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Bifree locally inverse semigroups as expansions

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Abstract

In [Studia Sci. Math. Hungar. 41 (2004) 39–58] we constructed for a completely simple semigroup C an expansion $\mathcal{S}(C)$, which is isomorphic to the Birget–Rhodes expansion $C^{\mathbf{Pr}}$ [J. Algebra 120 (1989) 284–300], if C is a group. Analogous to the fact, proven in [J. Algebra 120 (1989) 284–300], that $C^{\mathbf{Pr}}$ contains a copy of the free inverse semigroup in case C is the free group on X, we show that $\mathcal{S}(C)$ contains a copy of the bifree locally inverse semigroup, if C is the bifree completely simple semigroup on X. As a consequence, among other things, we obtain a new proof of a result due to F. Pastijn [Trans. Amer. Math. Soc. 273 (1982) 631–655] which says that each locally inverse semigroup divides a perfect rectangular band of E-unitary inverse monoids.

1. Introduction

Following Birget and Rhodes [9] an *expansion* of a semigroup S is informally speaking, a way of writing S as a homomorphic image of another semigroup $\overline{S}$, such that the latter and the homomorphism $\eta_S : \overline{S} \to S$ have some nice properties. One of the major features of the Birget–Rhodes group expansion $G^{\mathbf{Pr}}$ [9] is the property that if G is the free group on a set X, then $G^{\mathbf{Pr}}$ contains a copy of the free inverse semigroup on X. So, in the context of varieties, $G^{\mathbf{Pr}}$ shifts free objects from the variety of groups up to free objects in the variety of inverse semigroups.

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Motivated by a recent paper due to Kellendonk and Lawson [13], in [7] we constructed a kind of expansion $\mathcal{S}(C)$ for a completely simple semigroup C, which generalizes some important properties of $G^{\mathbf{Pr}}$. For example, as a consequence of the definition $G^{\mathbf{Pr}}$ is embeddable into a semidirect product of a semilattice by G. Analogously $\mathcal{S}(C)$ embeds into a restricted semidirect product of a semilattice by C. The aim of this paper is to show that also the shifting property for free objects has an analogon, even in the theory of e-varieties, which was introduced by Hall [10] and independently by Kačourek and Szendrei [12] to study regular semigroups from a universal algebraic viewpoint.

An *e-variety* is a class of regular semigroups closed under taking direct products regular subsemigroups and homomorphic images. In this theory the bifree object on a set X is the natural counterpart of the free object in a usual variety. Its definition, first given for orthodox semigroups in [12], reads as follows: let X be a nonempty set and $X' = \{x'\colon x \in X\}$ be a disjoint copy of X, $x \mapsto x'$ being a bijection. Let S be a regular semigroup. A mapping $\theta : X \cup X' \to S$ is *matched* if $x'\theta$ is an inverse of $x\theta$ in S for each $x \in X$. Let $\mathcal{V}$ be a class of regular semigroups. A semigroup $F \in \mathcal{V}$ together with a matched mapping $\iota : X \cup X' \to F$ is a *bifree object on* X in $\mathcal{V}$ if for any $S \in \mathcal{V}$ and any matched mapping $\theta : X \cup X' \to S$ there is a unique homomorphism $\bar{\theta} : F \to S$ extending θ, that is, $\iota\bar{\theta} = \theta$.

It was proven in [12] that in each e-variety of orthodox semigroups bifree objects exist and are unique up to isomorphism. For the nonorthodox case Yeh [26] has shown that bifree objects exist in an e-variety if and only if it is contained in the e-variety of all E-solid or all locally inverse semigroups. Three models of the bifree locally inverse semigroup $BF\mathcal{LI}(X)$ on a set X have been obtained up to now by Auinger [1,2].

Let $BF\mathcal{CS}(X)$ denote the bifree completely simple semigroup on X. In our main result we show that the expansion $\mathcal{S}(BF\mathcal{CS}(X))$ contains a copy of $BF\mathcal{LI}(X)$. From this we infer that the latter is embeddable into a Rees matrix semigroup over an E-unitary inverse monoid as well as into a restricted semidirect product of a semilattice by $BF\mathcal{CS}(X)$. Further, since $\mathcal{S}(BF\mathcal{CS}(X))$ is a perfect rectangular band of E-unitary inverse monoids, we recapture a well-known deep result of Pastijn [21], which says that each locally inverse semigroup divides a perfect rectangular band of E-unitary inverse monoids.

2. Preliminaries

For the standard notions and notations in semigroup theory, the reader is referred to the textbooks of Howie [11], Lawson [14], and Petrich [23]. In particular, if s belongs to a semigroup S, then s' denotes an inverse of s in S, and $V(s)$ denotes the set of all inverses of s.

A semigroup S is called *locally inverse*, if for each idempotent $e \in S$ the submonoid eSe is an inverse semigroup. For important papers on the subject see, e.g., Pastijn [20,21], and Nambooripad [19]. It was shown by Trotter [24] that a regular semigroup S is locally inverse if and only if for any $s, t \in S$ the set $sV(s)V(V(t)tsV(s))V(t)t$ is independent of the choice of the inverses s', t', $(t'tss')'$. This gives rise to define a binary operation $\wedge$, the so-called *sandwich operation* on S by $s \wedge t = ss'(t'tss')'t't$ for all $s, t \in S$, see, e.g., Auinger [1] and Yeh [26].

A semigroup S is called a (*perfect*) *rectangular band* $I \times \Lambda$ *of inverse semigroups* (*monoids*) $S_{i\lambda}$, $(i,\lambda) \in I \times \Lambda$, if S is the disjoint union of the inverse semigroups (monoids) $S_{i\lambda}$, and if $S_{i\lambda}S_{j\mu} \subseteq S_{i\mu}$ $(S_{i\lambda}S_{j\mu} = S_{i\mu})$ for all $(i,\lambda), (j,\mu) \in I \times \Lambda$. Obviously each such semigroup S is locally inverse. The structure of rectangular bands of inverse semigroups (monoids) was developed by Pastijn [20], by Pastijn and Petrich [22], and from a universal algebraic viewpoint by Meakin [16–18]. In particular, it was shown in [22] that a rectangular band of inverse semigroups is perfect if and only if each $s \in S_{i\lambda}$ has a (necessarily) unique inverse in $S_{j\mu}$. Note that a rectangular band of groups is automatically perfect whence completely simple.

Let C be a completely simple semigroup which acts on a regular semigroup T by endomorphisms on the left via $s \mapsto {}^u s$, $s \in T$, $u \in C$. Define a multiplication on the set $T *_{rr} C = \{(s,u) \in T \times C\colon {}^{uu'}s = s$, for some $u' \in V(u)\}$, by $(s,u)(t,v) = (s\,{}^u t, uv)$. Then $T *_{rr} C$ is a regular semigroup, termed a *restricted semidirect product of* T *by* C. Note in particular that ${}^{uu'}s = s$ implies ${}^{uu^*}s = s$ for all $u^* \in V(u)$, since ${}^{uu^*}s = {}^{uu^*}({}^{uu'}s) = {}^{uu'}s = s$. The restricted semidirect product of a regular semigroup by a completely simple semigroup was introduced by Auinger and Polák [4] as a straightforward generalization of the λ-semidirect product for inverse semigroups, introduced in [5]; for an excellent survey on the subject, concerning inverse semigroups, see also the textbook [14]. It was later generalized to locally $\mathcal{R}$-unipotent semigroups in the second component in [6].

The following construction stems from [7]:

Result 0. Let $C = \bigcup\{G_{i\lambda}\colon (i,\lambda) \in I \times \Lambda\}$ be a completely simple semigroup, where $1_{i\lambda}$ denotes the identity element of the maximal subgroup $G_{i\lambda}$. Let $F(C)$ be the free semigroup on the alphabet C, whose multiplication shall be denoted by $\cdot$. For $w = u_1 \cdot \dots \cdot u_n \in F(C)$ let $\overline{w} \in C$ be defined by $\overline{w} = u_1 \dots u_n$. Let finally ρ' denote the congruence on $F(C)$ which is generated by the pairs

(1) $(1_{i\mu} \cdot u, u)$, for all $u \in \bigcup\{G_{i\lambda}\colon \lambda \in \Lambda\}$,
(2) $(u \cdot 1_{j\lambda}, u)$, for all $u \in \bigcup\{G_{i\lambda}\colon i \in I\}$,
(3) $(u \cdot u' \cdot v, u \cdot u'v)$, for all $u, v \in C$, $u' \in V(u)$,
(4) $(v \cdot u' \cdot u, vu' \cdot u)$, for all $u, v \in C$, $u' \in V(u)$.

