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On two geometric constructions of $U(\mathfrak{sl}_n)$ and its representations ☆

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Abstract

Ginzburg and Nakajima have given two different geometric constructions of quotients of the universal enveloping algebra of $\mathfrak{sl}_n$ and its irreducible finite-dimensional highest weight representations using the convolution product in the Borel–Moore homology of flag varieties and quiver varieties, respectively. The purpose of this paper is to explain the precise relationship between the two constructions. In particular, we show that while the two yield different quotients of the universal enveloping algebra, they produce the same representations and the natural bases which arise in both constructions are the same. We also examine how this relationship can be used to translate the crystal structure on irreducible components of quiver varieties, defined by Kashiwara and Saito, to a crystal structure on the varieties appearing in Ginzburg's construction, thus recovering results of Malkin.

Keywords: Geometric representation theory; Convolution product; Lie algebras; Crystal bases; Quiver varieties

Introduction

The universal enveloping algebra of $\mathfrak{sl}_n$ and its finite-dimensional highest weight representations have been constructed geometrically in two different ways by Ginzburg [3] and Nakajima [10] (Nakajima's construction works for more general Kac–Moody algebras). Both constructions use a convolution product in homology. In Ginzburg's construction, the varieties

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involved are flag varieties and their cotangent bundles while in Nakajima's construction they are varieties attached to the quiver (oriented graph) whose underlying graph is the Dynkin graph of $\mathfrak{sl}_n$. Both realizations produce a natural basis of the representations given by the fundamental classes of the irreducible components of the varieties involved. In [8] Nakajima conjectured a specific relationship between the two varieties and this conjecture was later proved by Maffei [5]. In the current paper we review this relationship and use it to examine the representation theoretic constructions in the two settings and show that while the quotients of the universal enveloping algebra obtained are different, there is a natural homomorphism between the two and the natural bases in representations produced by the two constructions are in fact the same. Nakajima's construction using the convolution product was in fact motivated by Ginzburg's construction and thus it is not surprising that we find that the quiver variety construction is in some sense a generalization of the flag variety construction to arbitrary (simply-laced) type. It was certainly expected by experts that the two bases obtained are the same. However, the author is not aware of a proof in the literature of the coincidence of the two bases and the precise relationship between the different constructions of the universal enveloping algebra (which are, in fact, slightly different in the two cases).

Finally, we use the relation between the two constructions to define the structure of a crystal graph on the irreducible components of the Spaltenstein varieties appearing in Ginzburg's construction by analogy with the already existing theory for quiver varieties developed by Kashiwara and Saito. In doing this, we recover the crystal structure on irreducible components of Spaltenstein varieties introduced by Malkin in [6]. We now explain the contents of the paper in some detail.

Fix a positive integer d and let

$$\mathcal{F} = \left\{0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d\right\}$$

be the set of all n-step flags in $\mathbb{C}^d$. Let $N = \{x \in \operatorname{End}(\mathbb{C}^d) \mid x^n = 0\}$. The cotangent bundle to $\mathcal{F}$ is isomorphic to

$$M = \left\{(x, F) \in N \times \mathcal{F} \;\middle|\; x(F_i) \subset F_{i-1}\right\}.$$

We have the natural projection $\mu : M \to N$ and for $x \in N$ we define

$$Z = M \times_N M = \left\{(m_1, m_2) \in M \times M \;\middle|\; \mu(m_1) = \mu(m_2)\right\},$$

$$\mathcal{F}_x = \mu^{-1}(x).$$

Using the convolution product (see Section 2), we give the top-dimensional Borel–Moore homology $H_{\text{top}}(Z)$ the structure of an algebra and $H_{\text{top}}(\mathcal{F}_x)$ the structure of a module over this algebra. Let I_d be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, a two-sided ideal of finite codimension in the enveloping algebra $U(\mathfrak{sl}_n)$. Here $\mathbb{C}^n$ is the natural $\mathfrak{sl}_n$-module. Then in [2,3] it is shown that $H_{\text{top}}(Z) \cong U(\mathfrak{sl}_n)/I_d$ and that under this isomorphism, $H_{\text{top}}(\mathcal{F}_x)$ is the irreducible highest weight $\mathfrak{sl}_n$-module of highest weight $w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1}$ where ω_i are the fundamental weights of $\mathfrak{sl}_n$ and w_i is the number of $(i \times i)$-Jordan blocks in the Jordan normal form of x.

Now, in [10], Nakajima constructs the same representations in a similar way using a convolution product in the homology of quiver varieties. In [5], Maffei showed that the varieties of Nakajima's construction are isomorphic to the following. Let S_x be a transversal slice in N to the $GL(\mathbb{C}^d)$-orbit through x (see Section 5). Then let

$$M' = \mu^{-1}(S_x),$$
$$Z' = M' \times_{S_x} M'.$$

Then, translated via the isomorphism of [5], a result of [10] is that, under the convolution product we have $H_{\mathrm{top}}(Z') \cong U(\mathfrak{sl}_n)/J$ and $H_{\mathrm{top}}(\mathcal{F}_x)$ is the same irreducible highest weight module as in Ginzburg's construction (see Theorems 4.5 and 4.7). Here J is a certain ideal of finite codimension in $U(\mathfrak{sl}_n)$ that is different from I_d in general. Thus the two constructions yield different quotients of the universal enveloping algebra but the same representation.

Since $Z' \subset Z$ and $M' \subset M$, we have a natural restriction with support morphism $H_{\mathrm{top}}(Z) \to H_{\mathrm{top}}(Z')$. The main result of this paper (see Theorem 5.5) is that the following diagram is commutative:

$$\begin{array}{ccccc}
H_{\mathrm{top}}(Z) \otimes H_{\mathrm{top}}(\mathcal{F}_x) & \longrightarrow & H_{\mathrm{top}}(Z') \otimes H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{\cong} & \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}\big(Z(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w})\big) \otimes \bigoplus_{\mathbf{v}} H_{\mathrm{top}}\big(\mathfrak{L}(\mathbf{v},\mathbf{w})\big) \\
\downarrow & & \downarrow & & \downarrow \\
H_{\mathrm{top}}(\mathcal{F}_x) & \longrightarrow & H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{\cong} & \bigoplus_{\mathbf{v}} H_{\mathrm{top}}\big(\mathfrak{L}(\mathbf{v},\mathbf{w})\big)
\end{array}$$

Here the rightmost term in each row involves the Nakajima quiver varieties (see Section 4 for definitions). We are also able to conclude that the natural bases of representations produced by both Ginzburg's and Nakajima's constructions coincide. We thus obtain a precise relation between the two approaches.

Recently, a relation has been established between a construction closely related to that of Ginzburg and another geometric approach of Mirković–Vilonen in terms of the affine Grassmannian [1]. It would be interesting to examine the connection between the quiver variety and Mirković–Vilonen realizations of finite-dimensional representations of Lie algebras.

The organization of the paper is as follows. In Sections 1 and 2 we recall the definition of $\mathfrak{sl}_n$ and the convolution product in Borel–Moore homology. In Sections 3 and 4 we review Ginzburg's and Nakajima's constructions of $U(\mathfrak{sl}_n)$ and its representations. Then in Section 5 we describe the precise relationship between the two constructions. Finally, in Section 6 we define the structure of a crystal on the irreducible components of $\mathcal{F}_x$.

1. Preliminaries

Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of type A_{n-1}. Then $\mathfrak{g}$ is the space of all traceless $n \times n$ matrices. Let $\{e_k, f_k\}_{k=1}^{n-1}$ be the set of Chevalley generators. The Cartan subalgebra $\mathfrak{h}$ is spanned by the matrices

$$h_k = e_{k,k} - e_{k+1,k+1}, \quad 1 \leqslant k \leqslant n-1,$$

where $e_{k,l}$ is the matrix with a one in entry (k,l) and zeroes everywhere else. Thus the dual space $\mathfrak{h}^*$ is spanned by the simple roots

$$\alpha_k = \epsilon_k - \epsilon_{k+1}, \quad 1 \leqslant k \leqslant n-1,$$

where $\epsilon_k(e_{l,l}) = \delta_{kl}$ and the fundamental weights are given by

$$\omega_k = \epsilon_1 + \cdots + \epsilon_k, \quad 1 \leqslant k \leqslant n-1.$$

Consider a dominant weight $\mathbf{w} = w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1}$. Then

$$\mathbf{w} = \lambda_1\epsilon_1 + \cdots + \lambda_{n-1}\epsilon_{n-1},$$

where $\lambda_k = w_k + \cdots + w_{n-1}$ and so $\mathbf{w}$ corresponds to a partition $\lambda(\mathbf{w}) = (\lambda_1 \geqslant \cdots \geqslant \lambda_{n-1})$. We say that a highest weight $\mathbf{w}$ is a partition of d if $|\lambda(\mathbf{w})| = \lambda_1 + \cdots + \lambda_{n-1} = d$ or, equivalently, if $\sum_{k=1}^{n} k w_k = d$.

2. Convolution algebra in homology

In this section we give a brief overview of the convolution algebra in homology. The reader interested in further details should consult [2].

