$, $a_1 > 0$, $a_5 > 0$. The most important and interesting case in physics is $\sigma = 2$. In [8–10, 13, 14, 17], when $\sigma = 2$, the global existence of solutions, finite dimensional global attractor, Gevery regularity of solutions, exponential attractor, number of determining nodes, and inertial forms have been studied. In [8] Duan and Holmes obtained the global existence for the Cauchy problem of Eq. (1.2) under the condition of $4a_4a_5 > (b_3 - b_5)^2$ for the global existence of the initial value problem. Guo and Wang [18] considered the 2D derivative Ginzburg–Landau equation

$$u_t = \rho u + (1 + i\nu) \Delta u - (1 + i\mu)|u|^{2\sigma} u + \alpha \lambda_1 \cdot \nabla(|u|^2 u)$$

$$+ \beta(\lambda_2 \cdot \nabla u)|u|^2, \quad (1.3)$$

where $\rho > 0$, $\alpha$, $\beta$, $\nu$, $\mu$ are real numbers, and $\lambda_1$, $\lambda_2$ are real constant vectors. They studied the existence of a finite dimensional global attractor of Eq. (1.3) with periodic boundary conditions by assuming that there exists a positive number $\delta > 0$ so that

$$\frac{1}{\sqrt{1 + \left(\frac{\mu - \nu \delta^2}{1 + \delta^2}\right)^2}} \geq \sigma \geq 3. \quad (1.4)$$
Later the results on the existence of solution of (1.3) were improved in [12], where the authors proved the existence of global $H^2$ solutions to the Cauchy problem of

$$u_t = \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \overline{u}_x + \alpha_5 u^2 \overline{u}_y - \alpha_6 |u|^{2\sigma} u,$$

where $\alpha_0 > 0$, $\alpha_j = a_j + i b_j$, $1 \leq j \leq 6$, $a_1 > 0$, $a_6 > 0$, $\sigma > 0$. They assumed that $\sigma \geq \frac{1 + \sqrt{10}}{2}$ when $b_1 b_6 > 0$, or that there exists a positive number $\delta > 0$ such that

$$\sqrt{1 + \frac{(b_1 \delta - b_6)^2}{(1 + \delta)(a_1 \delta + a_6)}} \geq \sigma \geq \frac{1 + \sqrt{10}}{2}$$

if $b_6 = 0$, or $b_1 b_6 < 0$ and $\sigma a_1 a_6 \geq \max((b_2 - b_4)^2, (b_3 - b_5)^2)$.

In this paper, we consider the initial boundary value problem for the 2D derivative Ginzburg–Landau equation (see [4, 5])

$$u_t = \gamma u + (1 + i \nu) \Delta u - (1 + i \mu)|u|^{2\sigma} u + \lambda_1 \cdot \nabla(|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2, \quad t \geq 0, \quad x \in \Omega,$$

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

$$u(t, x) = 0, \quad t \geq 0, \quad x \in \partial \Omega,$$

where $\gamma > 0$, $\nu$, $\mu$ are real numbers, $\lambda_1 = (\lambda_{11}, \lambda_{12})$, $\lambda_2 = (\lambda_{21}, \lambda_{22})$ are constant vectors with complex components, and $\Omega \subset \mathbb{R}^2$ is a regular domain, bounded or not. When $\Omega = \mathbb{R}^2$ we do not need the boundary condition (1.9). Here we treat the Dirichlet boundary problem; the Cauchy and other boundary (such as Neumann and periodic boundaries) value problems can be discussed in the same manner. If we redefine the space scale and the scale for $u$, we may normalize $a_j, a_6$ in (1.5) to 1 and thus reduce Eq. (1.5) to the form of (1.7). The main purpose of this paper is to study the existence and uniqueness of the global (in time) solution for the problem (1.7)–(1.9) with the initial condition belonging to $W^{1, p}_0(\Omega)$, $p > 2$.

Our assumptions on $\sigma$, $\nu$, and $\mu$ are

(A1) either (i) $\sigma > 2$ or (ii) $\sigma = 2$, $|\lambda_1|$ and $|\lambda_2|$ are suitably small;

(A2) $-1 - \nu \mu < \frac{i(\sigma + 1)}{\sigma} |\nu - \mu|$. 
Main Theorem. Assume that \( \sigma, \nu, \mu \) satisfy (A1) and (A2). Then for any \( u_0 \in W_0^{1, p}(\Omega) \), \( p > 2, (1.7)-(1.9) \) possess a unique solution

\[
u \in C([0, \infty) ; W_0^{1, p}(\Omega)) \cap C((0, \infty) ; W^{2,p}(\Omega)).
\]

When \( \sigma \) is an integer, \( u \in C^\infty((0, \infty) \times \Omega) \).

When \( \Omega \) is a bounded domain, the result is also true for \( p_0 = \max\left(\frac{3}{2}, \frac{\nu + \frac{1}{\nu}}{\nu}\right) \) \(< p \leq 2 \).

Remark. (1) The inequality of (A2) is the same as in [2, (15)], which indicates the “soft” turbulence for Eq. (1.1). The first and the third quadrants of the \((\nu, \mu)\)-plane, which indicate the modulational stability of (1.1), are contained in the area between the pair of hyperbolae of (A2).

(2) The assumptions (A1) and (A2), the class of initial data for the global existence, are more general than that used in [12, 18]. Note that, for any \( \sigma > 0 \), the left half inequalities of (1.4) and (1.6) commensurate in fact with two strips in the \((\nu, \mu)\)-plane

\[|\nu| < \frac{\nu + 1}{\sigma} \quad \text{or} \quad |\mu| < \frac{\nu + 1}{\sigma},\]

which are contained in the area between the pair of hyperbolae of (A2).

The rest of this paper is arranged as follows. First we prove in Section 2 the local existence of solutions by putting (1.7)-(1.9) into a functional setting. Next, in Section 3, adapting methods from [21], we establish some a priori estimates in \( L^p \) and then in \( H^1 \) space. Finally, in Section 4 we can use the properties of the analytical semigroup to obtain the global existence and regularity of solutions.