Put $\mathcal{S}(C) = F(C)/\rho'$. Then $\mathcal{S}(C)$ is a perfect rectangular band $I \times \Lambda$ of the E-unitary inverse monoids $S_{i\lambda} = \{w\rho'\colon \overline{w} \in G_{i\lambda}\}$. Moreover, the mapping $\eta_C : S \to C$, $\eta_C : w\rho' \mapsto \overline{w}$ is a surjective homomorphism with the property that the inverse images of idempotents are semilattices.

3. A model of the bifree locally inverse semigroup

We begin with an observation on locally inverse semigroups which is crucial and simplifies our construction considerably. It will be utilized in the proof of the main theorem of the paper, Theorem 9.

Proposition 1. *Let S be a locally inverse semigroup, and let $u = s_1 \dots s_n \in S$. Let further $\{k_i\colon\ 0 \leqslant i \leqslant m\}$ be a set of natural numbers such that $1 = k_0 < k_1 < \dots < k_m = n$, and let $(s_{k_i} \dots s_{k_{i+1}})'$ be an inverse of $s_{k_i} \dots s_{k_{i+1}}$, $0 \leqslant i \leqslant m-1$. Then the product $u' = (s_{k_{m-1}} \dots s_{k_m})' s_{k_{m-1}} (s_{k_{m-2}} \dots s_{k_{m-1}})' \dots s_{k_1} (s_{k_0} \dots s_{k_1})'$ is an inverse of u.*

Proof. We proceed by induction on n. For $n = 1$ the assertion is trivial. Assume that the assertion is satisfied for $n \geqslant 1$, and let $u = s_1 \dots s_{n+1}$ and $1 = k_0 < k_1 < \dots < k_m = n + 1$. Let $s'_{k_1} \in V(s_{k_1})$ and put

$$e = s'_{k_1} s_{k_1} \dots s_{n+1} (s_{k_{m-1}} \dots s_{n+1})' s_{k_{m-1}} (s_{k_{m-2}} \dots s_{k_{m-1}})' \dots s_{k_2} (s_{k_1} \dots s_{k_2})' s_{k_1} \quad \text{and}$$
$$f = s'_{k_1} s_{k_1} (s_1 \dots s_{k_1})' s_1 \dots s_{k_1}.$$

Then e and f are idempotents, belonging to the submonoid $s'_{k_1} s_{k_1} S s'_{k_1} s_{k_1}$. The former follows from the fact that $(s_{k_{m-1}} \dots s_{n+1})' s_{k_{m-1}} (s_{k_{m-2}} \dots s_{k_{m-1}})' \dots s_{k_2} (s_{k_1} \dots s_{k_2})'$ is an inverse of $s_{k_1} \dots s_{n+1}$ by hypothesis, the latter by an easy direct calculation. Put $(s_{k_{m-1}} \dots s_{k_m})' s_{k_{m-1}} (s_{k_{m-2}} \dots s_{k_{m-1}})' \dots s_{k_1} (s_{k_0} \dots s_{k_1})'$. We compute

$$uu'u = s_1 \dots s_{k_1} e f s'_{k_1} s_{k_1} \dots s_{n+1} = s_1 \dots s_{k_1} f e s'_{k_1} s_{k_1} \dots s_{n+1} = u.$$

Similarly it follows that $u'uu' = u'$, completing the proof. □

We continue with a simplified computation rule for the sandwich element.

Proposition 2. *Let S be a locally inverse semigroup, let $s, t \in S$, and let $(st)' \in V(st)$. Then $s \wedge t$ is equal to $s(ts)'t$.*

Proof. By definition, $s \wedge t = ss'(t'tss')'t't$, where s', t', and $(t'tss')'$ are arbitrary inverses of s, t, and $t'tss'$. Now an easy direct calculation yields that $s(ts)'t$ is an inverse of $t'tss'$, whence $s \wedge t = ss's(ts)'tt't = s(ts)'t$ follows. □

In particular, Proposition 2 implies that the element $s(ts)'t$ is independent of the choice of the inverse $(ts)' \in V(ts)$. Moreover, Proposition 1 in connection with Proposition 2 tells us that for $u = s_1 \dots s_n$ the element $u' = s'_n (s_n \wedge s_{n-1}) \dots s'_2 (s_2 \wedge s_1) s'_1$ is an inverse of u.

It was proven in [26] that, given a locally inverse semigroup S, and a subset $A \subseteq S$ such that $V(a) \cap A$ is nonempty for each $a \in A$, then there exists the least locally inverse subsemigroup U of S containing A. In fact, by Proposition 1 and the remark behind Proposition 2, we see that the elements of U are just the products $s_1 \dots s_n$, where $s_i \in A$ or $s_i = a \wedge b$, for some $a, b \in A$.

We collect some properties of the sandwich operation $\wedge$ which partly can be found in [3].

Proposition 3. *Let S be a locally inverse semigroup, and let $s, t \in S$. Then the following holds*:

(i) *$s \wedge t$ is an idempotent of S.*

(ii) $ss'(s \wedge t) = s \wedge t = (s \wedge t)t't$, *for all* $s' \in V(s)$, $t' \in V(t)$.
(iii) $s(t \wedge s)t = st$, *for all* $s, t \in S$.
(iv) *For each* $s \in S$ *the set* $\{s \wedge t\colon\ t \in S\}$ ($\{t \wedge s\colon\ t \in S\}$) *is a right* (*left*) *normal subband of* S.

Proof. The assertions (i)–(iii) directly follow from the definition of $\wedge$ or more easier from Proposition 2.

We prove (iv). Let $s, t_1, t_2, t_3 \in S$, and let $s' \in V(s)$. Then $(s \wedge t_1)ss'$ and $(s \wedge t_2)ss'$ are idempotents belonging to the submonoid $ss'Sss'$. We compute

$$\begin{aligned}(s \wedge t_1)(s \wedge t_2)(s \wedge t_3) &= \big((s \wedge t_1)ss'\big)\big((s \wedge t_2)ss'\big)(s \wedge t_3)\\ &= \big((s \wedge t_2)ss'\big)\big((s \wedge t_1)ss'\big)(s \wedge t_3)\\ &= (s \wedge t_2)(s \wedge t_1)(s \wedge t_3). \quad \square\end{aligned}$$

We recall some notations and results due to Auinger [2]:

X	a nonempty set;
$X' = \{x'\colon\ x \in X\}$	a disjoint copy of X;
F	the free semigroup on $X \cup X' \cup \{(a \wedge b)\colon\ a, b \in X \cup X'\}$;
λw	the first letter from $X \cup X'$ in $w \in F$;
$w\rho$	the last letter from $X \cup X'$ in $w \in F$.

In addition, we shall write I instead of $X \cup X'$, and $(I \wedge I)$ instead of $\{(a \wedge b)\colon\ a, b \in X \cup X'\}$. If $a = x' \in X'$, then a' shall denote the element $x \in X$. A word $w \in F$ is called *reduced* if it does not contain a subword of one of the following forms:

(1) $a(b \wedge a)$,
(2) $(b \wedge a)b$,
(3) $(a \wedge b)(a \wedge c)$,
(4) $(b \wedge a)(c \wedge a)$,
(5) aa',

where $a, b, c \in I$. For a word $w \in F$ we denote the number of its letters from $I \cup (I \wedge I)$ by $f(w)$. Further let $\mathbf{s}(w)$ denote the uniquely determined word which is obtained from w by a successive application of the following reductions:

(1) $a(b \wedge a) \to a$,
(2) $(b \wedge a)b \to b$,
(3) $(a \wedge b)(a \wedge c) \to (a \wedge c)$,
(4) $(b \wedge a)(c \wedge a) \to (b \wedge a)$,
(5) $aa' \to (a \wedge a')$,

where $a, b, c \in I$. $\mathbf{s}(w)$ is called *reduced*. On the set $\mathbf{s}(F)$ of all reduced words an associative binary operation $\odot$ may be defined by $u \odot v = \mathbf{s}(uv)$. The following theorem was proven in [2].