In this paper $H_*(Z)$ will denote the Borel–Moore homology with $\mathbb{C}$-coefficients of a locally-compact space Z. Thus, by definition, if Z is a closed subset of a smooth, oriented manifold M, then

$$H_k(Z) = H^{\dim_{\mathbb{R}} M - k}(M, M \setminus Z).$$

If Z and Z' are closed subsets of a smooth variety M, we have a $\cup$-product map

$$H^k(M, M \setminus Z) \times H^l(M, M \setminus Z') \to H^{k+l}\big(M, M \setminus (Z \cap Z')\big).$$

Thus we construct the intersection pairing in Borel–Moore homology

$$\cap : H_k(Z) \times H_l(Z') \to H_{k+l-d}(Z \cap Z'), \quad d = \dim_{\mathbb{R}} M.$$

Let M_1, M_2 and M_3 be smooth, oriented manifolds and $p_{kl} : M_1 \times M_2 \times M_3 \to M_k \times M_l$ be the obvious projections. Let $Z \subset M_1 \times M_2$ and $Z' \subset M_2 \times M_3$ be closed subvarieties and assume that the map

$$p_{13} : p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \to M_1 \times M_3$$

is proper and denote its image by $Z \circ Z'$. The operation of convolution

$$\star : H_k(Z) \times H_l(Z') \to H_{k+l-d}(Z \circ Z'), \quad d = \dim_{\mathbb{R}} M_2,$$

is defined by

$$c \star c' = (p_{13})_*\big(p_{12}^* c \cap p_{23}^* c'\big),$$

where $p_{12}^* c$ means $c \boxtimes [M_3]$, etc.

Now, let M be a smooth manifold and $\mu : M \to N$ be a proper morphism. Let

$$Z = M \times_N M = \big\{(m_1, m_2) \in M \times M \;\big|\; \mu(m_1) = \mu(m_2)\big\} \subset M \times M.$$

Then $Z \circ Z = Z$ and so convolution makes $H_*(Z)$ a finite-dimensional associative $\mathbb{C}$-algebra with unit.

For $x \in N$, let $M_x = \mu^{-1}(x)$. We also identify M_x with the variety $M_x \times \mathrm{pt}$. Then setting $M_1 = M_2 = M$ and $M_3 = \mathrm{pt}$, we have $Z \circ M_x = M_x$ and convolution makes $H_*(M_x)$ a $H_*(Z)$-module.

3. Ginzburg's construction

We recall here Ginzburg's construction of the enveloping algebra $U(\mathfrak{sl}_n)$ and its irreducible highest weight representations. Proofs omitted here can be found in [3] or [2].

Fix an integer $d \geqslant 1$. Let

$$\mathcal{F} = \{0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d\}$$

be the set of all n-step partial flags in $\mathbb{C}^d$. The space $\mathcal{F}$ is a disjoint union of smooth compact manifolds with connected components parameterized by compositions

$$\mathbf{d} = (d_1 + d_2 + \cdots + d_n = d), \quad d_i \in \mathbb{Z}_{\geqslant 0}.$$

The connected component of $\mathcal{F}$ corresponding to $\mathbf{d}$ is

$$\mathcal{F}_{\mathbf{d}} = \{F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d) \mid \dim F_i/F_{i-1} = d_i\},$$

and

$$\dim_{\mathbb{C}} \mathcal{F}_{\mathbf{d}} = \frac{d!}{d_1! d_2! \cdots d_n!}.$$

Let

$$N = \{x \in \operatorname{End}(\mathbb{C}^d) \mid x^n = 0\}.$$

Then

$$T^*\mathcal{F} \cong M = \{(x, F) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1},\ 1 \leqslant i \leqslant n\}.$$

The above decomposition of $\mathcal{F}$ yields a decomposition of M given by $M = \bigsqcup_{\mathbf{d}} M_{\mathbf{d}}$ where $M_{\mathbf{d}} = T^*\mathcal{F}_{\mathbf{d}}$ for an n-step composition $\mathbf{d}$ of d.

The natural projections give rise to the diagram

$$N \xleftarrow{\mu} M \xrightarrow{\pi} \mathcal{F}.$$

We have a natural action of $GL_d(\mathbb{C})$ on $\mathcal{F}$, N (by conjugation) and M and the projections commute with this action.

For $x \in N$, let $\mathcal{F}_x = \mu^{-1}(x)$. It has connected components $\mathcal{F}_{\mathbf{d},x}$ given by $\mathcal{F}_{\mathbf{d},x} = \mathcal{F}_{\mathbf{d}} \cap \mathcal{F}_x$. Define

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M \mid \mu(m_1) = \mu(m_2)\} \subset M \times M.$$

We use the convention that under the isomorphism

$$T^*\mathcal{F} \times T^*\mathcal{F} \cong T^*(\mathcal{F} \times \mathcal{F}),$$

the standard symplectic form on the right-hand side corresponds to $\omega_1 - \omega_2$ where ω_1 and ω_2 are the symplectic forms on the first and second factors of the left-hand side, respectively.

Proposition 3.1. *The variety Z is the union of the conormal bundles to the $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$. The closures of these conormal bundles are precisely the irreducible components of Z.*

Proposition 3.2. *We have $Z \circ Z = Z$. Thus $H_*(Z)$ is an associative algebra with unit and $H_*(\mathcal{F}_x)$ is an $H_*(Z)$-module for any $x \in N$.*

Proposition 3.3. *All irreducible components of Z contained in $M_{\mathbf{d}^1} \times M_{\mathbf{d}^2}$ are half-dimensional. That is, they have complex dimension*

$$\begin{aligned}\frac{1}{2}\dim_{\mathbb{C}}(M_{\mathbf{d}^1} \times M_{\mathbf{d}^2}) &= \frac{1}{2}\left(2\frac{d^1!}{d_1^1!d_2^1!\cdots d_n^1!} + 2\frac{d^2!}{d_1^2!d_2^2!\cdots d_n^2!}\right) \\ &= \frac{d^1!}{d_1^1!d_2^1!\cdots d_n^1!} + \frac{d^2!}{d_1^2!d_2^2!\cdots d_n^2!}.\end{aligned}$$

Let $H_{\mathrm{top}}(Z)$ be the vector subspace of $H_*(Z)$ spanned by the fundamental classes of the irreducible components of Z and let $H_{\mathrm{top}}(\mathcal{F}_x)$ be the vector subspace of $H_*(\mathcal{F}_x)$ spanned by the fundamental classes of the irreducible components of $\mathcal{F}_x$.

Proposition 3.4. *The homology group $H_{\mathrm{top}}(Z)$ is a subalgebra of $H_*(Z)$ and $H_{\mathrm{top}}(\mathcal{F}_x)$ is an $H_{\mathrm{top}}(Z)$-stable subspace of $H_*(\mathcal{F}_x)$.*

Now, for a composition $\mathbf{d}$ we have the diagonal subvariety $\Delta \subset \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ which is a $GL_d(\mathbb{C})$-orbit. We define

$$H_k = \sum_{\mathbf{d}} (d_k - d_{k+1})\big[T_\Delta^*(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})\big],$$

where $T_O^*(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ denotes the conormal bundle to a $GL_d(\mathbb{C})$-orbit $O \subset \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$. Note that under the sign convention for the symplectic form mentioned above, the conormal bundle $T_\Delta^*(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is the diagonal in $T^*\mathcal{F}_{\mathbf{d}} \times T^*\mathcal{F}_{\mathbf{d}}$.

Now, for a composition $\mathbf{d} = (d_1 + \cdots + d_n)$ and $1 \leqslant k \leqslant n-1$, let

$$\begin{aligned}\mathbf{d}_k^+ &= d_1 + \cdots + d_{k-1} + (d_k + 1) + (d_{k+1} - 1) + d_{k+2} + \cdots + d_n, \\ \mathbf{d}_k^- &= d_1 + \cdots + d_{k-1} + (d_k - 1) + (d_{k+1} + 1) + d_{k+2} + \cdots + d_n,\end{aligned}$$

provided that these are compositions (that is, all terms are $\geqslant 0$). Otherwise, we define $\mathbf{d}_k^\pm = \nabla$, the ghost composition.

If $1 \leqslant k \leqslant n-1$ and $\mathbf{d} = (d_1 + \cdots + d_n)$ is a composition such that $\mathbf{d}_k^+ \neq \nabla$, respectively $\mathbf{d}_k^- \neq \nabla$, we define

$$\begin{aligned}Y_{\mathbf{d}_k^+,\mathbf{d}} &= \big\{(F', F) \in \mathcal{F}_{\mathbf{d}_k^+} \times \mathcal{F}_{\mathbf{d}} \;\big|\; F_l = F_l' \ \forall l \neq k,\ F_k \subset F_k',\ \dim(F_k'/F_k) = 1\big\}, \\ Y_{\mathbf{d}_k^-,\mathbf{d}} &= \big\{(F', F) \in \mathcal{F}_{\mathbf{d}_k^-} \times \mathcal{F}_{\mathbf{d}} \;\big|\; F_l = F_l' \ \forall l \neq k,\ F_k' \subset F_k,\ \dim(F_k/F_k') = 1\big\}.\end{aligned}$$

Note that each $Y_{\mathbf{d}_k^\pm,\mathbf{d}}$ is a $GL_d(\mathbb{C})$-orbit in $\mathcal{F}_{\mathbf{d}_k^\pm} \times \mathcal{F}_{\mathbf{d}}$ of minimal dimension and thus is a smooth closed subvariety. Let

$$E_k = \sum_{\mathbf{d}} \big[T^*_{Y_{\mathbf{d}_k^+,\mathbf{d}}} (\mathcal{F}_{\mathbf{d}_k^+} \times \mathcal{F}_{\mathbf{d}}) \big], \tag{3.1}$$

$$F_k = \sum_{\mathbf{d}} (-1)^{s_k(\mathbf{d}_k^+,\mathbf{d})} \big[T^*_{Y_{\mathbf{d}_k^-,\mathbf{d}}} (\mathcal{F}_{\mathbf{d}_k^-} \times \mathcal{F}_{\mathbf{d}}) \big], \tag{3.2}$$

where $s_k(\mathbf{d}_k^+, \mathbf{d}) = \frac{1}{2}(\dim_{\mathbb{C}} M_{\mathbf{d}_k^+} - \dim_{\mathbb{C}} M_{\mathbf{d}})$.