2. LOCAL EXISTENCE OF SOLUTION

In the sequel, \( W^{m,p}(\Omega) \) and \( W_0^{m,p}(\Omega) \) denote the usual Sobolev spaces, \( H^k(\Omega) = W^{m-k}(\Omega) \) and \( H_0^k(\Omega) = W_0^{m-k}(\Omega) \). Let \( X_p = L^p(\Omega), 1 < p < \infty \). We define a linear operator in \( X_p \) by

\[B_p u = -(1 + i\nu)\Delta u, \quad D(B_p) = W^{2,p} \cap W_0^{1,p}(\Omega). \]

Let \( A_p = B_p + \gamma_0 \), where \( \gamma_0 \) is any fixed positive number when \( \Omega \) is unbounded and \( \gamma_0 = 0 \) when \( \Omega \) is bounded. Then \( A_p \) is a strong elliptic operator on \( \Omega \), \( 0 \in \rho(A_p) \), the resolvent of \( A_p \), and \( -A_p \) generates a uniformly bounded analytic semigroup \( S(t) = e^{-A_p t} \) for \( t \geq 0 \) (see [24, Theorem 7.3.5, 2.5.2]).
Moreover, if

\[ 2 \leq p < \frac{2\sqrt{1 + \nu^2}}{\sqrt{1 + \nu^2} - 1} \quad \text{or} \quad \frac{2\sqrt{1 + \nu^2}}{\sqrt{1 + \nu^2} + 1} < p \leq 2 \quad (2.2) \]

then the semigroup generated by \(-B_p\) is contractive. To see this we need only to show that \(-B_p\) is a dissipative operator; then by the Lumer–Phillips theorem we see that \(-B_p\) generates a semigroup of contraction on \(X_p\) [24, Theorem 1.4.3].

By the definition of a dissipative operator, we have to show that, for any \(u \in D(B_p)\), there exists a \(u^* \in \mathcal{F}(u)\), \(\Re \langle B_p u, u^* \rangle \geq 0\), where \(\mathcal{F}\) is the dual map of \(X_p^\prime\), that is, \(\mathcal{F}(u) = \{u^* \mid u^* \in X_p^\prime, \langle u, u^* \rangle = \|u\|_{X_p^\prime}^2 = \|u^*\|_{X_p^\prime}^2\}\).

It is obvious that \(u^* \in \mathcal{F}(u)\) if and only if \(u^* = |u|^{p-2}u/\|u\|_{X_p^\prime}^2\). For any \(u \in D(B_p)\) with \(\|u\|_{X_p} = 1\),

\[ \Re \langle B_p u, u^* \rangle = -\Re \int (1 + i \nu)|u|^{p-2} \overline{u}\Delta u \, dx \]

\[ = \Re \int (1 + i \nu)\nabla(|u|^{p-2} \overline{u})\nabla u \, dx \]

\[ = \frac{1}{2} \Re \int |u|^{p-4} (p|u|^2|\nabla u|^2 + (1 + i \nu)(p - 2)|\nabla u|^2) \, dx \]

\[ = \frac{1}{4} \int |u|^{p-4} \sum_{j=1}^2 \left( \overline{u} \partial_j u, u \partial_j \overline{u} \right) M(\nu, p) \left( \frac{u \partial_j \overline{u}}{\overline{u} \partial_j u} \right) \, dx, \quad (2.3) \]

where

\[ M(\nu, p) = M(\nu, p)'' = \begin{pmatrix} p & (1 + i \nu)(p - 2) \\ (1 - i \nu)(p - 2) & p \end{pmatrix}. \quad (2.4) \]

When (2.2) is satisfied, the smaller eigenvalue of \(M(\nu, p)\)

\[ \lambda_M(\nu, p) = p - |p - 2|\sqrt{1 + \nu^2} > 0; \quad (2.5) \]

therefore \(M(\nu, p)\) is positively definite and we have

\[ \int |u|^{p-4} \sum_{j=1}^2 \left( \overline{u} \partial_j u, u \partial_j \overline{u} \right) M(\nu, p) \left( \frac{u \partial_j \overline{u}}{\overline{u} \partial_j u} \right) \, dx \geq \lambda_M(\nu, p) \int |u|^{p-2} |\nabla u|^2 \, dx. \quad (2.6) \]
Thus \( \text{Re}(B_p u, u^*) \geq 0 \). This ideal turns out to be very useful in controlling the integrals reduced by the nonlinear terms involving derivatives.

Therefore we can define the fractional powers of \( A_p, A_p^{1/2} \), with the domain of definition \( D(A_p^\beta) \), for any \( 0 < \beta < 1 \). \( S(t) \) has the following properties \cite{19, 24}:

\[
W_0^{2\beta, p}(\Omega) \subset D(A_p^\beta) \subset W^{2\beta, p}(\Omega), \quad 0 \leq \beta \leq 1,
\]

\[
D(A_p^{1/2}) = W_0^{1, p}(\Omega) \hookrightarrow L^p(\Omega), \quad p > 2,
\]

\[
D(A_p^\beta) \subset H^{2\beta}(\Omega) \hookrightarrow L^p(\Omega), \quad p = 2 \quad \text{and} \quad \beta > 1/2,
\]

\[
\|S(t)u\|_{X_p} \leq M\|u\|_{X_p}, \quad \forall u \in X_p, \quad t \geq 0,
\]

\[
\|A_p^\beta S(t)u\|_{X_p} \leq M\|u\|_{X_p}, \quad \forall u \in X_p, \quad \beta > 0, \quad t > 0,
\]

\[
\|A_p^\beta S(t)u\|_{X_p} \leq M\|u\|_{D(A_p^\beta)}, \quad \forall u \in D(A_p^\beta), \quad \beta \geq 0, \quad t \geq 0.
\]

When (2.2) is satisfied, the \( M^\gamma \)’s are 1. Let

\[
F(u) = (\gamma + \gamma_0)u + (1 + i\mu)|u|^2u + (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2.
\]

(2.13)

Then we can put (1.7)–(1.9) into a functional setting

\[
u_t + A_p u = F(u), \quad u(0) = u_0.
\]

(2.14)

It is not difficult to see that the nonlinear map \( F(u) \) is locally Lipschitz continuous from \( D(A_p^{1/2}) \) into \( X_p \) and from \( D(A_p) \) into \( D(A_p^{1/2}) \) for \( p > 2 \).