Theorem 4. *The semigroup* $(\mathbf{s}(F), \odot)$ *together with the matched mapping* $\iota : X \cup X' \to \mathbf{s}(F)$*,* $x \mapsto x$*,* $x' \mapsto x'$ *is a model of* $BF\mathcal{CS}(X)$*.*

In what follows, the letter $BF\mathcal{CS}(X)$ solely stands for the representation, given in Theorem 4. In fact, $BF\mathcal{CS}(X)$ is a perfect rectangular band $I \times I$ of groups G_{ab}, $a, b \in I$, where $(a \wedge b)$ is the identity element of G_{ab}. A word $w = a_1 \dots a_n \in \mathbf{s}(F)$ belongs to G_{ab} if and only if $\lambda w = a$ and $w\rho = b$. Further for $w = a_1 \dots a_n \in \mathbf{s}(F)$ the uniquely determined inverse of w in the maximal subgroup G_{cd} is

$$\mathbf{s}\big((c \wedge a_n\rho)\bar{a}_n(\lambda a_n \wedge a_{n-1}\rho)\bar{a}_{n-1} \dots (\lambda a_2 \wedge a_1\rho)\bar{a}_1(\lambda a_1 \wedge d)\big),$$

where $\bar{a}_i = a_i'$, if $a_i \in I$, and $\bar{a}_i = a_i$, if $a_i \in (I \wedge I)$.

Note that our point of view slightly differs from the one in [2], in that we consider $BF\mathcal{CS}(X)$ as a semigroup rather than a binary semigroup. For our purpose there is no need to work with the sandwich operation on the whole of $BF\mathcal{CS}(X)$.

The rest of the section is devoted to show that $\mathcal{S}(BF\mathcal{CS}(X))$ contains a subsemigroup which is isomorphic to the bifree locally inverse semigroup on X. Let S be a locally inverse semigroup, and let $\theta : I \to S$ be a matched mapping. For $a \in I$ let $\hat{a}$ be the image of a under θ. Note in particular that by the definition of a matched mapping, $\hat{a}' \in V(\hat{a})$ for $a \in I$. For each word $w = u_1 \cdot \dots \cdot u_n$ of the free semigroup $F(BF\mathcal{CS}(X))$ on the alphabet $BF\mathcal{CS}(X)$ let $(w)\pi$ be the word in $F(FB\mathcal{CS}(X))$ which is obtained from w by successively replacing all subwords, which are of the form $u(\lambda v \wedge a) \cdot v$, respectively $v \cdot (a \wedge v\rho)u$, $a \in I$, by $u \cdot v$ $[v]$, respectively by $v \cdot u$ $[v]$ (in case u is empty). It is easy to see that the order in which the replacing process takes place does not affect the final result, whence $(w)\pi$ is well-defined. Let now $(w)\psi$ be the element of S, which is obtained by substituting each letter a from I occurring in a factor u_i of $(w)\pi$, by $\hat{a}$, and each letter $(a \wedge b)$ from $(I \wedge I)$ occurring in a factor u_i of $(w)\pi$ by $\hat{a} \wedge \hat{b}$. Obviously ψ is a mapping from $F(BF\mathcal{CS}(X))$ into S. Note in particular that if $u_i \in (I \wedge I)$, then

$$\begin{aligned}(w)\psi &= (u_1 \cdot \dots \cdot u_{i-1} \cdot u_i \odot u_{i+1} \cdot u_{i+2} \cdot \dots \cdot u_n)\psi \\ &= (u_1 \cdot \dots \cdot u_{i-2} \cdot u_{i-1} \odot u_i \cdot u_{i+1} \cdot \dots \cdot u_n)\psi,\end{aligned}$$

and if $u_i = av$, respectively $u_i = va$, $a \in I$, we have

$$(w)\psi = \big(u_1 \cdot \dots \cdot u_{i-1} \odot (a \wedge a')\big)\psi(u_i \cdot \dots \cdot u_n)\psi,$$

respectively

$$(w)\psi = (u_1 \cdot \dots \cdot u_i)\psi\big((a' \wedge a) \odot u_{i+1} \cdot \dots \cdot u_n\big)\psi.$$

We will use these facts in the sequel without further reference.

Let ρ' denote the congruence defined in Result 0, where $C = BF\mathcal{CS}(X)$. In what follows we show that ρ' is contained in $\ker\psi$, which enables us to define a mapping

$\hat{\psi}: \mathcal{S}(BFCS(X)) \to S$, by $w\rho' \mapsto w\psi$, $w \in F(BFCS(X))$. With respect to the generating set of ρ' given in Result 0, and the way in which ρ' is built up from this set (see, e.g., the textbook [11]), it suffices to establish the following equalities:

(1) $(u_1 \cdot (\lambda w \wedge a) \cdot w \cdot u_2)\psi = (u_1 \cdot w \cdot u_2)\psi$,
(2) $(u_1 \cdot w \cdot (a \wedge w\rho) \cdot u_2)\psi = (u_1 \cdot w \cdot u_2)\psi$,
(3) $(u_1 \cdot u \cdot u' \cdot w \cdot u_2)\psi = (u_1 \cdot u \cdot u' \odot w \cdot u_2)\psi$,
(4) $(u_1 \cdot w \cdot u' \cdot u \cdot u_2)\psi = (u_1 \cdot w \odot u' \cdot u \cdot u_2)\psi$,

for all $a \in I$, $u, u_1, u_2, w \in \mathbf{s}(F)$, $u' \in V(u)$. By the remark behind the definition of ψ, and since $(\lambda w \wedge a) \odot w = w = w \odot (a \wedge w\rho)$ for each $a \in I$, we directly see that (1) and (2) are satisfied. To prove (3) and (4) we need some prerequisites. In the sequel the following notation comes in handy. For $a \in (I \wedge I)$ let $\hat{a}$ denote the element $\widehat{\lambda a} \wedge \widehat{a\rho} \in S$. If $u = a_1 \ldots a_n \in \mathbf{s}(F)$, then $\hat{u}$ shall denote the element $\hat{a}_1 \ldots \hat{a}_n \in S$.

We continue with a slight modification of an important definition in [2]. For $a \in I \cup (I \wedge I)$ let $\bar{a} = a'$ if $a \in I$ and $\bar{a} = a$ if $a \in (I \wedge I)$. For $u = a_1 \ldots a_n \in \mathbf{s}(F)$ let $\bar{u} = \mathbf{s}(\bar{a}_n(\lambda a_n \wedge a_{n-1}\rho)\bar{a}_{n-1} \ldots (\lambda a_2 \wedge a_1\rho)\bar{a}_1)$. Note that $\bar{u}$ is obtained from $\bar{a}_n(\lambda a_n \wedge a_{n-1}\rho)\bar{a}_{n-1} \ldots (\lambda a_2 \wedge a_1\rho)\bar{a}_1$ by deleting each letter $\bar{a}_i$, where $a_i \in (I \wedge I)$, and by deleting the letter $(\lambda a_{i+1} \wedge a_i\rho)$, if either $a_i \in I$, $a_{i+1} \in (I \wedge I)$, and $\lambda a_{i+1} = a_i'$, or if $a_i \in (I \wedge I)$, $a_{i+1} \in I$, and $a_i\rho = a_{i+1}'$. In particular, no reduction of the form $aa' \to (a \wedge a')$, $a \in I$, occurs in performing $\bar{u}$. This important fact can easily be seen, by checking some significant examples, keeping in mind that u is reduced. If for example $u = ab(b' \wedge c)(d \wedge e)$, then

$$\bar{u} = \mathbf{s}\big((d \wedge e)(d \wedge c)(b' \wedge c)(b' \wedge b)b'(b \wedge a)a'\big) = (d \wedge c)b'(b \wedge a)a'.$$

Lemma 5. *Let $u \in \mathbf{s}(F)$. Then $(u)\psi(\bar{u})\psi(u)\psi = (u)\psi$.*

Proof. We prove the assertion by induction on $f(u) = n$. For $n = 1$ the assertion is trivial. Assume that it is true for some $n \geqslant 1$, and let $u = a_1 \ldots a_{n+1}$. Put $v = a_2 \ldots a_{n+1}$. We distinguish two main cases.