Theorem 3.5. [3] *The map*

$$e_k \mapsto E_k, \qquad f_k \mapsto F_k, \qquad h_k \mapsto H_k,$$

extends to a surjective algebra homomorphism $U(\mathfrak{sl}_n) \twoheadrightarrow H_{\text{top}}(Z)$. Under this homomorphism, $H_{\text{top}}(\mathcal{F}_x)$ is the irreducible highest weight module of highest weight $w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1}$ where ω_i are the fundamental weights and w_i is the number of $(i \times i)$-Jordan blocks in the Jordan normal form of x.

Remark 3.6. Note that the sign appearing in (3.2) does not appear in [2,3]. This arises from the fact that Theorem 2.7.26(iii) in [2] should read $[Z_{12}] \star [Z_{23}] = (-1)^{\dim F} \chi(F) \cdot [Z_{13}]$ (see [10, Lemma 8.5]).

Let I_d be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, a two-sided ideal of finite codimension in the enveloping algebra $U(\mathfrak{sl}_n)$. Here $\mathbb{C}^n$ is the natural $\mathfrak{sl}_n$-module.

Theorem 3.7. [2, Proposition 4.2.5] *The homomorphism of Theorem* 3.5 *yields an algebra isomorphism*

$$U(\mathfrak{sl}_n)/I_d \cong H_{\text{top}}(Z).$$

It is known that the simple $\mathfrak{sl}_n$-modules that occur with non-zero multiplicity in the decomposition of $(\mathbb{C}^n)^{\otimes d}$ are precisely those modules whose highest weight is a partition of d.

4. Nakajima's construction

In this section, we will review the description of the quiver varieties presented in [10]. Further details may be found in [8,10]. We only discuss the case corresponding to the Lie algebra $\mathfrak{sl}_n$. Note that we use a different stability condition that the one used in [8,10] and so our definitions differ slightly from the ones that appear there. One can translate between the two stability conditions by taking transposes of the maps appearing in the definitions of the quiver varieties. See [9] for a discussion of various choices of stability condition.

As before, let $\mathfrak{g} = \mathfrak{sl}_n$ be the simple Lie algebra of type A_{n-1}. Let $I = \{1, \ldots, n-1\}$ be the set of vertices of the Dynkin graph of $\mathfrak{g}$ with the set of oriented edges given by

$$H = \big\{ h_{k,l} \bigm| k, l \in I,\ |k-l| = 1 \big\}.$$

For two adjacent vertices k and l, $h_{k,l}$ is the oriented edge from vertex k to vertex l. We denote the outgoing and incoming vertices of $h \in H$ by $\text{out}(h)$ and $\text{in}(h)$, respectively. Thus $\text{out}(h_{k,l}) = k$

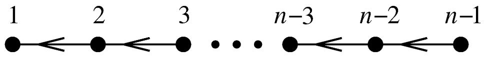

Fig. 1. The quiver of type A_{n-1}.

and $\mathrm{in}(h_{k,l}) = l$. Define the involution $\bar{}: H \to H$ as the function that interchanges $h_{k,l}$ and $h_{l,k}$. Fix the orientation $\Omega = \{h_{k,k-1} \mid 2 \leqslant k \leqslant n-1\}$. We picture this quiver as in Fig. 1.

Let $V = \bigoplus_{k \in I} V_k$ and $W = \bigoplus_{k \in I} W_k$ be two finite-dimensional complex I-graded vector spaces with graded dimensions

$$\mathbf{v} = (\dim V_1, \dim V_2, \dots, \dim V_{n-1}),$$
$$\mathbf{w} = (\dim W_1, \dim W_2, \dots, \dim W_{n-1}).$$

Then we define

$$\mathbf{M}(\mathbf{v}, \mathbf{w}) = \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) \oplus \bigoplus_{k \in I} \mathrm{Hom}(W_k, V_k) \oplus \bigoplus_{k \in I} \mathrm{Hom}(V_k, W_k).$$

The above three components of an element of $\mathbf{M}(\mathbf{v}, \mathbf{w})$ will be denoted by $B = (B_h)$, $i = (i_k)$ and $j = (j_k)$. We associate elements in the weight lattice of $\mathfrak{g}$ to the dimensions vectors $\mathbf{v} = (v_1, \dots, v_{n-1})$ and $\mathbf{w} = (w_1, \dots, w_{k-1})$ as follows:

$$\alpha_{\mathbf{v}} = \sum_{k \in I} v_k \alpha_k, \qquad \omega_{\mathbf{w}} = \sum_{k \in I} w_k \omega_k,$$

where α_k and ω_k are the simple roots and fundamental weights, respectively.

Now, let

$$G_{\mathbf{v}} = \prod_{k \in I} GL(V_k)$$

act on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ by

$$g(B, i, j) = \left(gBg^{-1}, gi, jg^{-1}\right),$$

where $gBg^{-1} = (B'_h) = (g_{\mathrm{in}(h)} B_h g^{-1}_{\mathrm{out}(h)})$, $gi = (i'_k) = (g_k i_k)$ and $jg^{-1} = (j'_k) = (j_k g_k^{-1})$. Let $\epsilon : H \to \{\pm 1\}$ be given by

$$\epsilon(h) = \begin{cases} +1 & \text{if } h \in \Omega, \\ -1 & \text{if } h \in \bar{\Omega}. \end{cases}$$

Define a map $\mu : \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \bigoplus_{k \in I} \mathrm{End}(V_k, V_k)$ with kth component given by

$$\mu_k(B, i, j) = \sum_{h \in H:\ \mathrm{in}(h) = k} \epsilon(h) B_h B_{\bar{h}} + i_k j_k.$$

Let $A(\mu^{-1}(0))$ be the coordinate ring of the affine algebraic variety $\mu^{-1}(0)$ and define

$$\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)/G = \mathrm{Spec}\, A\left(\mu^{-1}(0)\right)^G.$$

This is the affine algebro-geometric quotient of $\mu^{-1}(0)$ by G. It is an affine algebraic variety and its geometric points are closed $G_{\mathbf{v}}$-orbits.

We say that a collection $S=(S_k)$ of subspaces $S_k \subset V_k$ is B-stable if $B_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$ for all $h \in H$. We say that a point of $\mu^{-1}(0)$ is stable if any B-stable collection of subspaces S containing the image of i is equal to all of V. We let $\mu^{-1}(0)^s$ denote the set of stable points.

Proposition 4.1. *The stabilizer in $G_{\mathbf{v}}$ of any point in $\mu^{-1}(0)^s$ is trivial.*

We then define

$$\mathfrak{M}(\mathbf{v},\mathbf{w}) = \mu^{-1}(0)^s / G_{\mathbf{v}},$$

which is diffeomorphic to an affine algebraic manifold. We know (see [10, Corollary 3.12]) that

$$\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v},\mathbf{w}) = \mathbf{v}\cdot(2\mathbf{w} - C\mathbf{v}),$$

where C is the Cartan matrix of $\mathfrak{sl}_n$.

For $(B,i,j) \in \mu^{-1}(0)^s$, we denote the corresponding orbit in $\mathfrak{M}(\mathbf{v},\mathbf{w})$ by $[B,i,j]$ and if the orbit through (B,i,j) is closed, we denote the corresponding point of $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ by the same notation.

We have a map

$$\pi : \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w}),$$

which sends an orbit $[B,i,j]$ to the unique closed orbit $[B_0,i_0,j_0]$ contained in the closure of $G(B,i,j)$. Let $\mathfrak{L}(\mathbf{v},\mathbf{w}) = \pi^{-1}(0)$.

Proposition 4.2. *The subvariety $\mathfrak{L}(\mathbf{v},\mathbf{w}) \subset \mathfrak{M}(\mathbf{v},\mathbf{w})$ is half-dimensional and is homotopic to $\mathfrak{M}(\mathbf{v},\mathbf{w})$.*

Actually, under a natural symplectic form on $\mathfrak{M}(\mathbf{v},\mathbf{w})$, the subvariety $\mathfrak{L}(\mathbf{v},\mathbf{w})$ is Lagrangian. It will be useful in the sequel to also consider the following direct construction of $\mathfrak{L}(\mathbf{v},\mathbf{w})$. Let

$$\Lambda(\mathbf{v},\mathbf{w}) = \bigl\{(B,i,j) \in \mu^{-1}(0) \bigm| j=0,\ B \text{ is nilpotent}\bigr\},$$

where B nilpotent means that there exists $N \geqslant 1$ such that for any sequence $h_1, h_2, \ldots, h_N$ in H satisfying $\mathrm{in}(h_k) = \mathrm{out}(h_{k+1})$, the composition $B_{h_N} \cdots B_{h_2} B_{h_1} : V_{\mathrm{out}(h_1)} \to V_{\mathrm{in}(h_N)}$ is zero. Furthermore, define

$$\Lambda(\mathbf{v},\mathbf{w})^s = \bigl\{(B,i,j) \in \Lambda(\mathbf{v},\mathbf{w}) \bigm| (B,i,j) \in \mu^{-1}(0)^s\bigr\}.$$

Then we have the following lemma.