Thanks to the theory of operator semigroups in \cite{19, 24} we have

**THEOREM 2.1.** Let \( \sigma > 0, p > 2 \). If \( u_0 \in D(A_p^{1/2}) \), then there exists a unique solution of the abstract Cauchy problem (2.14) on an interval \([0, T_\ast)\),

\[
u(t) \in C([0, T_\ast), D(A_p^{1/2})) \cap C((0, T_\ast), D(A_p)).
\]

(2.15)

If \( T_\ast < +\infty \), then

\[
\lim_{t \to T_\ast} \|u(t)\|_{D(A_p^{1/2})} = +\infty.
\]

(2.16)

We remark that, when \( \Omega \) is a bounded domain, we can prove the local existence for a wider class of initial data. Let

\[
p_0 = p_0(\sigma) = \max \left\{ \frac{3}{2}, \frac{4\sigma + 2}{2\sigma + 3} \right\}
\]
and $p_0 < p < 2$. Note that $F(u)$ is no longer locally Lipschitz continuous from $D(A^{1/2}) = W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. However, we can find a $p_1$ with $1 < p_1 < p$ such that

$$F(u) : W_0^{1,p}(\Omega) = D(A^{1/2}) \to L^{p_1}(\Omega)$$

is locally Lipschitz continuous.

In fact, since $\frac{1}{2} < p_0 < p < 2$,

$$2 \varepsilon_0 \equiv \min\left\{ p, \frac{3}{3 - p}, \frac{2p}{(2 - p)(2\sigma + 1)} \right\} - 1 > 0.$$ We take $p_1 = 1 + \varepsilon_0$; then

$$1 < p_1 < \min\left\{ p, \frac{3}{3 - p}, \frac{2p}{(2 - p)(2\sigma + 1)} \right\}.$$ Thus

$$D(A^{1/2}) = W_0^{1,p}(\Omega) \to L^{p((2\sigma + 1)/p - p_1)}(\Omega),$$

and

$$\|F(u)\|_{L^{p_1}} \leq C(R^{2\sigma} + R^2 + 1)\|u\|_{W^{1,p}}, \quad (2.17)$$

$$\|F(u_1) - F(u_2)\|_{L^{p_1}} \leq C(R^{2\sigma} + R^2 + 1)\|u_1 - u_2\|_{W^{1,p}}, \quad (2.18)$$

for $u, u_1, u_2 \in B_R(0)$, the ball of radius $R$ centered at $0$ in $W_0^{1,p}(\Omega)$.

By virtue of (2.11), (2.12), (2.17), and (2.18) we have

**Theorem 2.2.** Assume that $\Omega$ is a bounded domain, $\sigma > 0$, $p_0 < p < 2$. Then for any $u_0 \in W_0^{1,p}(\Omega)$, there exists a unique solution of (2.14) on an interval $[0, T_*]$ such that (2.15) and (2.16) hold.

**Proof.** We write (2.14) in an integral form

$$u(t) = e^{A\tau}u_0 + \int_0^t e^{-A(t-s)}F(u(s)) \, ds. \quad (2.19)$$

Note that there is a $\theta_1 \in (0, 1)$ such that

$$\|v\|_{W^{1,p}} \leq C\|v\|_{W^{1,\theta_1}}^{\theta_1}\|v\|_{W^{2,p_1}}^{\theta_1}, \quad \forall v \in W^{2,p_1}(\Omega),$$

by (2.11) and (2.12), and

$$\|e^{-A\tau}v\|_{W^{1,p}} \leq C\|e^{-A\tau}v\|_{L^{p_1}}^{\frac{1}{\theta_1}}\|e^{-A\tau}v\|_{W^{2,p_1}}^{\frac{\theta_1}{p_1}} \leq Ct^{-\theta_1}\|v\|_{L^{p_1}}, \quad \forall v \in L^{p_1}(\Omega). \quad (2.20)$$
For any \( u_0 \in W_0^{1,p}(\Omega) \), we define a mapping on

\[
\mathcal{Y} = \left\{ v \in C([0, T], W_0^{1,p}(\Omega)) \mid \sup_{[0, T]} \|v\|_{W^{1,p}} \leq M\|u_0\| + 1 \right\}
\]

by

\[
u(t) = \mathcal{F}v(t) = e^{-A\rho t}u_0 + \int_0^t e^{-A\rho(t-s)}F(v(s)) \, ds,
\]

\[\quad t \in [0, T], \quad v \in \mathcal{Y}. \quad (2.21)\]

We can find \( T > 0 \) such that \( \mathcal{F} \) is a contractive mapping on \( \mathcal{Y} \). In fact, for \( v \in \mathcal{Y} \), by (2.11), (2.17), and (2.20),

\[
\|u\|_{W^{1,p}} \leq M\|u_0\|_{W^{1,p}} + \int_0^t \|A_p^{1/2}e^{-A\rho(t-s)}F(v(s))\|_{L^p} \, ds
\]

\[
\leq M\|u_0\|_{W^{1,p}} + \int_0^t C(t-s)^{-\theta_1}\|F(v(s))\|_{L^p} \, ds
\]

\[
\leq M\|u_0\|_{W^{1,p}} + \int_0^t C(t-s)^{-\theta_1}(R^{2\sigma} + R^2 + 1)R \, ds
\]

\[
\leq M\|u_0\|_{W^{1,p}} + CT^{1-\theta_1}(R^{2\sigma} + R^2 + 1)R.
\]

Thus, when \( T \) is small, \( \mathcal{F} \) maps \( \mathcal{Y} \) into itself.

For any \( v_1, v_2 \in \mathcal{Y} \),

\[
\left\| \mathcal{F}v_1 - \mathcal{F}v_2 \right\|_{W^{1,p}} \leq \int_0^t \|A_p^{1/2}e^{-A\rho(t-s)}(F(v_1(s)) - F(v_2(s)))\|_{L^p} \, ds
\]

\[
\leq \int_0^t C(t-s)^{-\theta_1}(R^{2\sigma} + R^2 + 1)\|v_1(s) - v_2(s)\|_{W^{1,p}} \, ds
\]

\[
\leq CT^{1-\theta_1}(R^{2\sigma} + R^2 + 1) \sup_{[0, T]} \|v_1 - v_2\|_{W^{1,p}}.
\]

Thus, if we take \( T \) small so that \( C(R^{2\sigma} + R^2 + 1)T^{1-\theta_1} \leq \frac{1}{2} \), then \( \mathcal{F} \) is contraction on \( \mathcal{Y} \). There is a unique fixed point of \( \mathcal{F} \) in \( \mathcal{Y} \), which is a solution of (2.19).

Now we show that (2.15) holds.