Let first $a_1 \in I$. By the above remark, we have $\bar{u} = \bar{v}(\lambda a_2 \wedge a_1)a_1'$ or $\bar{u} = \bar{v}a_1'$. In the first case it follows

$$\begin{aligned}(u)\psi(\bar{u})\psi(u\psi) &= \hat{a}_1(v)\psi(\bar{v})\psi(\widehat{\lambda a_2} \wedge \hat{a}_1)\hat{a}_1'\hat{a}_1(v)\psi\\ &= \hat{a}_1(v)\psi(\bar{v})\psi(\widehat{\lambda a_2} \wedge \hat{a}_1)(v)\psi\\ &= \hat{a}_1(v)\psi(\bar{v})\psi(\widehat{\lambda a_2} \wedge \hat{a}_1)\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi.\end{aligned}$$

Put $e = (\widehat{\lambda a_2} \wedge \hat{a}_1)\widehat{\lambda a_2}(\widehat{\lambda a_2})'$. Then e and $(v)\psi(\bar{v})\psi$ are idempotents belonging to the submonoid $\widehat{\lambda a_2}(\widehat{\lambda a_2})'S\widehat{\lambda a_2}(\widehat{\lambda a_2})'$. The former follows by Proposition 3(i) combined with (ii), the latter by the induction hypothesis. We proceed

$$\begin{aligned}(u)\psi(\bar{u})\psi(u)\psi &= \hat{a}_1(v)\psi(\bar{v})\psi e(v)\psi\\ &= \hat{a}_1 e(v)\psi(\bar{v})\psi(v)\psi\end{aligned}$$

$$= \hat{a}_1\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi(\bar{v})\psi(v)\psi \quad \text{by Proposition 3(iii)}$$
$$= \hat{a}_1(v)\psi \quad \text{by hypothesis}$$
$$= (u)\psi.$$

If on the other hand $\bar{u} = \bar{v}a_1'$, then $\lambda a_2 = a_1'$, and we get

$$\begin{aligned}(u)\psi(\bar{u})\psi(u)\psi &= \hat{a}_1(v)\psi(\bar{v})\psi(v)\psi\hat{a}_1'\hat{a}_1(v)\psi \\ &= \hat{a}_1(v)\psi(\bar{v})\psi\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi \\ &= \hat{a}_1(v)\psi(\bar{v})\psi(v)\psi \\ &= \hat{a}_1(v)\psi \quad \text{by hypothesis} \\ &= (u)\psi.\end{aligned}$$

Let now $a_1 \in (I \wedge I)$. Again by the above remark, we have $\bar{u} = \bar{v}(\lambda a_2 \wedge a_1\rho)$ or $\bar{u} = \bar{v}$. In the first case we compute

$$\begin{aligned}(u)\psi(\bar{u})\psi(u)\psi &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(\widehat{\lambda a_2} \wedge \widehat{a_1\rho})(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(\widehat{\lambda a_2} \wedge \widehat{a_1\rho})(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi.\end{aligned}$$

Put $f = (\widehat{\lambda a_2} \wedge \widehat{a_1\rho})(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})\widehat{\lambda a_2}(\widehat{\lambda a_2})'$. Then f and $(v)\psi(\bar{v})\psi$ are idempotents belonging to $\widehat{\lambda a_2}(\widehat{\lambda a_2})'S\widehat{\lambda a_2}(\widehat{\lambda a_2})'$ and it follows

$$\begin{aligned}(u)\psi(\bar{u})\psi(u)\psi &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi f(v)\psi \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})f(v)\psi(\bar{v})\psi(v)\psi \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(\widehat{\lambda a_2} \wedge \widehat{a_1\rho})(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi(\bar{v})\psi(v)\psi \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(\widehat{\lambda a_2} \wedge \widehat{a_1\rho})\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi(\bar{v})\psi(v)\psi \\ &\qquad \text{by Proposition 3(iv)} \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(\widehat{a_1\rho})'\widehat{a_1\rho}\widehat{\lambda a_2}(\widehat{\lambda a_2})'(v)\psi(\bar{v})\psi(v)\psi \\ &\qquad \text{by Proposition 3(iii)} \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(v)\psi = (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi = (u)\psi.\end{aligned}$$

If on the other hand $\bar{u} = \bar{v}$, then $a_2 \in I$ and $a_1\rho = a_2'$, whence we get

$$\begin{aligned}(u)\psi(\bar{u})\psi(u)\psi &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi \\ &= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(\widehat{a_1\rho})'\widehat{a_1\rho}(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi.\end{aligned}$$

Put $g = (\widehat{a_1\rho})'\widehat{a_1\rho}(\widehat{\lambda a_1} \wedge \widehat{a_1\rho})$. Then g and $(v)\psi(\bar{v})\psi$ are idempotents in the submonoid $(\widehat{a_1\rho})'\widehat{a_1\rho}S(\widehat{a_1\rho})'\widehat{a_1\rho}$, which implies

$$(u)\psi(\bar{u})\psi(u)\psi = (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi g(v)\psi = (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})g(v)\psi(\bar{v})\psi(v)\psi$$
$$= (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi(\bar{v})\psi(v)\psi = (\widehat{\lambda a_1} \wedge \widehat{a_1\rho})(v)\psi = (u)\psi. \quad \square$$

Lemma 6. *Let $u = a_1 \dots a_n$, where $n \geqslant 2$, and let $a_1, a_n \in I$. Put $v = a_2 \dots a_n$. Then $(u \cdot \bar{u} \cdot w)\psi = (u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w)\psi$, for each $w \in \mathbf{s}(F)$.*

Proof. Let $w = b_1 \dots b_m$ and put $w_1 = (\lambda a_2 \wedge a_1) \odot a_1' \odot w$. Assume first that $a_1 = b_1$. In this case we have $w_1 = (\lambda a_2 \wedge a_1) \odot w$. Let $M = \{l\colon\ 2 \leqslant l,\ a_i = b_i$ and $a_i \in (I \wedge I)$ for all $i \in \{2, \dots, l\}\}$. We distinguish two main cases.

(1) The set M is not empty. Let k be the maximum of M. We obtain

$$(u \cdot \bar{v} \cdot w_1)\pi = \big(u \cdot \overline{a_{k+1} \dots a_n} \odot (\lambda a_{k+1} \wedge a_k\rho) \cdot b_{k+1} \dots b_m\big)\pi.$$

Now, if

(i) $a_{k+1} \in I$ and $a_k\rho = a_{k+1}'$, we get

$$\overline{a_{k+1} \dots a_n} \odot (\lambda a_{k+1} \wedge a_k\rho) = \overline{a_{k+1} \dots a_n},$$

whence

$$(u \cdot \bar{v} \cdot w_1)\pi = u \cdot \overline{a_{k+1} \dots a_n} \cdot b_{k+1} \dots b_m$$

follows. Further, if

(ii) $\lambda a_{k+1} = \lambda b_{k+1}$, we also have

$$(u \cdot \bar{v} \cdot w_1)\pi = u \cdot \overline{a_{k+1} \dots a_n} \cdot b_{k+1} \dots b_m.$$

In all the other cases $(u \cdot \bar{v} \cdot w_1)\pi$ is equal to

$$u \cdot \overline{a_{k+1} \dots a_n}(\lambda a_{k+1} \wedge a_k\rho) \cdot b_{k+1} \dots b_m.$$

We compute $(u \cdot \bar{u} \cdot w)\psi$. Note that since $a_k \in (I \wedge I)$, we have $\bar{u} = \overline{a_{k+1} \dots a_n}\ \overline{a_1 \dots a_k}$ if and only if $a_{k+1} \in I$ and $a_k\rho = a_{k+1}'$, and $\bar{u} = \overline{a_{k+1} \dots a_n}(\lambda a_{k+1} \wedge a_k\rho)\overline{a_1 \dots a_k}$ otherwise. Since

$$\widehat{\overline{a_{k+1} \dots a_n}}(\hat{a}_{k+1} \wedge \widehat{a_{k+1}'}) = \widehat{\overline{a_{k+1} \dots a_n}}\,\hat{a}_{k+1}\widehat{a_{k+1}'} = \widehat{\overline{a_{k+1} \dots a_n}},$$

in any case we get

$$\hat{\bar{u}} = \widehat{\overline{a_{k+1} \dots a_n}}(\widehat{\lambda a_{k+1}} \wedge \widehat{a_k\rho})\widehat{\overline{a_1 \dots a_k}}.$$