Lemma 4.3. *We have*

$$\mathfrak{L}(\mathbf{v},\mathbf{w}) = \Lambda(\mathbf{v},\mathbf{w})^s / G_{\mathbf{v}}.$$

If $V' = (V'_k)$ is a collection of subspaces of $V = (V_k)$, we have a natural inclusion map $\mathfrak{M}_0(\mathbf{v}', \mathbf{w}) \hookrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. Thus, for vector spaces V^1, V^2, W, we can consider the projections $\pi : \mathfrak{M}(\mathbf{v}^k, \mathbf{w}) \to \mathfrak{M}_0(\mathbf{v}^k, \mathbf{w})$ as maps to $\mathfrak{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$. We then define

$$Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) = \{(x^1, x^2) \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \mid \pi(x_1) = \pi(x_2)\}.$$

Since $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \circ Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w}) \subset Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})$, we have the convolution product

$$H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \otimes H_*(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})) \to H_*(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})).$$

All of the irreducible components of $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ have the same dimension. Let $H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ denote the top degree part of $H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$. It has a natural basis $\{[X]\}$ where X runs over the irreducible components of $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$.

Proposition 4.4. *The convolution product makes the direct sum* $\bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ *into an associative algebra, and* $\bigoplus_{\mathbf{v}} H_*(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ *is a left* $\bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$*-module. In addition, the top degree part* $\bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ *is a subalgebra, and* $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ *is a* $\bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$*-stable submodule.*

Let $\Delta(\mathbf{v}, \mathbf{w})$ denote the diagonal in $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$. Then its fundamental class $[\Delta(\mathbf{v}, \mathbf{w})]$ is in $H_{\mathrm{top}}(Z(\mathbf{v}, \mathbf{v}; \mathbf{w}))$. Left and right multiplication by $[\Delta(\mathbf{v}, \mathbf{w})]$ define projections

$$[\Delta(\mathbf{v}, \mathbf{w})]\cdot : \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \to \bigoplus_{\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}, \mathbf{v}^2; \mathbf{w})),$$

$$\cdot[\Delta(\mathbf{v}, \mathbf{w})] : \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \to \bigoplus_{\mathbf{v}^1} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}; \mathbf{w})).$$

For $k \in I$, define the *Hecke correspondence* $\mathfrak{B}_k(\mathbf{v}, \mathbf{w})$ to be the variety of all (B, i, j, S) (modulo the $G_{\mathbf{v}}$-action) such that $(B, i, j) \in \mu^{-1}(0)^s$ and S is a B-invariant subspace contained in the kernel of j such that $\dim S = \mathbf{e}^k$ where $\mathbf{e}^k$ has k-component equal to one and all other components equal to zero. We consider (B, i, j, S) as a point in $Z(\mathbf{v} - \mathbf{e}^k, \mathbf{v}; \mathbf{w})$ by taking the quotient by the subspace S in the first factor. Then $\mathfrak{B}_k(\mathbf{v}, \mathbf{w})$ is an irreducible component of $Z(\mathbf{v} - \mathbf{e}^k, \mathbf{v}; \mathbf{w})$. Let $\omega : \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \to \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ be the map that interchanges the two factors. Then define

$$E_k = \sum_{\mathbf{v}} [\mathfrak{B}_k(\mathbf{v}, \mathbf{w})] \in \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})), \tag{4.1}$$

$$F_k = \sum_{\mathbf{v}} (-1)^{r_k(\mathbf{v},\mathbf{w})} [\omega(\mathfrak{B}_k(\mathbf{v}, \mathbf{w}))] \in \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})), \tag{4.2}$$

$$H_k = \sum_{\mathbf{v}} \langle h_k, \omega_{\mathbf{w}} - \alpha_{\mathbf{v}} \rangle [\Delta(\mathbf{v}, \mathbf{w})], \tag{4.3}$$

where $r_k(\mathbf{v}, \mathbf{w}) = \frac{1}{2}(\dim \mathfrak{M}_{\mathbb{C}}(\mathbf{v} - \mathbf{e}^k, \mathbf{w}) - \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}, \mathbf{w})) = -\mathbf{e}^k \cdot (\mathbf{w} - C\mathbf{v}) - 1$. Here C is the Cartan matrix of $\mathfrak{sl}_n$. Note that since we are restricting ourselves to the Lie algebra $\mathfrak{sl}_n$, the varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$ are only non-empty for a finite number of $\mathbf{v}$ and so the above elements are well defined.

Theorem 4.5. [10] *There exists a unique surjective algebra homomorphism*

$$\Phi : U(\mathfrak{sl}_n) \twoheadrightarrow \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}\big(Z\big(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w}\big)\big)$$

such that

$$\Phi(h_k) = H_k, \quad \Phi(e_k) = E_k, \quad \Phi(f_k) = F_k.$$

Under this homomorphism, $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v},\mathbf{w}))$ is the irreducible integrable highest weight module with highest weight $\omega_{\mathbf{w}}$. The class $[\mathfrak{L}(\mathbf{0},\mathbf{w})]$ is a highest weight vector.

Remark 4.6. The result in [10] is actually in terms of the modified universal enveloping algebra. In the more general case of a Kac–Moody algebra with symmetric Cartan matrix, this language is more natural. However, in our case of $\mathfrak{sl}_n$, since for a fixed $\mathbf{w}$ the quiver varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$ are non-empty only for a finite number of $\mathbf{v}$, we can avoid the use of the modified universal enveloping algebra.

Let $J_{\mathbf{w}}$ be the annihilator in $U(\mathfrak{sl}_n)$ of $\bigoplus_{\mathbf{v}} L(\omega_{\mathbf{w}} - \alpha_{\mathbf{v}})$, where the sum is over all $\mathbf{v}$ such that $\omega_{\mathbf{w}} - \alpha_{\mathbf{v}}$ is dominant integral and is a weight of $L(\omega_{\mathbf{w}})$. Here $L(\lambda)$ is the irreducible integrable highest weight representation of highest weight λ.

Theorem 4.7. [10, Theorem 10.15] *The homomorphism of Theorem* 4.5 *yields an algebra isomorphism*

$$U(\mathfrak{sl}_n)/J_{\mathbf{w}} \cong \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}\big(Z\big(\mathbf{v}^1,\mathbf{v}^2;\mathbf{w}\big)\big).$$

5. A comparison of the two constructions

We now describe the precise relationship between the constructions of Ginzburg and Nakajima.

We begin by recalling a result of Maffei [5]. Let $x \in N$ and let $\{x, y, h\}$ be an $\mathfrak{sl}_2$ triple in $GL(\mathbb{C}^d)$. We define the *transversal slice* to the orbit O_x of x in N at the point x to be

$$S_x = \big\{u \in N \bigm| [u - x, y] = 0\big\}.$$

We allow $\{0, 0, 0\}$ to be an $\mathfrak{sl}_2$ triple. Thus we have $S_0 = N$.

Now, the orbits of the action of $GL(\mathbb{C}^d)$ on N are determined by partitions of d. Corresponding to a partition λ is the orbit consisting of all those matrices whose Jordan blocks have sizes λ_i. We let O_λ denote the orbit corresponding to the partition λ.

Let $\mu_{\mathbf{d}} : M_{\mathbf{d}} \to N$ denote the restriction of the map μ to $M_{\mathbf{d}}$. Then let $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n)$ be a permutation of $\mathbf{d}$ and define the partition $\lambda_{\mathbf{d}} = 1^{\alpha_1-\alpha_2}2^{\alpha_2-\alpha_3}\cdots n^{\alpha_n}$. Then $\lambda_{\mathbf{d}}$ is a partition of d and if $(x, F) \in M_{\mathbf{d}}$, then $x \in \bar{O}_{\lambda_{\mathbf{d}}}$. Furthermore, the map $\mu_{\mathbf{d}} : M_{\mathbf{d}} \to \bar{O}_{\lambda_{\mathbf{d}}}$ is a resolution of singularities and is an isomorphism over $O_{\lambda_{\mathbf{d}}}$. Define

$$S_{\mathbf{d},x} = S_x \cap \bar{O}_{\lambda_{\mathbf{d}}}, \qquad \tilde{S}_{\mathbf{d},x} = \mu_{\mathbf{d}}^{-1}(S_{\mathbf{d},x}) = \mu_{\mathbf{d}}^{-1}(S_x).$$

Now, for $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geqslant 0})^{n-1}$ define $\mathbf{a} = \mathbf{a}(\mathbf{v}, \mathbf{w}) = (a_1, \dots, a_n)$ by

$$\begin{aligned} a_1 &= w_1 + \dots + w_{n-1} - v_1, \qquad a_n = v_{n-1}, \\ a_k &= w_k + \dots + w_{n-1} - v_k + v_{k-1}, \quad 2 \leqslant k \leqslant n-1. \end{aligned} \tag{5.1}$$

Note that $\sum_{k=1}^{n} a_k = d = \sum_{k=1}^{n-1} k w_k$ and that for a fixed d and $\mathbf{w}$, the above map is a bijection between $(n-1)$-tuples of integers $\mathbf{v}$ and n-tuples of integers $\mathbf{a}$ such that $\sum_i a_i = d$. Furthermore, let $\mathfrak{M}^1(\mathbf{v}, \mathbf{w}) = \pi(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$.