Let \( p_2 = 2/(2 - p_1) > 2 \). Then \( W^{2,p_2}(\Omega) \) is \( W^{1,p_1}(\Omega) \) and

\[
\|v\|_{W^{1,p_1}} \leq C\|v\|_{L^2}^{1-\theta_2}\|v\|_{W^{2,p_2}}^{\theta_2}, \quad v \in W^{2,p_2}(\Omega),
\]

\[
\|v\|_{W^{1,p_1}} \leq C\|v\|_{L^2}^{1-\theta_2}\|v\|_{W^{2,p_2}}^{\theta_2}, \quad v \in W^{2,p_2}(\Omega),
\]
where \( \theta_2 = 1/p_1 + p_1/2 - \frac{1}{2} \in (0, 1) \). By (2.11) and (2.12),
\[
\begin{align*}
\|e^{-A_{\rho,t}}v\|_{W^{1,p_2}} & \leq C\|e^{-A_{\rho,t}}v\|_{L^{p_1}}^{1-\theta_2}\|e^{-A_{\rho,t}}v\|_{W^{1,p_1}}^{\theta_2} \\
& \leq Ct^{-\theta_2}\|v\|_{L^{p_1}}, \quad \forall v \in L^{p_1}(\Omega).
\end{align*}
\]
From (2.19),
\[
\begin{align*}
\|u(t)\|_{W^{1,p_2}} & \leq C\|e^{-A_{\rho,t}}u_0\|_{W^{1,p_2}} + \int_0^t \|e^{-A_{\rho,t}(t-s)}F(u(s))\|_{W^{1,p_2}} ds \\
& \leq Ct^{-\theta_1}\|u_0\|_{W^{1,p}} + \int_0^t C(t-s)^{-\theta_2}\|F(u(s))\|_{L^{p_1}} ds \\
& \leq Ct^{-\theta_1}\|u_0\|_{W^{1,p}} + C(R^{\alpha} + R^{2} + 1)RT^{1-\theta_2},
\end{align*}
\]
where because
\[
\begin{align*}
\|e^{-A_{\rho,t}}u_0\|_{W^{1,p_2}} & \leq C\|e^{-A_{\rho,t}}u_0\|_{W^{1,p}}^{1-\theta_3}C\|e^{-A_{\rho,t}}u_0\|_{W^{1,p}}^{\theta_3} \\
& \leq Ct^{-\theta_1}\|u_0\|_{W^{1,p}}, \quad \theta_3 = \frac{1}{p} - \frac{1}{p_2}.
\end{align*}
\]
Hence, for any \( t_0 \in (0, T) \), \( u(t_0) \in W^{1,p_2}(\Omega) \), \( p_2 > 2 \). Viewing \( t_0 > 0 \) as an initial time and \( u(t_0) \) as an initial function, by Theorem 2.1 we see that \( u(t) \in C([0, T_\rho], D(A^{1/2}_{\rho})) \cap C([t_0, T_\rho], D(A_{\rho})) \). By the uniqueness of the solution and the arbitrariness of \( t_0 \), we see that (2.15) and (2.16) hold true.

3. A PRIORI ESTIMATES

In this section we first show that \( u(t) \) is bounded in \( L^p \) space \((p \) satisfies (3.1) below) rather than merely in \( L^2 \) space. Then we prove \( V_\beta(u(t)) \) (see (3.12) below) is bounded and thus \( u(t) \) remains bounded in \( H^1 \). We remark that the condition (3.1) which ensures the boundedness of \( u(t) \) in \( L^p(\Omega) \) is the same as (2.2), which guarantees that \( -B_p \) generates an analytical semigroup \( e^{-B_{\rho,t}} \) of contraction.

Lemma 3.1. Assume that \( p, \nu \) satisfy
\[
2 \leq p < \frac{2\sqrt{1 + \nu^2}}{\sqrt{1 + \nu^2} - 1} \quad \text{or} \quad \frac{2\sqrt{1 + \nu^2}}{\sqrt{1 + \nu^2} + 1} < p \leq 2. \quad (3.1)
\]
If \( \sigma > 2 \), then we have for any \( t \in [0, T] \),
\[
\|u\|_p^\sigma \leq C(T, \|u_0\|_p),
\]
\[
\int_0^t \|u(s)\|_{\sigma + \frac{2}{p}}^\sigma \, ds \leq C(T, \|u_0\|_p),
\]
\[
\int_0^t \|\nabla u(s)\|_{\sigma}^\sigma \, ds \leq C(T, \|u_0\|_p),
\]
where \( \| \cdot \|_q \) is the norm of \( L^q(\Omega) \), \( 1 \leq q \leq \infty \).

If \( \sigma = 2 \) and
\[
b_0 = \sqrt{\pi} \max \left\{ \text{Im} \left( \lambda_{11} + \lambda_{12} \right), \left| \lambda_{21} + \lambda_{22} \right| \right\} \leq \lambda_M (\nu, p),
\]
we have (3.2) too. If (3.5) holds strictly, i.e., \( b_0 < \lambda_M (\nu, p) \), then (3.3) also holds. If \( b_0 < \lambda_M (\nu, 2) = 2 \), (3.4) holds also true.

**Proof.** Direct calculation yields
\[
\frac{1}{p} \frac{d}{dt} \|u\|_p^\sigma = \text{Re} \int |u|^{\sigma - 2} \bar{u} \Delta u, dx
\]
\[
= \text{Re} \int |u|^{\sigma - 2} \bar{u} \left( (1 + \mu i) \Delta u - (1 + i \nu) |u|^{\sigma} u \right.
\]
\[
+ (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^{2} \big) \, dx
\]
\[
= \gamma \|u\|_p^\sigma - \|u\|_{\sigma + \frac{2}{2}}^{\sigma + \frac{2}{2}} + I_1 + I_2,
\]
where
\[
I_1 = \text{Re} \int (1 + \nu) |u|^{\sigma - 2} \bar{u} \Delta u \, dx
\]
\[
\leq - \frac{\lambda_M(\nu, p)}{4} \int |u|^{\sigma - 2} |\nabla u|^2 \, dx,
\]
\[
I_2 = \text{Re} \int |u|^{\sigma - 2} \bar{u} \left( (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^{2} \right) \, dx.
\]
Note that
\[
(\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^{2}
\]
\[
= \alpha_1 |u|^2 u_{x_1} + \alpha_2 |u|^2 u_{x_2} + \alpha_3 u^2 \bar{u}_{x_1} + \alpha_4 u^2 \bar{u}_{x_2},
\]
where \( \alpha_1 = 2 \lambda_{11} + \lambda_{21}, \) \( \alpha_2 = 2 \lambda_{12} + \lambda_{22}, \) \( \alpha_3 = \lambda_{11}, \) \( \alpha_4 = \lambda_{12}. \) Let \( a_j = \text{Re } \alpha_j, b_j = \text{Im } \alpha_j; \) then \( b_0 = \sqrt{2} \max(|b_1 - b_3|, |b_2 - b_4|) \) and