Further from $(u \cdot \bar{u} \cdot w)\pi = u \cdot \bar{u} \cdot w$ we infer

$$(u \cdot \bar{u} \cdot w)\psi = \hat{u}\widehat{\overline{a_{k+1} \dots a_n}}(\widehat{\lambda a_{k+1}} \wedge \hat{a}_k)\widehat{\overline{a_1 \dots a_k}}\,\hat{a}_1 \dots \hat{a}_k\hat{b}_{k+1} \dots \hat{b}_m.$$

Put

$$e = \widehat{(a_k\rho)}'\widehat{a_k\rho}\hat{a}_{k+1}\ldots\hat{a}_n\,\overline{\widehat{a_{k+1}\ldots a_n}}(\widehat{\lambda a_{k+1}} \wedge \widehat{a_k\rho}),$$

and put

$$f = \widehat{(a_k\rho)}'\widehat{a_k\rho}\,\overline{\widehat{a_1\ldots a_k}}\,\hat{a}_1\ldots\hat{a}_k.$$

Then e and f are idempotents belonging to $(\widehat{a_k\rho})'\widehat{a_k\rho}S(\widehat{a_k\rho})'\widehat{a_k\rho}$, and we obtain

$$\begin{aligned}(u\cdot\bar{u}\cdot w)\psi &= \hat{a}_1\ldots\hat{a}_k\,ef\,\hat{b}_{k+1}\ldots\hat{b}_m = \hat{a}_1\ldots\hat{a}_k\,fe\,\hat{b}_{k+1}\ldots\hat{b}_m\\ &= \hat{u}\overline{\widehat{a_{k+1}\ldots a_n}}(\widehat{\lambda a_{k+1}}\wedge\widehat{a_k\rho})\hat{b}_{k+1}\ldots\hat{b}_m \quad \text{by Lemma 5.}\end{aligned}$$

Now, if none of (i) and (ii) is satisfied or if (i) is satisfied, then $(u\cdot\bar{u}\cdot w)\psi = (u\cdot\bar{v}\cdot w_1)\psi$ directly follows from the above. It remains to show the assertion in case (ii). Put

$$g = \hat{a}_{k+1}\ldots\hat{a}_n\overline{\widehat{a_{k+1}\ldots a_n}}\widehat{\lambda a_{k+1}}(\widehat{\lambda a_{k+1}})',$$

and put

$$h = (\widehat{\lambda a_{k+1}}\wedge\widehat{a_k\rho})\widehat{\lambda a_{k+1}}(\widehat{\lambda a_{k+1}})'.$$

We compute

$$\begin{aligned}(u\cdot\bar{u}\cdot w)\psi &= \hat{u}\overline{\widehat{a_{k+1}\ldots a_n}}(\widehat{\lambda a_{k+1}}\wedge\widehat{a_k\rho})\hat{b}_{k+1}\ldots\hat{b}_m\\ &= \hat{u}\overline{\widehat{a_{k+1}\ldots a_n}}(\widehat{\lambda a_{k+1}}\wedge\hat{a}_k\rho)\widehat{\lambda a_{k+1}}(\widehat{\lambda a_{k+1}})'\hat{b}_{k+1}\hat{b}_m\\ &= \hat{a}_1\ldots\hat{a}_k gh\hat{b}_{k+1}\ldots\hat{b}_m = \hat{a}_1\ldots\hat{a}_k hg\hat{b}_{k+1}\ldots\hat{b}_m\\ &= \hat{a}_1\ldots\hat{a}_k(\widehat{a_k\rho})'\widehat{a_k\rho}(\widehat{\lambda a_{k+1}}\wedge\widehat{a_k\rho})\widehat{\lambda a_{k+1}}(\widehat{\lambda a_{k+1}})'g\hat{b}_{k+1}\ldots\hat{b}_m\\ &= \hat{a}_1\ldots\hat{a}_k(\widehat{a_k\rho})'\widehat{a_k\rho}\widehat{\lambda a_{k+1}}(\widehat{\lambda a_{k+1}})'g\hat{b}_{k+1}\ldots\hat{b}_m \quad \text{by Lemma 3(iii)}\\ &= \hat{a}_1\ldots\hat{a}_n\overline{\widehat{a_{k+1}\ldots a_n}}\hat{b}_{k+1}\ldots\hat{b}_m\\ &= (u\cdot\bar{v}\cdot w_1)\psi,\end{aligned}$$

completing the proof in case $M \neq \emptyset$.

(2) The set M is empty. We then have either $a_2 = b_2$ and $a_2 \in I$ or $a_2 \neq b_2$. Assume first that $a_2 = b_2$ and $a_2 \in I$. It follows $(u\cdot\bar{v}\cdot w_1)\psi = (u\cdot\bar{v}\cdot a_2b_3\ldots b_{m+1})\psi = \hat{u}\hat{\bar{v}}\hat{a}_2\hat{b}_3\ldots\hat{b}_m$. On the other hand, we get

$$\begin{aligned}(u\cdot\bar{u}\cdot w)\psi &= \hat{u}\hat{\bar{v}}\hat{a}_1\hat{a}_2\hat{b}_3\ldots\hat{b}_m = \hat{u}\hat{\bar{v}}(\hat{a}_2\wedge\hat{a}_1)\hat{a}_1'\hat{a}_1\hat{a}_2\hat{b}_3\ldots\hat{b}_m\\ &= \hat{u}\hat{\bar{v}}(\hat{a}_2\wedge\hat{a}_1)\hat{a}_2\hat{b}_3\ldots\hat{b}_m.\end{aligned}$$

Put $e = \hat{v}\hat{\bar{v}}$, and put $f = (\hat{a}_2 \wedge \hat{a}_1)\hat{a}_2\hat{a}_2'$. Then e and f are idempotents belonging to the submonoid $\hat{a}_2\hat{a}_2' S \hat{a}_2\hat{a}_2'$, and we compute

$$\begin{aligned}(u \cdot \bar{u} \cdot w)\psi &= \hat{a}_1 e f \hat{a}_2 \hat{b}_3 \dots \hat{b}_m = \hat{a}_1 f e \hat{a}_2 \hat{b}_3 \dots \hat{b}_m = \hat{a}_1(\hat{a}_2 \wedge \hat{a}_1)\hat{v}\hat{\bar{v}}\hat{a}_2\hat{b}_3 \dots \hat{b}_m \\ &= \hat{a}_1\hat{a}_2 \dots \hat{a}_n \hat{\bar{v}} \hat{a}_2 \hat{b}_3 \dots \hat{b}_m \quad \text{by Lemma 3(iii)} \\ &= (u \cdot \bar{v} \cdot w_1)\psi.\end{aligned}$$

Assume now that $a_2 \neq b_2$. Note that $(u \cdot \bar{v} \cdot w_1)\pi = u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot b_2 \dots b_m$ in this case. If additionally $\lambda a_2 \neq \lambda b_2$, we get $(u \cdot \bar{v} \cdot w_1)\psi = \hat{u}\hat{\bar{v}}(\widehat{\lambda a_2} \wedge \hat{a}_1)\hat{b}_2 \dots \hat{b}_m$. On the other hand, $(u \cdot \bar{u} \cdot w)\psi = \hat{u}\hat{\bar{v}}(\widehat{\lambda a_2} \wedge \hat{a}_1)\hat{a}_1'\hat{a}_1\hat{b}_2 \dots \hat{b}_m$, which implies the assertion, since $(\widehat{\lambda a_2} \wedge \hat{a}_1)\hat{a}_1'\hat{a}_1 = \widehat{\lambda a_2} \wedge \hat{a}_1$.