Theorem 5.1. [5] *Let $\mathbf{v}$, $\mathbf{w}$, d and $\mathbf{a} = \mathbf{a}(\mathbf{v}, \mathbf{w})$ be as above and let $x \in N$ be a nilpotent element of type $1^{w_1} 2^{w_2} \cdots (n-1)^{w_{n-1}}$. Then there exists an isomorphism $\theta : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \xrightarrow{\cong} \tilde{S}_{\mathbf{a},x}$ and $\theta_1 : \mathfrak{M}^1(\mathbf{v}, \mathbf{w}) \xrightarrow{\cong} S_{\mathbf{a},x}$ such that $\theta_1(0) = x$ and the following diagram commutes*:

$$\begin{array}{ccc} \mathfrak{M}(\mathbf{v}, \mathbf{w}) & \xrightarrow{\theta} & \tilde{S}_{\mathbf{a},x} \\ \pi \downarrow & & \downarrow \mu_{\mathbf{a}} \\ \mathfrak{M}^1(\mathbf{v}, \mathbf{w}) & \xrightarrow{\theta_1} & S_{\mathbf{a},x} \end{array}$$

Note that by Theorem 5.1, if we restrict θ to $\mathfrak{L}(\mathbf{v}, \mathbf{w})$, we obtain an isomorphism $\mathfrak{L}(\mathbf{v}, \mathbf{w}) \cong \mathcal{F}_{\mathbf{a},x}$ which we will also denote by θ. This restriction is fairly simple to describe as we now show.

We define a *path* to be an ordered set of edges $(h_1, \dots, h_N)$ such that $\mathrm{in}(h_i) = \mathrm{out}(h_{i+1})$. Then let $\mathcal{P}$ be the set of all paths that head left and then right. That is,

$$\mathcal{P} = \{(h_{k,k-1}, h_{k-1,k-2}, \dots, h_{l+1,l}, h_{l,l+1}, \dots, h_{m-1,m}) \mid 1 \leqslant l \leqslant m, k \leqslant n-1\}.$$

For $p = (h_1, \dots, h_N) \in \mathcal{P}$, let $\mathrm{in}(p) = \mathrm{in}(h_N)$ be the incoming vertex of the last edge in p and let $\mathrm{out}(p) = \mathrm{out}(h_1)$ be the outgoing vertex of the first edge in p. We define $\mathrm{ord}(p)$ to be the number of edges heading to the left. That is, $\mathrm{ord}(p) = \#\{h_i \in p \mid h_i \in \Omega\}$. Furthermore we let $B_p = B_{h_N} \cdots B_{h_1}$ be the obvious composition of maps.

Now, for $1 \leqslant m \leqslant k \leqslant n-1$, let $\iota_k^m : W_k^{(m)} \cong W_k$ be an isomorphism to a copy of W_k. Then for $1 \leqslant k \leqslant n-1$, let

$$\phi_k = \bigoplus_{p \in \mathcal{P},\, \mathrm{in}(p) = k} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p) - \mathrm{ord}(p)} : \bigoplus_{l=1}^{n-1} \bigoplus_{m \leqslant k,l} W_l^{(m)} \to V_k. \tag{5.2}$$

Let $d = \sum_{k=1}^{n-1} k w_k$ and identify $\bigoplus_{m,k:\, m \leqslant k} W_k^{(m)}$ with $\mathbb{C}^d$. Then $\theta : \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w}) \to \mathcal{F}$ sends the point $[B, i, j]$ to the flag $F = (0 = F_0 \subset \dots \subset F_n = \mathbb{C}^d)$ where $F_k = \ker \phi_k$. Note that θ is well defined since the kernel of ϕ_k does not change under the action of $G_{\mathbf{v}}$. For $1 \leqslant k \leqslant n-1$, define

$$W^{\leqslant k} = \bigoplus_{m,l:\, m \leqslant l,k} W_l^{(m)}.$$

Note that we always have

$$F_k = \ker \phi_k \subset W^{\leqslant k}.$$

Corollary 5.2. *The image of the map* $\theta : \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w}) \to \mathcal{F}$ *lies in* $\mathcal{F}_x$ *where* $x \in N$ *is the map given in block form by* $W_k^{(m)} \xrightarrow{\cong} W_k^{(m-1)}$ *(and* $x(W_k^{(1)}) = 0$*). Furthermore* $\theta : \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w}) \to \mathcal{F}_x$ *is an isomorphism and* $\theta(\mathfrak{L}(\mathbf{v}, \mathbf{w})) = \mathcal{F}_{\mathbf{a},x}$ *where* $\mathbf{a} = \mathbf{a}(\mathbf{v}, \mathbf{w})$ *is defined by* (5.1).

Proposition 5.3. *Let* $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geqslant 0})^{n-1}$, $\mathbf{a} = \mathbf{a}(\mathbf{v}, \mathbf{w})$, $x \in N$ *a nilpotent element of type* $1^{w_1} 2^{w_2} \cdots (n-1)^{w_{n-1}}$*, and* $1 \leqslant k \leqslant n-1$*. Then*

$$(\theta \times \theta)\big(\mathfrak{B}_k(\mathbf{v}, \mathbf{w})\big) = \big(T^*_{Y_{\mathbf{a}_k^+, \mathbf{a}}}(\mathcal{F}_{\mathbf{a}_k^+} \times \mathcal{F}_{\mathbf{a}})\big) \cap (\tilde{S}_{\mathbf{a}_k^+, x} \times \tilde{S}_{\mathbf{a}, x}). \tag{5.3}$$

Proof. The right side of (5.3) is equal to

$$\big\{(F', F) \in \tilde{S}_{\mathbf{a}_k^+, x} \times \tilde{S}_{\mathbf{a}, x} \bigm| F_l = F'_l \ \forall l \neq k,\ F_k \subset F'_k,\ \dim(F'_k/F_k) = 1\big\}. \tag{5.4}$$

Recall that

$$\mathfrak{B}_k(\mathbf{v}, \mathbf{w}) = \big\{(B, i, j, S) \bigm| (B, i, j) \in \mu^{-1}(0)^s,\ S \subset V,\ j(S) = 0,\ S\ B\text{-invariant}, \\ \dim S = \mathbf{e}^k\big\}/G_{\mathbf{v}}.$$

We consider this as a subset of $\mathfrak{M}(\mathbf{v} - \mathbf{e}^k, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ by taking the quotient by the subspace S in the first factor. We know by Theorem 5.1 that

$$\theta : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \xrightarrow{\cong} \tilde{S}_{\mathbf{a},x}, \qquad \theta : \mathfrak{M}\big(\mathbf{v} - \mathbf{e}^k, \mathbf{w}\big) \xrightarrow{\cong} \tilde{S}_{\mathbf{a}_k^+, x}.$$

Thus, it suffices to show that a choice of B-invariant subspace S of V_k corresponds to a choice of F'_k such that $F_k \subset F'_k \subset x^{-1}(F_{k-1})$. We first do this for the case where $W = W_1$. Then $i = i_1$ and $j = j_1$. In this case, the isomorphism between quiver varieties and flag varieties is particularly simple (see [5,8]). The isomorphism is given by $\theta : [B, i, j] \mapsto (x, F)$ where

$$x = ji, \qquad F = (0 \subset \ker i \subset \ker B_{12} i \subset \cdots \subset \ker B_{n-2,n-1} \cdots B_{12} i \subset W).$$

That is, $F_l = \ker B_{l-1,l} \cdots B_{12} i$. Now, let $S \subset V_k$ be a B-invariant subspace contained in the kernel of j with $\dim S = 1$ and let (B', i', j') be the point of $\mathfrak{M}(\mathbf{v} - \mathbf{e}^k, \mathbf{w})$ obtained from (B, i, j) by taking the quotient by the subspace S. Now, since S is B-invariant, we have that $S \in \ker B_{k,k-1} \cap \ker B_{k,k+1}$. Here we adopt the convention that $B_{1,0} = 0$ and $B_{n-1,n} = 0$. Let $p : V_k \to V_k/S$ be the canonical projection. Then $\theta([B', i', j']) = (x, F')$ where $x = ji$ and

$$F'_l = \ker B_{l-1,l} \cdots B_{12} i = F_l, \quad l < k,$$
$$F'_l = \ker B_{l-1,l} \cdots B_{k,k+1} p B_{k-1,k} \cdots B_{12} i, \quad l \geqslant k.$$

Now, since $S \subset \ker B_{k,k+1}$, we have that $B_{k,k+1} p = B_{k,k+1}$. Thus, for $l > k$, $F'_l = F_l$. Also,

$$F'_k = \ker p B_{k-1,k} \cdots B_{12} i \supset \ker B_{k-1,k} \cdots B_{12} i = F_k.$$

Thus it remains to show that $F'_k \subset x^{-1}(F_{k-1})$. Now,

$$\begin{aligned} x^{-1}(F_{k-1}) &= x^{-1}(\ker B_{k-2,k-1} \cdots B_{12} i) \\ &= \ker(B_{k-2,k-1} \cdots B_{12} i x) \\ &= \ker(B_{k-2,k-1} \cdots B_{12} i j i). \end{aligned}$$

Now, since $(B, i, j) \in \mu^{-1}(0)$, we have that $ij = B_{21} B_{12}$ and $B_{l-1,l} B_{l,l-1} = B_{l+1,l} B_{l+1,l}$ for $2 \leqslant l \leqslant n-2$. Thus,

$$\begin{aligned} B_{k-2,k-1} \cdots B_{12} i j i &= B_{k-2,k-1} \cdots B_{12} B_{21} B_{12} i \\ &\ \vdots \\ &= B_{k,k-1} B_{k-1,k} B_{k-2,k-1} \cdots B_{12} i. \end{aligned}$$

Thus,

$$x^{-1}(F_{k-1}) = \ker(B_{k,k-1} B_{k-1,k} B_{k-2,k-1} \cdots B_{12} i).$$

Now, since $S \subset \ker B_{k,k-1}$, we have

$$F'_k = \ker(p B_{k-1,k} \cdots B_{12} i) \subset \ker(B_{k,k-1} B_{k-1,k} \cdots B_{12} i) = x^{-1}(F_{k-1}).$$

We have shown that every choice of subspace S corresponds to a flag F' satisfying the conditions in (5.4). It is easy to see that such a flag F' comes from a subspace S as follows. We have that $F_k \subset F'_k$. We take S to be the subspace of V_k such that

$$\ker(p B_{k-1,k} \cdots B_{12} i) = F'_k$$

for the projection $p : V_k \to V_k / S$. Thus we have proven the proposition in the special case $W = W_1$.