\[
I_2 = I_{21} + I_{22} + I_{23} + I_{24},
\]

where

\[
I_{21} = \text{Re } \int \alpha_1 |u|^{\rho - 2} \bar{u} \cdot |u|^2 u_s \, dx
\]
\[
= \frac{a_1}{2} \int |u|^\rho (|u|^2)_s \, dx \, dy - b_1 \text{Im } \int |u|^\rho \bar{u} u_s \, dx
\]
\[
= -b_1 \text{Im } \int |u|^\rho \bar{u} u_s \, dx,
\]

\[
I_{22} = \text{Re } \int \alpha_2 |u|^{\rho - 2} \bar{u} \cdot |u|^2 u_s \, dx
\]
\[
= -b_2 \text{Im } \int |u|^\rho \bar{u} u_s \, dx,
\]

\[
I_{23} = \text{Re } \int \alpha_3 |u|^{\rho - 2} \bar{u} \cdot u_s \, dx
\]
\[
= \text{Re } \int \bar{u}_3 |u|^\rho \bar{u} \cdot u_s \, dx
\]
\[
= b_3 \text{Im } \int |u|^\rho \bar{u} u_s \, dx,
\]

\[
I_{24} = \text{Re } \int \alpha_4 |u|^{\rho - 2} \bar{u} \cdot u_s \, dx
\]
\[
= b_4 \text{Im } \int |u|^\rho \bar{u} u_s \, dx.
\]

Therefore

\[
|I_2| \leq |b_1 - b_3| \int |u|^{\rho + 1} |u_s| \, dx + |b_2 - b_4| \int |u|^{\rho + 1} |u_s| \, dx
\]
\[
\leq b_0 \int |u|^{\rho + 1} |\nabla u| \, dx.
\]
If $\sigma > 2$, by the Hölder inequality we have

$$b_0 \int |u|^{p+1} |\nabla u| \, dx$$

$$\leq \frac{\lambda_M(\nu, p)}{8} \int |u|^{p-2} |\nabla u|^2 \, dx + \frac{2b_0^2}{\lambda_M(\nu, p)} \int |u|^p \, dx$$

$$\leq \frac{\lambda_M(\nu, p)}{8} \int |u|^{p-2} |\nabla u|^2 \, dx + \frac{1}{2} \int |u|^{p+2\sigma} \, dx + C \int |u|^p \, dx,$$

where (2.3)–(2.6) and (3.1) are used. Thus we have

$$\frac{1}{p} \frac{d}{dt} \|u\|_p^p \leq (\gamma + C)\|u\|_p^p - \frac{1}{2} \|u\|_{p+2\sigma}^{p+2\sigma} - \frac{\lambda_M(\nu, p)}{8} \int |u|^{p-2} |\nabla u|^2 \, dx. \quad (3.7)$$

Applying the Gronwall inequality we get

$$\|u\|_p^p \leq \|u_0\|_p^p e^{(\gamma + C)t}, \quad t \in [0, T]. \quad (3.8)$$

Integrating (3.7) over $[0, T]$ we have (3.3); taking $p = 2$ we obtain (3.4).

If $\sigma = 2$, for any $\varepsilon > 0$,

$$|I_2| \leq b_0 \int |u|^{p+1} |\nabla u| \, dx$$

$$\leq \frac{\varepsilon \lambda_M(\nu, p)}{4} \int |u|^{p-2} |\nabla u|^2 \, dx + \frac{2b_0^2}{\varepsilon \lambda_M(\nu, p)} \int |u|^4 \, dx$$

so we have instead of (3.7)

$$\frac{1}{p} \frac{d}{dt} \|u\|_p^p \leq \gamma \|u\|_p^p - \left(1 - \frac{b_0}{\varepsilon \lambda_M(\nu, p)}\right) \|u\|_{p+2\sigma}^{p+2\sigma}$$

$$- \frac{1}{4} (1 - \varepsilon) \lambda_M(\nu, p) \int |u|^{p-2} |\nabla u|^2 \, dx. \quad (3.9)$$

When $b_0 \leq \lambda_M(\nu, p)$, we take $\varepsilon = 1$ and apply the Gronwall inequality to get

$$\|u\|_p^p \leq \|u_0\|_p^p e^{\nu t}, \quad t \in [0, T].$$

When $b_0 < \lambda_M(\nu, p)$, we can choose an $\varepsilon \in (0, 1)$ such that $b_0 / \varepsilon \lambda_M(\nu, p) < 1$. Thus from (3.9) we get (3.3) and (3.4).

The proof of the lemma is completed.
Lemma 3.2. Under the assumptions of (A1) and (A2), we have
\[ \|\nabla u(t)\|_2^2 \leq C(T), \quad t \in [0, T]. \]

Proof. Using Eq. (1.7) and integration by parts we get
\[
\frac{d}{dt} \int \frac{1}{2} |\nabla u|^2 \, dx = \text{Re} \int \nabla \bar{u} \nabla u, \, dx \\
= \text{Re} \int \nabla \bar{u} \left( \gamma u + (1 + i\nu) \Delta u - (1 + i\mu)|u|^{2\sigma} u \\
+ (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^2 \right) \, dx \\
= \gamma \|\nabla u\|_2^2 - \|\Delta u\|_2^2 + \text{Re} \int (1 + i\mu)|u|^{2\sigma} \Delta \bar{u} \, dx \\
- \text{Re} \int \Delta \bar{u} \left( (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^2 \right) \, dx \\
= \gamma \|\nabla u\|_2^2 - \|\Delta u\|_2^2 + I_3 + I_4, \quad (3.10)
\]
where
\[ I_3 = \text{Re} \int (1 - i\mu)|u|^{2\sigma} \Delta u \, dx, \]
\[ |I_4| = \left| - \text{Re} \int \Delta \bar{u} \left( (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla)|u|^2 \right) \, dx \right| \]
\[ \leq (3|\lambda_1| + |\lambda_2|) \int |u|^2 |\nabla u| |\Delta u| \, dx. \]