Moreover, if $\lambda a_2 = \lambda b_2$, then $(u \cdot \bar{v} \cdot w_1)\psi = \hat{u}\hat{\bar{v}}\hat{b}_2 \dots \hat{b}_m$. On the other hand put $g = \hat{v}\hat{\bar{v}}\widehat{\lambda a_2}(\widehat{\lambda a_2})'$ and $h = (\widehat{\lambda a_2} \wedge \hat{a}_1)\widehat{\lambda a_2}(\widehat{\lambda a_2})'$. Then g and h are idempotents belonging to $\widehat{\lambda a_2}(\widehat{\lambda a_2})' S \widehat{\lambda a_2}(\widehat{\lambda a_2'})$, whence

$$\begin{aligned}(u \cdot \bar{u} \cdot w)\psi &= \hat{a}_1 g h \hat{b}_2 \dots \hat{b}_m = \hat{a}_1 h g \hat{b}_2 \dots \hat{b}_m = \hat{a}_1(\widehat{\lambda a_2} \wedge \hat{a}_1)\hat{v}\hat{\bar{v}}\widehat{\lambda a_2}(\widehat{\lambda a_2})'\hat{b}_2 \dots \hat{b}_m \\ &= \hat{u}\hat{\bar{v}}\hat{b}_2 \dots \hat{b}_m = (u \cdot \bar{v} \cdot w_1)\psi.\end{aligned}$$

It remains to handle the case $a_1 \neq b_1$. Here $a_1' \odot w$ is obtained from $a_1' w$ by possibly applying one reduction of the form $a(b \wedge a) \to a$. Keeping in mind the definition of π, we directly see that

$$(u \cdot \bar{u} \cdot w)\psi = \big(u \cdot \bar{v} \odot (\lambda a_2 \wedge a_1) \odot a_1' \cdot w\big)\psi = \big(u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w\big)\psi. \quad \square$$

Lemma 7. *Let $u = a_1 \dots a_n$, where $n \geqslant 2$ and $a_1, a_n \in I$. Let further $v = a_2 \dots a_n$. Then $(u_1 \cdot \bar{u} \cdot w \cdot u_2)\psi = (u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w \cdot u_2)\psi$ for all $u_1, u_2 \in \mathbf{s}(F)$.*

Proof. The assertion will be proved by induction on $f(w) = m$. Let $m = 1$ and $w = b \in I \cup (I \wedge I)$. Note first that since $\bar{u}$ ends with $a_1' \in I$ in any case we have $(u_1 \cdot u \cdot \bar{u} \cdot b \cdot u_2)\psi = (u_1 \cdot u \cdot \bar{u} \cdot b \odot u_2)\psi$, which is equal to $u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot b \odot u_2)\psi$ by Lemma 6.

Assume now that $a_1 = b$. We get

$$\big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot b \odot u_2\big)\psi = \big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \cdot u_2\big)\psi.$$

Since $(\lambda a_2 \wedge a_1) = (\lambda a_2 \wedge a_1) \odot a_1' \odot a_1$, the assertion follows.

On the other hand, if $a_1 \neq b$, we directly see that

$$\begin{aligned}(u_1 \cdot u \cdot \bar{u} \cdot b \cdot u_2)\psi &= \big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \cdot b \cdot u_2\big)\psi \\ &= \big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot b \cdot u_2\big)\psi.\end{aligned}$$

Assume that the assertion is true for $m \geqslant 1$ and let $w = b_1 \dots b_{m+1}$. If $b_{m+1} \in (I \wedge I)$, then

$$\begin{aligned}(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi &= (u_1 \cdot u \cdot \bar{u} \cdot b_1 \ldots b_m \cdot b_{m+1} \odot u_2)\psi \\ &= \big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot b_1 \ldots b_m \cdot b_{m+1} \odot u_2\big)\psi \\ &\qquad \text{by hypothesis} \\ &= \big(u_1 \cdot u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w \cdot u_2\big)\psi.\end{aligned}$$

Let finally $b_{m+1} \in I$ and put $w_1 = (\lambda a_2 \wedge a_1) \odot a_1' \odot w$. Since $a_1 \in I$, we get

$$(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi = \big(u_1 \odot (a_1 \wedge a_1')\big)\psi (u \cdot \bar{u} \cdot w)\psi \big((b_{m+1}' \wedge b_{m+1}) \odot u_2\big)\psi \quad \text{and}$$
$$(u_1 \cdot u \cdot \bar{v} \cdot w_1 \cdot u_2)\psi = \big(u_1 \odot (a_1 \wedge a_1')\big)\psi (u \cdot \bar{v} \cdot w_1)\psi \big((b_{m+1}' \wedge b_{m+1}) \odot u_2\big)\psi.$$

Since $(u \cdot \bar{u} \cdot w)\psi$ is equal to $(u \cdot \bar{v} \cdot w_1)\psi$ by Lemma 6, the assertion follows, completing the proof. □

The next proposition will be crucial for establishing equality (3).

Proposition 8. $(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi = (u_1 \cdot u \cdot \bar{u} \odot w \cdot u_2)\psi$, *for all* $u_1, u_2, u, w \in \mathbf{s}(F)$.

Proof. We prove the assertion by induction on $f(u) = n$. If $n = 1$, it follows by a simple direct case checking which is left to the reader.

Assume that the assertion is true for $n \geqslant 1$ and let $u = a_1 \ldots a_{n+1}$. Let first $a_1 \in (I \wedge I)$ and put $v = a_2 \ldots a_{n+1}$. It follows

$$\begin{aligned}(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi &= \big(u_1 \odot a_1 \cdot v \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1 \rho) \odot w \cdot u_2\big)\psi \\ &= \big(u_1 \odot a_1 \cdot v \cdot \bar{v} \odot (\lambda a_1 \wedge a_1 \rho) \odot w \cdot u_2\big)\psi \quad \text{by hypothesis} \\ &= (u_1 \cdot u \cdot \bar{u} \odot w \cdot u_2)\psi.\end{aligned}$$

Similarly if $a_{n+1} \in (I \wedge I)$, we obtain with $v = a_1 \ldots a_n$,

$$\begin{aligned}(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi &= (u_1 \cdot v \cdot \bar{v} \cdot w \cdot u_2)\psi \quad \text{by definition of } \pi \\ &= (u_1 \cdot v \cdot \bar{v} \odot w \cdot u_2)\psi \quad \text{by hypothesis} \\ &= (u_1 \cdot u \cdot \bar{u} \odot w \cdot u_2)\psi \quad \text{by definition of } \pi.\end{aligned}$$

It remains to handle the case $a_1, a_n \in I$. Put $v = a_2 \ldots a_{n+1}$. We compute

$$\begin{aligned}(u_1 \cdot u \cdot \bar{u} \cdot w \cdot u_2)\psi &= \big(u_1 \odot (a_2 \wedge a_1')\big)\psi (u \cdot \bar{u} \cdot w \cdot u_2)\psi \\ &= \big(u_1 \odot (a_1 \wedge a_1')\big)\psi \big(u \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w \cdot u_2\big)\psi \\ &\qquad \text{by Lemma 7} \\ &= \big(u_1 \odot (a_1 \wedge a_1')\big)\psi \big(a_1 \cdot v \cdot \bar{v} \cdot (\lambda a_2 \wedge a_1) \odot a_1' \odot w \cdot u_2\big)\psi\end{aligned}$$

$$\text{by definition of } v, \text{ and since } a_1 \in I$$
$$= \big(u_1 \odot (a_1 \wedge a_1')\big)\psi\big(a_1 \cdot v \cdot \bar{v} \odot (\lambda a_2 \wedge a_1) \odot a_1' \odot w \cdot u_2\big)\psi$$
$$\text{by hypothesis}$$
$$= \big(u_1 \odot (a_1 \wedge a_1')\big)\psi(u \cdot \bar{u} \odot w \cdot u_2)\psi$$
$$= (u_1 \cdot u \cdot \bar{u} \odot w \cdot u_2)\psi,$$

completing the proof. □

Now we are ready to establish equality (3). For this let $u, u_1, u_2, w \in \mathbf{s}(F)$, and let u' be the inverse of u in the maximal subgroup G_{ab} of $BF\mathcal{CS}(X)$. Then $u' = (a \wedge u\rho) \odot \bar{u} \odot (\lambda u \wedge b)$, whence it follows,

$$\begin{aligned}(u_1 \cdot u \cdot u' \cdot w \cdot u_2)\psi &= \big(u_1 \cdot u \cdot (a \wedge u\rho) \odot \bar{u} \odot (\lambda u \wedge b) \cdot w \cdot u_2\big)\psi \\ &= \big(u_1 \cdot u \cdot \bar{u} \cdot (\lambda u \wedge b) \odot w \cdot u_2\big)\psi \\ &= \big(u_1 \cdot u \cdot \bar{u} \odot (\lambda u \wedge b) \odot w \cdot u_2\big)\psi \quad \text{by Proposition 8} \\ &= (u_1 \cdot u \cdot u' \odot w \cdot u_2)\psi.\end{aligned}$$

Equality (4) holds by a dual argument.