For the general case, we recall Maffei's construction in [5]. For general W, Maffei constructs a map $\Lambda(\mathbf{v}, \mathbf{w}) \to \Lambda(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$, denoted $(B, i, j) \mapsto (\tilde{B}, \tilde{i}, \tilde{j})$, where $\tilde{\mathbf{w}} = c\mathbf{e}^1$ for some $c \in \mathbb{Z}_{\geqslant 0}$. Thus, if we show that a choice of a B-stable subspace S such that $\dim S = \mathbf{e}^k$ corresponds to a choice of $\tilde{B}$-stable subspace $\tilde{S}$ such that $\dim \tilde{S} = \mathbf{e}^k$ then we reduce the proof to the special case considered above. Now,

$$\tilde{V}_k = V_k \oplus W'_k, \quad \text{where } W'_k = \bigoplus_{l,m:\ 1 \leqslant m \leqslant l-k,\ k+1 \leqslant l \leqslant n-1} W_l^{(m)},$$

and $W_l^{(m)}$ is an isomorphic copy of W_l. For $1 \leqslant m \leqslant l-k$ and $k+1 \leqslant l \leqslant n-1$, we have (see [5])

$$\operatorname{pr}_{W_l^{(m)}} \tilde{B}_{k,k-1}|_{W_l^{(m)}} = \mathrm{Id}_{W_l},$$
$$\operatorname{pr}_{W_l^{(m)}} \tilde{B}_{k,k-1}|_{V_k} = 0,$$

where $\operatorname{pr}_{W_l^{(m)}}$ denotes the projection onto the subspace $W_l^{(m)}$. In particular, $\ker \tilde{B}_{k,k-1} \subset V_k$. Thus, since the subspace $\tilde{S} \subset \tilde{V}_k$ must be contained in $\ker \tilde{B}_{k,k-1}$, it must lie in V_k. The result then follows from Remark 19 of [5]. □

We now compare the Lie algebra action in the two settings. By [2, Section 3.7.14], S_x is transverse to the orbit O_x in N. Thus, there is an open neighborhood $U \subset N$ of S such that

$$U \cong (O_x \cap U) \times S.$$

Let $\tilde{U}_{\mathbf{d}} = \mu_{\mathbf{d}}^{-1}(U)$ and $M'_{\mathbf{d}} = \mu_{\mathbf{d}}^{-1}(S_x) = \mu_{\mathbf{d}}^{-1}(S_{\mathbf{d},x}) = \tilde{S}_{\mathbf{d},x}$. Then $\tilde{U}_{\mathbf{d}} \subset M_{\mathbf{d}}$ is an open neighborhood of $M'_{\mathbf{d}}$. Let $D = O_x \cap U$, a small neighborhood of x in O_x. By [2, Corollary 3.2.21],

$$\tilde{U}_{\mathbf{d}} \cong (O_x \cap U) \times M'_{\mathbf{d}}.$$

Then the two commutative diagrams

$$\begin{array}{ccc} M'_{\mathbf{d}} & \longrightarrow & M_{\mathbf{d}} \\ \downarrow{\scriptstyle \mu_{\mathbf{d}}} & & \downarrow{\scriptstyle \mu_{\mathbf{d}}} \\ S_x & \longrightarrow & N \end{array} \quad \text{and} \quad \begin{array}{ccccc} M'_{\mathbf{d}} & \xrightarrow{p \mapsto (x,p)} & D \times M'_{\mathbf{d}} & \xrightarrow{\cong} & \tilde{U}_{\mathbf{d}} \\ \downarrow{\scriptstyle \mu_{\mathbf{d}}} & & \downarrow{\scriptstyle \mathbf{1}_D \times \mu_{\mathbf{d}}} & & \downarrow{\scriptstyle \mu_{\mathbf{d}}} \\ S_x & \xrightarrow{y \mapsto (x,y)} & D \times S_x & \xrightarrow{\cong} & U \end{array}$$

are isomorphic, where the horizontal arrows in the left diagram are given by the natural inclusions. If we let $\tilde{U} = \mu^{-1}(U)$ and $M' = \mu^{-1}(S_x)$ then $\tilde{U} = \bigsqcup_{\mathbf{d}} \tilde{U}_{\mathbf{d}}$, $M' = \bigsqcup_{\mathbf{d}} M'_{\mathbf{d}}$ and $\tilde{U} \subset M$ is an open neighborhood of M'. Thus we have that the two commutative diagrams

$$\begin{array}{ccc} M' & \longrightarrow & M \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ S_x & \longrightarrow & N \end{array} \quad \text{and} \quad \begin{array}{ccccc} M' & \xrightarrow{p \mapsto (x,p)} & D \times M' & \xrightarrow{\cong} & \tilde{U} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathbf{1}_D \times \mu} & & \downarrow{\scriptstyle \mu} \\ S_x & \xrightarrow{y \mapsto (x,y)} & D \times S_x & \xrightarrow{\cong} & U \end{array}$$

are isomorphic. Let $Z' = M' \times_{S_x} M'$. Then by Theorem 5.1,

$$Z' \cong \bigsqcup_{\mathbf{v}^1,\mathbf{v}^2} Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}).$$

We then have the commutative diagram

$$\begin{array}{ccc} Z' = M' \times_{S_x} M' & \xrightarrow{i} & M \times_N M = Z \\ \downarrow & & \downarrow \\ M' \times M' & \xrightarrow{i} & M \times M, \end{array} \tag{5.5}$$

where the maps are the obvious inclusions. Diagram (5.5) is isomorphic to

$$\begin{array}{ccc} Z' = M' \times_{S_x} M' & \xrightarrow{i=p\mapsto(x,p)} & D_\Delta \times (M' \times_{S_x} M') \cong Z \cap (\tilde{U} \times \tilde{U}) \\ j \big\downarrow & & \big\downarrow \Delta\times j \\ M' \times M' & \xrightarrow{i=p\mapsto((x,x),p)} & (D \times D) \times (M' \times M'), \end{array} \tag{5.6}$$

where $\Delta : D_\Delta \to D \times D$ is the embedding of the diagonal.

Lemma 5.4. *The inverse image in $M' \times M'$ of an irreducible component of the variety Z is either empty or else is an irreducible component of the variety Z'.*

Proof. Let X be a (closed) irreducible component of Z. If X does not intersect the open subset $\tilde{U} \times \tilde{U} \subset M \times M$, then $i^{-1}(X) = \emptyset$, since $i(Z') \subset \tilde{U} \times \tilde{U}$. Now assume that $X_U = X \cap (\tilde{U} \times \tilde{U})$ is non-empty. Then X_U is an irreducible component of $Z \cap (\tilde{U} \times \tilde{U})$. Thus it must be of the form $X_{\mathbf{U}} \cong D_\Delta \times X'$ where X' is an irreducible component of $M' \times_{S_x} M' = Z'$. We then have $i^{-1}(X) = X'$ and the result follows. □

The diagram (5.5) gives rise to a restriction with support morphism

$$i^* : H_*(Z) \to H_*(Z'), \qquad c \mapsto c \cap [M' \times M'].$$

By Lemma 5.4, i^* takes $H_{\text{top}}(Z)$ to $H_{\text{top}}(Z')$. Furthermore, by Proposition 5.3 we have that

$$i^*\big(\big[T^*_{Y_{\mathbf{a}_k^+,\mathbf{a}}}(\mathcal{F}_{\mathbf{a}_k^+} \times \mathcal{F}_{\mathbf{a}})\big]\big) = \big[(\theta \times \theta)\big(\mathfrak{B}_k(\mathbf{v}, \mathbf{w})\big)\big], \tag{5.7}$$

where $\mathbf{a} = \mathbf{a}(\mathbf{v}, \mathbf{w})$.

Now, $\mathcal{F}_x = \mu^{-1}(x)$ can be viewed as a subvariety of M' or M. If $i : M' \to M$ is the inclusion, then the restriction with supports morphism $i^* : H_{\text{top}}(\mathcal{F}_x) \to H_{\text{top}}(\mathcal{F}_x)$ is an isomorphism, where the first and second $H_{\text{top}}(\mathcal{F}_x)$ are $H^0(M, M \setminus \mathcal{F}_x)$ and $H^0(M', M' \setminus \mathcal{F}_x)$, respectively.