We multiply (3.6) by \( \delta > 0 \), take \( p = 2\sigma + 2 \), and then add the resultant to (3.10) to obtain
\[
\frac{d}{dt} V_\delta(u(t)) \leq \gamma \left( \|\nabla u\|_2^2 + \delta \|u\|_{2^{2\sigma + 2}}^{2\sigma + 2} \right) - \left( \|\Delta u\|_2^2 + \delta \|u\|_{4\sigma + 2}^{4\sigma + 2} \right) \\
+ \text{Re} \int (1 + \delta + i(\delta\nu - \mu))|u|^{2\sigma} \Delta u \, dx \\
+ (3|\lambda_1| + |\lambda_2|) \int |u|^2 |\nabla u| |\Delta u| \, dx \\
+ \delta (3|\lambda_1| + |\lambda_2|) \int |u|^{2\sigma + 3} |\nabla u| \, dx, \quad (3.11)
\]
where $\delta$ is a positive number to be determined later, and
\[ V_\delta(u(t)) = \int \left( \frac{1}{2} |\nabla u|^2 + \frac{\delta}{2 \sigma + 2} |u|^{2\sigma + 2} \right) dx. \quad (3.12) \]

Similar to (2.3)–(2.6), for any $\alpha$ with
\[ |\alpha| < \frac{\sqrt{2\sigma + 1}}{\sigma}, \]
the smaller eigenvalue $\lambda_M(\alpha, 2\sigma + 2)$ of $M(\alpha, 2\sigma + 2)$ is positive and thus $M(\alpha, 2\sigma + 2)$ is definitely positive. Thus we have
\[
\text{Re} \int (1 + i\alpha) |u|^{2\sigma} \bar{u} \Delta u \, dx \\
= - \int |u|^{2\sigma - 2} \sum_{j=1}^{2} (\bar{u} \partial_j u, u \partial_j \bar{u}) M(\alpha, 2\sigma + 2) \left( \frac{u \partial_j \bar{u}}{\bar{u} \partial_j u} \right) \, dx \\
\leq - \lambda_M(\alpha, 2\sigma + 2) \int |u|^{2\sigma} |\nabla u|^2 \, dx,
\]
or equivalently,
\[
\text{Re} \int (1 + i\alpha) |u|^{2\sigma} \bar{u} \Delta u \, dx + \lambda_M(\alpha, 2\sigma + 2) \int |u|^{2\sigma} |\nabla u|^2 \, dx \leq 0. \quad (3.13)
\]
We multiply (3.13) by $-\eta$ ($\eta > 0$ to be chosen) and then add the resultant to (3.11) to get
\[
\frac{d}{dt} V_\delta(u(t)) \leq \gamma \left( |\nabla u|_2^2 + \delta |u|_{2\sigma + 2}^2 \right) - (1 - \kappa) \left( \|\Delta u\|_2^2 + \delta \|u\|_{4\sigma + 2}^2 \right) \\
- \eta \lambda_M(\alpha, 2\sigma + 2) \int |u|^{2\sigma} |\nabla u|^2 \, dx \\
+ \frac{1}{2} \text{Re} \int (|u|^{2\sigma} u, \Delta u) \cdot N \cdot \left( |u|^{2\sigma} \Delta \bar{u} \right) \, dx \\
+ (3|\lambda_1| + |\lambda_2|) \int |u|^2 |\nabla u| |\Delta u| \, dx \\
+ \delta (3|\lambda_1| + |\lambda_2|) \int |u|^{2\sigma + 3} |\nabla u| \, dx, \quad (3.14)
\]
where \(0 \leq \kappa < 1\) is to be determined,

\[
N = \overline{N}' = \begin{pmatrix}
-2\delta \kappa & 1 + \delta - \eta - i(\delta \nu - \mu - \alpha \eta) \\
1 + \delta - \eta + i(\delta \nu - \mu - \alpha \eta) & -2\kappa
\end{pmatrix}.
\]

**Claim.** When \(\sigma, \nu, \) and \(\mu\) satisfy \((A2)\), we can choose suitable \(\delta, \eta\) positive, \(\kappa\) in \((0, 1)\), and \(|\alpha| < \frac{2\sigma + 1}{\sigma}\) such that \(N\) is nonpositive. Whence

\[
\text{Re} \int (|u|^{2\sigma}u, \Delta u) \cdot N \cdot \left(\frac{|u|^{2\sigma}u}{\Delta u}\right) \, dx \leq 0.
\]

Now we assume that the claim is true. Then we need only to control the last two integrals of (3.14). By the Cauchy inequality,

\[
\begin{align*}
\int |u|^2 |\nabla u| |\Delta u| \, dx & \leq \varepsilon_1 \int |\Delta u|^2 \, dx + \frac{1}{4\varepsilon_1} \int |u|^4 |\nabla u|^2 \, dx, \\
\int |u|^{2\sigma + 3} |\nabla u| \, dx & \leq \varepsilon_2 \int |u|^{4\sigma + 2} \, dx + \frac{1}{4\varepsilon_2} \int |u|^4 |\nabla u|^2 \, dx,
\end{align*}
\]

where \(\varepsilon_1 > 0, \varepsilon_2 > 0\) are arbitrary. If \(\sigma > 2\), by the Young inequality we have

\[
\int |u|^4 |\nabla u|^2 \, dx
= \int |u|^4 |\nabla u|^{4/\sigma} |\nabla u|^{2(1-2/\sigma)} \, dx
\leq \varepsilon_3 \int |u|^{2\sigma} |\nabla u|^2 \, dx + \frac{\sigma - 2}{\sigma} \left(\frac{2}{\varepsilon_3}\right)^{2/(\sigma-2)} \int |\nabla u|^2 \, dx, \quad \forall \varepsilon_3 > 0.
\]

We can choose \(\varepsilon_1, \varepsilon_2, \) and \(\varepsilon_3 > 0\) sufficiently small so that

\[
\begin{align*}
\varepsilon_1 (3|\lambda_1| + |\lambda_2|) & \leq \frac{1}{2} (1 - \kappa), \\
\varepsilon_2 \delta (3|\lambda_1| + |\lambda_2|) & \leq \frac{1}{2} (1 - \kappa), \\
\varepsilon_3 \left(\frac{1}{4\varepsilon_1} + \frac{\delta}{4\varepsilon_2}\right) \left(3|\lambda_1| + |\lambda_2|\right) & \leq \eta \lambda_M (\alpha, 2\sigma + 2).
\end{align*}
\]
Therefore we obtain
\[
\frac{d}{dt} V_\delta(u(t)) \leq (\gamma + C)(\|\nabla u\|_{L^2}^2 + \delta \gamma \|u\|_{L^{2n+\frac{2}{\delta}}}^2) = (\gamma + C)V_\delta(u(t)).
\]
\(3.15\)