Summarizing, we have shown that $\hat{\psi} : \mathcal{S}(BF\mathcal{CS}(X)) \to S$, $w\rho' \mapsto w\psi$ is a uniquely defined mapping. Obviously $\hat{\psi}$ is not a homomorphism in general, since for $(a \wedge b)\rho', a\rho' \in \mathcal{S}(BF\mathcal{CS}(X))$, where $a, b \in I$, we get

$$\big((a \wedge b)\rho' a\rho'\big)\hat{\psi} = \big((a \wedge b) \cdot a\big)\psi = (a)\psi = \hat{a},$$

whereas $((a \wedge b)\rho')\hat{\psi}(a\rho')\hat{\psi} = (\hat{a} \wedge \hat{b})\hat{a}$, which is not equal to $\hat{a}$ in general. However, we may define a regular subsemigroup T of $\mathcal{S}(BF\mathcal{CS}(X))$ with the property that the restriction $\hat{\psi}|_T$ of $\hat{\psi}$ to T is a homomorphism.

Let T be the set of all words $a_1 \cdot \dots \cdot a_n \in F(\mathbf{s}(F))$, where a_1 and a_n belong to I, and where a_i either belongs to I, or $a_i \in (I \wedge I)$ and $a_{i-1} = (\lambda a_i)'$ and $(a_i\rho)' = a_{i+1}$, $i \in \{2, \dots, n-1\}$. Let further $T\rho'$ be the set of all $w\rho'$, $w \in T$. Then $T\rho'$ is a subsemigroup of $\mathcal{S}(BF\mathcal{CS}(X))$ and by the definition of ψ, $\hat{\psi}|_T$ is a homomorphism. In fact, we are ready now to formulate the main result of the paper.

Theorem 9. *The semigroup $T\rho'$ together with the matched mapping $\iota : I \to T\rho'$, $a \mapsto a\rho'$, is a model of the bifree locally inverse semigroup $BF\mathcal{LI}(X)$ on X.*

Proof. We show first that $T\rho'$ is regular whence it follows that $T\rho'$ is locally inverse as a subsemigroup of $\mathcal{S}(BF\mathcal{CS}(X))$. Let $a, b \in I$. Then $b' \odot (b \wedge a) \odot a'$ is an inverse of $a \odot b$ in $BF\mathcal{CS}(X)$, and it follows by [7, Proposition 1(i)] that

$$(a \cdot b)\rho' = \big(a \cdot b \cdot b' \odot (b \wedge a) \odot a' \cdot a \cdot b\big)\rho' = \big(a \cdot b \cdot b' \cdot (b \wedge a) \cdot a' \cdot a \cdot b\big)\rho',$$

and likewise

$$\begin{aligned}\big(b'\cdot(b\wedge a)\cdot a'\cdot a\cdot b\cdot b'\cdot(b\wedge a)\cdot a'\big)\rho' &= \big(b'\cdot(b\wedge a)\cdot a'\cdot a\odot b\cdot b'\cdot(b\wedge a)\cdot a'\big)\rho'\\ &= \big(b'\cdot(b\wedge a)\cdot a'\big)\rho',\end{aligned}$$

since $a\odot b\in V(b'\odot(b\wedge a)\odot a')$. Consequently $(b'\cdot(b\wedge a)\cdot a')\rho'$ is an inverse of $(a\cdot b)\rho'$ in $T\rho'$. Utilizing Proposition 1, we see that for $w\rho'=(a_1\cdot\cdots\cdot a_n)\rho'\in T\rho'$ the product

$$\big(a_n'\cdot(a_n\wedge a_{n-1}\rho)\cdot a_{n-1}'\cdot(\lambda a_{n-1}\wedge a_{n-2}\rho)\cdot a_{n-2}'\cdot\cdots\cdot(a_2\rho\wedge a_1)\cdot a_1'\big)\rho',$$

where the letter $(\lambda a_i\wedge a_{i-1}\rho)$ occurs, if and only if $a_{i-1},a_i\in I$, is an inverse of $w\rho'$ belonging to $T\rho'$. If for example $w\rho'=(a\cdot(a'\wedge b')\cdot b\cdot c)\rho'$, then $(c'\cdot(c\wedge b)\cdot b'\cdot a')\rho'$ is an inverse of $w\rho'$ which lies in $T\rho'$. Hence $T\rho'$ is regular.

It remains to show that $\bar\theta:=\hat\psi|_T$ is the unique homomorphism extending θ. Since $(a)\iota\bar\theta=(a\rho')\hat\psi|_T=a\psi=\hat a=a\theta$, for $a\in I$, we obviously have that $\bar\theta$ extends θ. Further, since $a\rho'\wedge b\rho'=(a\cdot a'\cdot(a\wedge b)\cdot b'\cdot b)\rho'$, by the above, we observe that $T\rho'$ is multiplicatively generated by the elements $a\rho'$ and $a\rho'\wedge b\rho'$, $a,b\in I$. It is well known [26] and easily follows from Proposition 2, that the operation $\wedge$ is preserved by any homomorphism between locally inverse semigroups. Consequently $\bar\theta$ is unique, completing the proof. □

4. Two applications

This section is devoted to infer two consequences of Theorem 9. First, we obtain an embedding of $BF\mathcal{LI}(X)$ into a Rees matrix semigroup over an inverse monoid, and second we show that $BF\mathcal{LI}(X)$ is embeddable into a restricted semidirect product of a semilattice by $BF\mathcal{CS}(X)$. A similar representation is due to Auinger [2].

We know from the results in [7] that $\mathcal{S}(BF\mathcal{CS}(X))$ is a perfect rectangular band $I\times I$ of E-unitary inverse monoids M_{ab}, $a,b\in I$, where $(u_1\cdot\cdots\cdot u_n)\rho'$ belongs to M_{ab} if and only if $\lambda u_1=a$ and $u_n\rho=b$. Further, the identity element of M_{ab} is $(a\wedge b)\rho'$. If N denotes the subsemigroup of $\mathcal{S}(BF\mathcal{CS}(X))$ which is generated by the elements $a\rho'$, where $a\in I\cup(I\wedge I)$, then N is a perfect rectangular band $I\times I$ of inverse monoids N_{ab}. In particular, for $(a_1\cdot\cdots\cdot a_n)\rho'\in N_{ab}$, the element

$$\big((c\wedge a_n\rho)\cdot\bar a_n\cdot(\lambda a_n\wedge a_{n-1}\rho)\cdot\cdots\cdot\bar a_2\cdot(\lambda a_2\wedge a_1\rho)\cdot\bar a_1\cdot(\lambda a_1\wedge d)\big)\rho',$$

where $\bar a_i=a_i'$ if $a_i\in I$ and $\bar a_i=a_i$, if $a_i\in(I\wedge I)$, is the uniquely determined inverse of $(a_1\cdot\cdots\cdot a_n)\rho'$ in N_{cd}. This follows from a similar argument as used in the proof of Theorem 9.