Theorem 5.5.

(1) *The morphism $i^* : H_{\text{top}}(Z) \to H_{\text{top}}(Z') \cong \bigoplus_{\mathbf{v}^1,\mathbf{v}^2} H_{\text{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ is an algebra homomorphism* (*with respect to the convolution product*).
(2) *The following diagram, where $x \in N$ is a nilpotent element of type $1^{w_1} 2^{w_2} \cdots (n-1)^{w_{n-1}}$ and whose vertical maps are given by convolution, commutes*:

$$\begin{array}{ccccc}
H_{\mathrm{top}}(Z) \otimes H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{i^* \otimes i^*} & H_{\mathrm{top}}(Z') \otimes H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{\cong} & \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{\mathrm{top}}\big(Z\big(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}\big)\big) \otimes \bigoplus_{\mathbf{v}} H_{\mathrm{top}}\big(\mathfrak{L}(\mathbf{v}, \mathbf{w})\big) \\
\downarrow & & \downarrow & & \downarrow \\
H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{i^*} & H_{\mathrm{top}}(\mathcal{F}_x) & \xrightarrow{\cong} & \bigoplus_{\mathbf{v}} H_{\mathrm{top}}\big(\mathfrak{L}(\mathbf{v}, \mathbf{w})\big)
\end{array}$$

Proof. Note that the two rightmost horizontal maps are the isomorphisms induced by the map θ of Theorem 5.1. We prove only the first part of the theorem. The second part in analogous. We have a sequence of embeddings

$$M' \times M' \hookrightarrow \tilde{U} \times \tilde{U} \hookrightarrow M \times M.$$

So i^* factors as

$$i^* : H_{\mathrm{top}}(Z) \to H_{\mathrm{top}}\big(Z \cap (\tilde{U} \times \tilde{U})\big) \to H_{\mathrm{top}}(Z').$$

The first map is the restriction to an open subset and thus commutes with convolution by base locality (cf. [2, Section 2.7.45]). The second map is induced by the embedding

$$Z' \hookrightarrow Z \cap (\tilde{U} \times \tilde{U}).$$

By the above results, this is isomorphic to the natural embedding

$$Z' \hookrightarrow D_\Delta \times Z', \qquad z \mapsto (x, z).$$

The corresponding map

$$i^* : H_{\mathrm{top}}(D_\Delta \times Z') \to H_{\mathrm{top}}(Z')$$

commutes with convolution by the Künneth formula for convolution (cf. [2, Section 2.7.16]). □

Corollary 5.6. *If $c \in H_{\mathrm{top}}(F_x)$ and c' is the corresponding class in $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ (under the isomorphism θ) then we have*

$$\begin{aligned} E_k^{\mathrm{Gin}} c &= E_k^{\mathrm{Nak}} c' \quad \text{and} \\ F_k^{\mathrm{Gin}} c &= F_k^{\mathrm{Nak}} c' \quad \text{for all } k. \end{aligned}$$

Here the superscripts Gin *and* Nak *correspond to the actions defined by Ginzburg and Nakajima, respectively.*

Proof. The result follows from (5.7) and the fact that since $\tilde{U}_{\mathbf{d}} \cong (O_x \cap U) \times M'_{\mathbf{d}}$ we have

$$\begin{aligned} \dim_{\mathbb{C}} M_{\mathbf{d}^1} - \dim_{\mathbb{C}} M_{\mathbf{d}^2} &= \big(\dim_{\mathbb{C}}(O_x \cap U) + \dim_{\mathbb{C}} M'_{\mathbf{d}^1}\big) - \big(\dim_{\mathbb{C}}(O_x \cap U) + \dim_{\mathbb{C}} M'_{\mathbf{d}^2}\big) \\ &= \dim_{\mathbb{C}} M'_{\mathbf{d}^1} - \dim_{\mathbb{C}} M'_{\mathbf{d}^2}. \end{aligned}$$

Thus the signs appearing in (3.2) and (4.2) are the same. □

We see from Corollary 5.6 that the Ginzburg and Nakajima constructions yield the same representations, with the same bases, given by the fundamental classes of the irreducible components of $\mathcal{F}_x \cong \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w})$. However, note that the corresponding quotients of the universal enveloping algebra constructed via convolution is different (compare Theorems 3.7 and 4.7). To see that these two quotients are indeed different, it suffices to consider the case of $\mathfrak{sl}_3$ with $\mathbf{w} = (1, 1)$ (so $\omega_{\mathbf{w}} = \omega_1 + \omega_2$ and $d = 3$). Then the weight $3\omega_1$ corresponds to a partition of d but is not a weight of $L(\omega_{\mathbf{w}})$ (since the tableau of shape (21) with all three entries equal to 1 is not semistandard).

6. Crystal structure on flag varieties

Kashiwara and Saito have introduced the structure of a crystal on the set of irreducible components of Nakajima's quiver varieties. In this section, we recall this construction and use the isomorphism of Section 5 to define a crystal structure on the flag varieties (or, more precisely, on the set of irreducible components of the Spaltenstein varieties $\mathcal{F}_x$). In this way we recover the crystal structure defined by Malkin (see [6]). In fact, Malkin and Nakajima have defined a tensor product quiver variety (see [7,11]). One would expect that the relationship between the two constructions examined in this paper could be extended to this setting and one would recover the tensor product crystal structure defined in [6]. However, we will restrict ourselves to the case of a single representation here.

We first review the realization of the crystal graph via quiver varieties. See [4,12] for proofs omitted here. Note that, as mentioned in Section 4, we are using a different stability condition and thus our definitions differ slightly from those in [4,12].

Let $\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{v}'' \in (\mathbb{Z}_{\geqslant 0})^I$ be such that $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$. Consider the maps

$$\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w}) \xleftarrow{p_1} \tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \xrightarrow{p_2} \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \xrightarrow{p_3} \Lambda(\mathbf{v}, \mathbf{w}), \tag{6.1}$$

where the notation is as follows. A point of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(B, i) \in \Lambda(\mathbf{v}, \mathbf{w})$ together with an I-graded, B-stable subspace S of V such that $\dim S = \mathbf{v}''$. A point of $\tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point (B, i, S) of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ together with a collection of isomorphisms $R''_k : V''_k \cong S_k$ and $R'_k : V'_k \cong V_k/S_k$ for each $k \in I$. Then we define $p_2(B, i, S, R', R'') = (B, i, S)$, $p_3(B, i, S) = (B, i)$ and $p_1(B, i, S, R', R'') = (B'', B', i')$ where B'', B', i' are determined by

$$R''_{\mathrm{in}(h)} B''_h = B_h R''_{\mathrm{out}(h)} : V''_{\mathrm{out}(h)} \to S_{\mathrm{in}(h)},$$
$$R'_k i'_k = \bar{i}_k : W_k \to V_k/S_k,$$
$$R'_{\mathrm{in}(h)} B'_h = B_h R'_{\mathrm{out}(h)} : V'_{\mathrm{out}(h)} \to V_{\mathrm{in}(h)}/S_{\mathrm{in}(h)},$$

where $\bar{i}_k$ denotes the composition of the map i_k with the canonical projection $V_k \to V_k/S_k$. It follows that B' and B'' are nilpotent.

Lemma 6.1. [8, Lemma 10.3] *One has*

$$(p_3 \circ p_2)^{-1}\big(\Lambda(\mathbf{v}, \mathbf{w})^s\big) \subset p_1^{-1}\big(\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w})^s\big).$$

Thus, we can restrict (6.1) to stable points, forget the $\Lambda(\mathbf{v}'', \mathbf{0})$-factor and consider the quotient by $G_{\mathbf{v}}$, $G_{\mathbf{v}'}$. This yields the diagram

$$\mathcal{L}(\mathbf{v}', \mathbf{w}) \xleftarrow{\pi_1} \mathcal{L}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \xrightarrow{\pi_2} \mathcal{L}(\mathbf{v}, \mathbf{w}), \tag{6.2}$$

where

$$\mathcal{L}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \stackrel{\text{def}}{=} \{(B, i, S) \in \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \mid (B, i) \in \Lambda(\mathbf{v}, \mathbf{w})^s\}/G_{\mathbf{v}}.$$

For $k \in I$ define $\varepsilon_k : \Lambda(\mathbf{v}, \mathbf{w}) \to \mathbb{Z}_{\geqslant 0}$ by

$$\varepsilon_k((B, i)) = \dim_{\mathbb{C}} \ker\left(V_k \xrightarrow{(B_h)} \bigoplus_{h:\ \mathrm{out}(h)=k} V_{\mathrm{in}(h)}\right).$$

Then, for $c \in \mathbb{Z}_{\geqslant 0}$, let

$$\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c} = \{[B, i] \in \mathcal{L}(\mathbf{v}, \mathbf{w}) \mid \varepsilon_k((B, i)) = c\},$$

where $[B, i]$ denotes the $G_{\mathbf{v}}$-orbit through the point (B, i). $\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c}$ is a locally closed subvariety of $\mathcal{L}(\mathbf{v}, \mathbf{w})$.