By the Gronwall inequality we have
\[
V_\delta(u(t)) \leq V(u_0)e^{\gamma t + Ct}, \quad t \in [0, T].
\]
\(3.16\)

When \(\sigma = 2, |\lambda_1| \text{ and } |\lambda_2| \) are sufficiently small, we can find \(\varepsilon_1, \varepsilon_2 > 0\) such that
\[
(\varepsilon_1 + \varepsilon_2)(3|\lambda_1| + |\lambda_2|) \leq (1 - \kappa),
\]
\[
\left(\frac{1}{4\varepsilon_1} + \frac{\delta}{4\varepsilon_2}\right)(3|\lambda_1| + |\lambda_2|) \leq \eta \lambda_M(\alpha, 2\sigma + 2),
\]
so (3.15), (3.16) are also valid.

The only thing left is to prove the claim. It is clear that \(N\) is nonpositive if and only if
\[
(1 + \delta - \eta)^2 + (\delta v - \mu - \alpha \eta)^2 \leq 4\delta \kappa^2,
\]
\(3.17\)

which means
\[
\text{dist}(z(\delta), \eta(1 + i\alpha)) \leq 2\sqrt{\delta} \kappa,
\]
\(3.18\)

where \(z(\delta) = 1 + i\mu + \delta(1 - i\nu)\). Let \(\Sigma = \{\eta(1 + i\alpha) \mid \eta \geq 0, |\alpha| < \frac{1}{4\varepsilon_2}\}\). Then the claim is equivalent to find suitable \(\delta, \eta, \alpha, \kappa\) such that
\[
\text{dist}(z(\delta), \Sigma) \leq 2\sqrt{\delta} \kappa.
\]
\(3.19\)

We show this in three cases.

Case (I). \(\nu \mu \geq 0\). It suffices to take \(\delta = \frac{\mu}{\kappa}, \eta = 1 + \delta, \) and \(\alpha = 0\) if \(\nu \neq 0\); or \(\delta > 0\) sufficiently large, \(\eta = 1 + \delta, \) and \(\alpha = \frac{\mu}{1 + \sigma}\) if \(\nu = 0\); then \(z(\delta) \in \Sigma\).

Case (II). \(\nu \mu < 0\) with \(|\nu| < \frac{\sqrt{\delta} + 1}{\delta}\) or \(|\mu| < \frac{\sqrt{\delta} + 1}{\delta}\). In this case we can take \(\delta\) sufficiently large or small, \(\eta = 1 + \delta, \) and \(\alpha = \frac{\mu}{1 + \sigma} - \frac{\delta}{1 + \sigma} \nu\); then \(z(\delta) \in \Sigma\).

Case (III). \(\nu \mu < 0\) with both \(|\nu| \geq \frac{\sqrt{\delta} + 1}{\delta}\) and \(|\mu| \geq \frac{\sqrt{\delta} + 1}{\delta}\). In this case \(z(\delta) = (1 + \delta)(1 \pm i \delta |\nu| + |\mu|)\). When \(|\nu| = \frac{\sqrt{\delta} + 1}{\delta}\) and \(|\mu| = \frac{\sqrt{\delta} + 1}{\delta}\), \(z(\delta)\) is on the boundary of \(\Sigma\) for any \(\delta > 0\). So we can take \(\delta\) sufficiently large, \(\eta = 1 + \delta, \alpha = \frac{\mu - \delta \nu}{\delta + 1 + \sigma}, 0 < \varepsilon < \kappa < 1\).
When \( \mu \leq -\frac{\sqrt{\alpha^2 + 1}}{\sigma} \) and \( \nu > \frac{\sqrt{\alpha^2 + 1}}{\sigma} \), we let \( \tau = \sqrt{\alpha} \); then

\[
h(\tau) = \text{dist}(z(\tau^2), \Sigma) - 2\kappa \tau
\]

\[
= \frac{\sigma \nu - \sqrt{2\sigma + 1}}{\sigma + 1} \tau^2 - 2\kappa \tau + \frac{\sigma \mu + \sqrt{2\sigma + 1}}{\sigma + 1}.
\]

When \( \tau_0 = \frac{\kappa(\sigma + 1)}{\nu \sigma - \sqrt{2\sigma + 1}} \), \( h(\tau_0) = \min_{\tau \in \mathbb{R}} h(\tau) \). It is clear that \( h(\tau_0) < 0 \) if and only if

\[
-\kappa^2 < \frac{\sigma^2}{(\sigma + 1)^2} \left( \mu + \frac{\sqrt{2\sigma + 1}}{\sigma} \right) \left( \nu - \frac{\sqrt{2\sigma + 1}}{\sigma} \right).
\]

(3.20)

Once

\[
-1 < \frac{\sigma^2}{(\sigma + 1)^2} \left( \mu + \frac{\sqrt{2\sigma + 1}}{\sigma} \right) \left( \nu - \frac{\sqrt{2\sigma + 1}}{\sigma} \right)
\]

(3.21)

holds, we can find a \( \kappa \in (0, 1) \) such that (3.20) is satisfied; thus \( h(\tau_0) < 0 \) and therefore \( \text{dist}(z(\tau_0), \Sigma) < 2\tau_0 \kappa \).

From (3.21) we get

\[
-1 - \nu \mu < \frac{\sqrt{2\sigma + 1}}{\sigma} |\nu - \mu|.
\]

(3.22)

When \( \nu \leq -\frac{\sqrt{\alpha^2 + 1}}{\sigma} \) and \( \mu > \frac{\sqrt{\alpha^2 + 1}}{\sigma} \), we can also obtain (3.22). Therefore (A2) implies the claim. The proof of Lemma 3.2 is finished.

4. GLOBAL EXISTENCE AND REGULARITY OF SOLUTION

We are now ready to prove the main theorem. We need only to prove that the solution of (1.7)–(1.9) satisfies

\[
\|u(t)\|_{D(A_{\tau}^{\frac{1}{2}})} \leq C(\|u_0\|_{D(A_{\tau}^{\frac{1}{2}})}, T), \quad t \in [0, T].
\]

for any \( T > 0 \).