As was shown by Pastijn [22] each perfect rectangular band $I\times\Lambda$ of inverse monoids $S_{i\lambda}$ can be embedded into a Rees matrix semigroup over an inverse monoid as follows. For $(i,\lambda)\in I\times\Lambda$ let $e_{i\lambda}$ denote the identity element of the submonoid $S_{i\lambda}$. We may assume that there is an element $0\in I\cap\Lambda$. Let $P=(p_{\lambda j})$ be defined by $p_{\lambda j}=e_{0\lambda}e_{j0}$. Then all elements $e_{0\lambda}e_{j0}$, $j\in I$, $\lambda\in\Lambda$ are units in S_{00} and S is isomorphic to the Rees matrix semigroup $\mathcal{M}[S_{00}; I, \Lambda; P]$ with sandwich matrix P, via $s_{i\lambda} \mapsto (i, e_{00}s_{i\lambda}e_{00}, \lambda)$, where $s_{i\lambda} \in S_{i\lambda}$.

We apply this result to our situation. Let N be defined as above. Choose $z \in I$. Then

$$N_{zz} = \bigl\{\bigl((z \wedge z) \cdot w \cdot (z \wedge z)\bigr)\rho'\colon\ w = a_1 \cdot \cdots \cdot a_n,\ a_i \in I \cup (I \wedge I)\bigr\}$$

is an inverse monoid with identity $(z \wedge z)\rho'$. Let $P = (p_{ab})$, $a, b \in I$ be defined by $p_{ab} = ((z \wedge a) \cdot (b \wedge z))\rho'$. Then P is an $I \times I$ matrix which entirely consists of units of N_{zz}. As an immediate consequence of Theorem 9, we obtain

Corollary 10. *The bifree locally inverse semigroup, represented by $T\rho'$, is embeddable into the Rees matrix semigroup $\mathcal{M}[N_{zz}; I, I; P]$ over the E-unitary inverse monoid N_{zz}, via $w\rho' \mapsto (a_1, ((z \wedge z) \cdot w \cdot (z \wedge z))\rho', a_n)$, where $w = a_1 \cdot \cdots \cdot a_n \in T$.*

We refer to some results of [7]. Let C be a completely simple semigroup and let $\langle E_C\rangle$ be the subsemigroup of C, generated by the set of idempotents E_C. We define a binary relation $\sim$ on C by $u \sim v \Leftrightarrow ue = v$, for some $e \in \langle E_C\rangle$. Obviously $\sim$ is an equivalence relation. Moreover, $\sim$ is a left congruence contained in Green's relation $\mathcal{R}$ on C. This gives rise to define a left action of C on $\mathcal{P}_{\text{fin}}(C/\sim)$, the $\cup$-semilattice of all finite subsets of $C/\sim$, via ${}^uB = \{\widetilde{uv}\colon\ \tilde{v} \in B\}$, $u \in C$, $B \in \mathcal{P}_{\text{fin}}(C/\sim)$, where $\tilde{z}$ denotes the $\sim$-class of $z \in C$. Hence we may define the restricted semidirect product $\mathcal{P}_{\text{fin}}(C/\sim) *_{rr} C$ with respect to this action. For $u \in C$ let $\mathcal{R}_u$ be the $\mathcal{R}$-class containing u. Put

$$\widetilde{C}^{\mathcal{R}} = \bigl\{(A, u) \in \mathcal{P}_{\text{fin}}(C/\sim) \times C\colon\ \widetilde{uu'},\ \tilde{u} \in A,\ \text{and } \tilde{v} \subseteq \mathcal{R}_u \text{ for each } \tilde{v} \in A\bigr\}.$$

Then $\widetilde{C}^{\mathcal{R}}$ with multiplication $(A, u)(B, v) = (A \cup {}^uB, uv)$ is a subsemigroup of $\mathcal{P}_{\text{fin}}(C/\sim) *_{rr} C$. Moreover, it was shown in [7] that $\mathcal{S}(C)$ is isomorphic to $\widetilde{C}^{\mathcal{R}}$ via

$$\tilde{\psi} : (u_1 \cdot \cdots \cdot u_n)\rho' \mapsto \bigl(\bigl\{\widetilde{u_1u_1'}\tilde{u}_1, \widetilde{u_1u_2}, \ldots, \widetilde{u_1 \ldots u_n}\bigr\}, u_1 \ldots u_n\bigr).$$

With respect to Theorem 9 we infer

Corollary 11. *The bifree locally inverse semigroup, represented by $T\rho'$, is embeddable into the restricted semidirect product $\mathcal{P}_{\text{fin}}(BF\mathcal{CS}(X)/\sim) *_{rr} BF\mathcal{CS}(X)$ via*

$$\tilde{\psi}|_{T\rho'}\colon\ (a_1 \cdot \cdots \cdot a_n)\rho' \mapsto \bigl(\bigl\{\widetilde{(a_1 \wedge a_1')}, \tilde{a}_1, a_1 \mathbin{\tilde{\odot}} a_2, \ldots, a_1 \mathbin{\widetilde{\odot \cdots \odot}} a_n\bigr\},\ a_1 \odot \cdots \odot a_n\bigr).$$

From Corollary 11 we may pass to an Auinger [2] like semidirect product representation of $BF\mathcal{LI}(X)$, since this special case admits to work with certain subsets of $BF\mathcal{CS}(X)$, rather than sets of $\sim$-classes, in the first component of the restricted semidirect product of Corollary 11. To specify this more precisely, let $\mathcal{P}_{\text{fin}}(\mathbf{s}(F))$ be the $\cup$-semilattice of all finite subsets of $BF\mathcal{CS}(X)$, the latter being represented as $\mathbf{s}(F)$. Since $\mathbf{s}(F)$ naturally acts on $\mathcal{P}_{\text{fin}}(\mathbf{s}(F))$ by multiplication on the left, we may form the restricted semidirect product

$\mathcal{P}_{\text{fin}}(\mathbf{s}(F)) *_{rr} \mathbf{s}(F)$. Let now $w = a_1 \cdot \cdots \cdot a_n \in T$. We know from Corollary 11 that $(a_1 \cdot \cdots \cdot a_n)\rho'$ is mapped onto

$$\left(\left\{\widetilde{\left(a_1 \wedge a_1'\right)}, \tilde{a}_1, \widetilde{a_1 \odot a_2}, \ldots, a_1 \widetilde{\odot \cdots \odot} a_n\right\}, a_1 \odot \cdots \odot a_n\right)$$

under $\tilde{\psi}|_{T\rho'}$. Put

$$\widetilde{A} = \left\{\widetilde{\left(a_1 \wedge a_1'\right)}, \tilde{a}_1, \widetilde{a_1 \odot a_2}, \ldots, a_1 \widetilde{\odot \cdots \odot} a_n\right\}.$$

We assign a subset $A \subseteq \mathbf{s}(F)$ to $\widetilde{A}$ as follows: let A be the set, consisting of all uniquely determined shortest members u of the $\sim$-classes occurring in $\widetilde{A}$ together with the elements $u \odot (u\rho)'$. In particular, by the shortest member of a $\sim$-class containing an idempotent $(a \wedge b)$, say, we mean $(a \wedge a')$. Obviously A is uniquely defined, which gives rise to define a mapping $\tau : (T\rho')\tilde{\psi}|_{T\rho'} \to \mathcal{P}_{\text{fin}}(\mathbf{s}(F)) *_{rr} \mathbf{s}(F)$ by $(\widetilde{A}, a_1 \odot \cdots \odot a_n) \mapsto (A, a_1 \odot \cdots \odot a_n)$. It is not hard to show that τ is an injective homomorphism.

We end this section with some concluding remarks. Utilizing a result of Szendrei [25], it was shown in [13] that $\mathcal{S}(C)$ is isomorphic to $C^{\mathbf{Pr}}$ in case C is a group. Since bifree and free groups are just the same, our considerations yield a new proof of the fact due to Birget and Rhodes [9] that $F\mathcal{G}(X)^{\mathbf{Pr}}$, ($F\mathcal{G}(X)$ the free group on X), contains a copy of the free inverse semigroup on X. Further, if we apply our construction to the free completely simple semigroup $F\mathcal{C}(X)$, we infer that $\mathcal{S}(F\mathcal{C}(X))$ contains a copy of the free (perfect) rectangular band of inverse semigroups (monoids). This was elaborated in [7]. In [8] we constructed expansions of inverse semigroups in a similar way, obtaining a factorization for dual prehomomorphisms; see also Lawson, Margolis, and Steinberg [15]. Summarizing, the concept of expansion as introduced in [9], together with its modifications, seems to be a powerful tool in semigroup theory.

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