Take \( \beta = \frac{1}{2} - \frac{1}{p} \); then \( D(A_{\beta}) \subset H^{2\beta}(\Omega) \hookrightarrow L^p(\Omega) \) (see (2.7)). From the constant variation formula of (2.14)

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s)) \, ds,
\]
and by (2.10)–(2.12) we have
\[ \| A^{1/2}_p u(t) \|_{L^p} \leq \| A^{1/2}_p u_0 \|_{L^p} + \int_0^t \| A^{1/2}_p S(t-s) F(u(s)) \|_{L^p} \, ds \]
\[ \leq M \| u_0 \|_{D(A^{1/2}_p)} + \int_0^t C \| A^{1/2}_p S(t-s) F(u(s)) \|_{H^2} \, ds \]
\[ \leq M \| u_0 \|_{D(A^{1/2}_p)} + \int_0^t C \| A^{\beta+1/2}_p S(t-s) F(u(s)) \|_{L^2} \, ds \]
\[ \leq M \| u_0 \|_{D(A^{1/2}_p)} + \int_0^t C (t-s)^{-\beta(\beta+1/2)} \| F(u(s)) \|_{L^2} \, ds. \]

Since \( \| u(t) \|_{H^1} \leq C \) from Lemma 3.2, from (2.13),
\[ \| F(u(t)) \|_{L^2} \leq C \left( \| u(t) \|_{L^2} + \| u(t) \|_{H^1}^{2\beta+1} + \| u(t) \|_{D(A^{1/2}_p)}^2 \| u(t) \|_{D(A^{1/2}_p)} \right) \]
\[ \leq C + C \| A^{1/2}_p u(t) \|_{L^p}; \]
thus
\[ \| A^{1/2}_p u(t) \|_{L^p} \leq M \| u_0 \|_{D(A^{1/2}_p)} \]
\[ + \int_0^t C (t-s)^{-\beta(\beta+1/2)} \left( 1 + \| A^{1/2}_p u(s) \|_{L^p} \right) \, ds. \]

Since \( \beta + 1/2 = 1 - 1/p < 1 \), by the Gronwall inequality with singularity [19] we have
\[ \| A^{1/2}_p u(t) \|_{L^p} \leq C(T), \quad t \in [0, T]. \] (4.1)

Now from (2.8), (2.13), and (4.1) we have
\[ \| A^{1/2}_p F(u(t)) \|_{X_p} \]
\[ \leq C \| \nabla F(u(t)) \|_{L^p} \]
\[ \leq C \left( \| u(t) \|_{D(A^{1/2}_p)} + \| u(t) \|_{D(A^{1/2}_p)}^{2\beta+1} + \| u(t) \|_{D(A^{1/2}_p)}^2 \| u(t) \|_{D(A^{1/2}_p)} \right) \]
\[ \leq C(T) \left( 1 + \| A_p u(t) \|_{L^p} \right), \]
so we derive from the constant variation formula and (2.11)
\[ \| A_p u(t) \|_{L^p} \]
\[ \leq \| A_p S(t) u_0 \|_{L^p} + \int_0^t \| A_p S(t-s) F(u(s)) \|_{L^p} \, ds \]
\[ \leq M r^{-1/2} \| u_0 \|_{D(A^{1/2}_p)} + \int_0^t C(T) (t-s)^{-1/2} \| A^{1/2}_p F(u(s)) \|_{L^p} \, ds \]
\[ \leq M r^{-1/2} \| u_0 \|_{D(A^{1/2}_p)} + \int_0^t C(T) (t-s)^{-1/2} \left( 1 + \| A_p u(s) \|_{L^p} \right) \, ds. \]
Thus by the Gronwall inequality we get
\[
\|A_p u(t)\|_{L^r} \leq C(T) + C(T) t^{-1/2}, \quad t \in (0, T].
\] (4.2)

From Eq. (2.14) we attain \( u \in C^1((0, T], X_p) \).

When \( \sigma \) is an integer, \( F(u) \) is differentiable, so we can show further regularity of the solution. In fact, for any small positive number \( t_0 \), we have already shown \( u(t) \in C([t_0, T]; D(A_p^s)) \), so
\[
\|A_p F(u(t))\|_{X_p} \leq C\left(\|u\|_{D(A_p^s)}^2 + \|u\|_{D(A_p^s)}^2 \right) \\
\leq C\left(1 + ||A_p^{3/2}u(t)||_{L^r}\right), \quad t \in [t_0, T],
\]
where \( C \) depends on \( t_0 \) and \( T \). Therefore from the variation formula
\[
u(t) = S(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s) F(u(s)) \, ds, \quad t \in [t_0, T],
\]
we obtain
\[
\|A_p^{3/2}u(t)\|_{L^r} \\
\leq \|A_p^{3/2}S(t - t_0)u(t_0)\|_{L^r} + \int_{t_0}^{t} \|A_p^{3/2}S(t - s) F(u(s))\|_{L^r} \, ds \\
\leq M(t - t_0)^{-1/2} ||u(t_0)||_{D(A_p^{s/2})} + \int_{t_0}^{t} C(t - s)^{-1/2} ||A_p F(u(s))||_{L^r} \, ds \\
\leq M(t - t_0)^{-1/2} ||u_0||_{D(A_p^{s/2})} + \int_{t_0}^{t} C(t - s)^{-1/2} \left(1 + ||A_p^{3/2}u||_{L^r}\right) ds, \\
t \in [t_0, T].
\]

By the Gronwall inequality we obtain
\[
\|A_p^{3/2}u(t)\|_{L^r} \leq C + C t^{-1/2}, \quad t \in [t_0, T].
\]

From Eq. (2.14) we have
\[
u \in C^1([t_0, T]; D(A_p^{s/2})).
\]

By using induction we can show that, for any integer \( j \geq 0 \) and \( t_j > 0 \), there exists a \( C_j \) such that
\[
\|A_p^{(j+3)/2}u(t)\|_{L^r} \leq C_j + C_j t_j^{-1/2}, \quad t \in [t_j, T].
\]

From Eq. (2.14) again we obtain
\[
u \in C^k([t_j, T]; D(A_p^{(j-k)/2})), \quad 1 \leq k \leq j.
\]
Therefore
\[ u \in C^\infty((0, \infty) \times \Omega). \]
This completes the proof of the main theorem.

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