Assume $\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c} \neq \emptyset$ and let $\mathbf{v}' = \mathbf{v} - c\mathbf{e}^k$ where $\mathbf{e}^k_l = \delta_{kl}$. Then

$$\pi_1^{-1}(\mathcal{L}(\mathbf{v}', \mathbf{w})_{k,0}) = \pi_2^{-1}(\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c}).$$

Let

$$\mathcal{L}(\mathbf{v}, \mathbf{w}; c\mathbf{e}^k)_{k,0} = \pi_1^{-1}(\mathcal{L}(\mathbf{v}', \mathbf{w})_{k,0}) = \pi_2^{-1}(\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c}).$$

We then have the following diagram:

$$\mathcal{L}(\mathbf{v}', \mathbf{w})_{k,0} \xleftarrow{\pi_1} \mathcal{L}(\mathbf{v}, \mathbf{w}; c\mathbf{e}^k)_{k,0} \xrightarrow{\pi_2} \mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c}. \tag{6.3}$$

The restriction of π_2 to $\mathcal{L}(\mathbf{v}, \mathbf{w}; c\mathbf{e}^k)_{k,0}$ is an isomorphism since the only possible choice for the subspace S of V is to have $S_l = 0$ for $l \neq k$ and S_k equal to the intersection of the kernels of B_h with $\mathrm{out}(h) = k$. Also, $\mathcal{L}(\mathbf{v}', \mathbf{w})_{k,0}$ is an open subvariety of $\mathcal{L}(\mathbf{v}', \mathbf{w})$.

Lemma 6.2. [12]

(1) *For any $k \in I$,*

$$\mathcal{L}(\mathbf{0}, \mathbf{w})_{k,c} = \begin{cases} \mathrm{pt} & \text{if } c = 0, \\ \emptyset & \text{if } c > 0. \end{cases}$$

(2) *Suppose $\mathcal{L}(\mathbf{v}, \mathbf{w})_{k,c} \neq \emptyset$ and $\mathbf{v}' = \mathbf{v} - c\mathbf{e}^k$. Then the fiber of the restriction of π_1 to $\mathcal{L}(\mathbf{v}, \mathbf{w}; c\mathbf{e}^k)_{k,0}$ is isomorphic to a Grassmannian variety.*

Corollary 6.3. *Suppose $\mathcal{L}(\mathbf{v},\mathbf{w})_{k,c} \neq \emptyset$. Then there is a* 1–1 *correspondence between the set of irreducible components of $\mathcal{L}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0}$ and the set of irreducible components of $\mathcal{L}(\mathbf{v},\mathbf{w})_{k,c}$.*

Let $\mathcal{B}(\mathbf{v},\mathbf{w})$ denote the set of irreducible components of $\mathcal{L}(\mathbf{v},\mathbf{w})$ and let $\mathcal{B}(\mathbf{w}) = \bigsqcup_{\mathbf{v}} \mathcal{B}(\mathbf{v},\mathbf{w})$. For $X \in \mathcal{B}(\mathbf{v},\mathbf{w})$, let $\varepsilon_k(X) = \varepsilon_k((B,i))$ for a generic point $[B,i] \in X$. Then for $c \in \mathbb{Z}_{\geqslant 0}$ define

$$\mathcal{B}(\mathbf{v},\mathbf{w})_{k,c} = \{X \in \mathcal{B}(\mathbf{v},\mathbf{w}) \mid \varepsilon_k(X) = c\}.$$

Then by Corollary 6.3, $\mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0} \cong \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c}$.

Suppose that $\bar{X} \in \mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0}$ corresponds to $X \in \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c}$ by the above isomorphism. Then we define maps

$$\tilde{f}_k^c : \mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0} \to \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c}, \quad \tilde{f}_k^c(\bar{X}) = X,$$
$$\tilde{e}_k^c : \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c} \to \mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0}, \quad \tilde{e}_k^c(X) = \bar{X}.$$

We then define the maps

$$\tilde{e}_k, \tilde{f}_k : \mathcal{B}(\mathbf{w}) \to \mathcal{B}(\mathbf{w}) \sqcup \{0\}$$

by

$$\tilde{e}_k : \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c} \xrightarrow{\tilde{e}_k^c} \mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0} \xrightarrow{\tilde{f}_k^{c-1}} \mathcal{B}(\mathbf{v}-\mathbf{e}^k,\mathbf{w})_{k,c-1},$$
$$\tilde{f}_k : \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c} \xrightarrow{\tilde{e}_k^c} \mathcal{B}(\mathbf{v}-c\mathbf{e}^k,\mathbf{w})_{k,0} \xrightarrow{\tilde{f}_k^{c+1}} \mathcal{B}(\mathbf{v}+\mathbf{e}^k,\mathbf{w})_{k,c+1}.$$

We set $\tilde{e}_k(X) = 0$ for $X \in \mathcal{B}(\mathbf{v},\mathbf{w})_{k,0}$ and $\tilde{f}_k(X) = 0$ for $X \in \mathcal{B}(\mathbf{v},\mathbf{w})_{k,c}$ with $\mathcal{B}(\mathbf{v},\mathbf{w})_{k,c+1} = \emptyset$. We also define

$$\mathrm{wt} : \mathcal{B}(\mathbf{w}) \to P, \quad \mathrm{wt}(X) = \omega_{\mathbf{w}} - \alpha_{\mathbf{v}} \quad \text{for } X \in \mathcal{B}(\mathbf{v},\mathbf{w}),$$
$$\varphi_k(X) = \varepsilon_k(X) + \langle h_k, \mathrm{wt}(X) \rangle.$$

Proposition 6.4. [12] *$\mathcal{B}(\mathbf{w})$ is a crystal and is isomorphic to the crystal of the highest weight $U_q(\mathfrak{g})$-module with highest weight $\omega_{\mathbf{w}}$.*

We now translate this structure to the language of flag varieties. We need the following results. We adopt the convention that $B_{1,0} = 0$ and $B_{n-1,n} = 0$.

Proposition 6.5. *We have*

$$B_{k,k-1} \circ \phi_k = \phi_{k-1} \circ x.$$

Proof. Recall that

$$\phi_{k-1} = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k-1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)}$$
$$\Rightarrow \quad \phi_{k-1} \circ x = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k-1,\, \mathrm{ord}(p)\geqslant 1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)+1}$$

and

$$\phi_k = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)}$$
$$\Rightarrow \quad B_{k,k-1} \circ \phi_k = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k} B_{k,k-1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)}.$$

Now, since $j = 0$, $\mu(B, i, j) = 0$ implies that

$$B_{l-1,l} B_{l,l-1} = B_{l+1,l} B_{l,l+1} \quad \text{for } 2 \leqslant l \leqslant n-2,$$
$$B_{2,1} B_{1,2} = 0, \qquad B_{n-2,n-1} B_{n-1,n-2} = 0.$$

Using these equations, one can see that

$$\{B_{k,k-1} B_p \mid p \in \mathcal{P},\ \mathrm{in}(p) = k\} = \{B_p \mid p \in \mathcal{P},\ \mathrm{in}(p) = k-1,\ \mathrm{ord}(p) \geqslant 1\}.$$

Therefore,

$$B_{k,k-1} \circ \phi_k = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k-1,\, \mathrm{ord}(p)\geqslant 1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-(\mathrm{ord}(p)-1)} = \phi_{k-1} \circ x. \qquad \square$$

Proposition 6.6. *We have*

$$B_{k,k+1} \circ \phi_k = \phi_{k+1}|_{W^{\leqslant k}}.$$

Proof. Let $\mathcal{P}'$ be the subset of $\mathcal{P}$ consisting of those paths that contain at least one edge belonging to $\bar{\Omega}$. Then

$$B_{k,k+1} \circ \phi_k = \bigoplus_{p\in\mathcal{P},\, \mathrm{in}(p)=k} B_{k,k+1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)}$$
$$= \bigoplus_{p\in\mathcal{P}',\, \mathrm{in}(p)=k+1} B_p i_{\mathrm{out}(p)} \iota_{\mathrm{out}(p)}^{\mathrm{out}(p)-\mathrm{ord}(p)}$$
$$= \phi_{k+1}|_{W^{\leqslant k}}. \qquad \square$$

Proposition 6.7. *One has*

$$\phi_k^{-1}(\ker B_{k,k-1} \cap \ker B_{k,k+1}) = x^{-1}(F_{k-1}) \cap F_{k+1}. \tag{6.4}$$

Proof. Since $F_{k-1} \subset W^{\leqslant k-1}$, we have that $x^{-1}(F_{k-1}) \subset W^{\leqslant k}$. Thus, using Propositions 6.5 and 6.6,

$$\begin{aligned} x^{-1}(F_{k-1}) \cap F_{k+1} &= x^{-1}(F_{k-1}) \cap \ker(\phi_{k+1}) \\ &= x^{-1}(F_{k-1}) \cap \ker(\phi_{k+1}|_{W^{\leqslant k}}) \\ &= x^{-1}(\ker \phi_{k-1}) \cap \ker(\phi_{k+1}|_{W^{\leqslant k}}) \\ &= \ker(\phi_{k-1} \circ x) \cap \ker(\phi_{k+1}|_{W^{\leqslant k}}) \\ &= \ker(B_{k,k-1} \circ \phi_k) \cap \ker(B_{k,k+1} \circ \phi_k) \\ &= \phi_k^{-1}(\ker B_{k,k-1} \cap \ker B_{k,k+1}). \qquad \square \end{aligned}$$

Note that

$$\ker B_{k,k-1} \cap \ker B_{k,k+1} = \ker\left(V_k \xrightarrow{(B_h)} \bigoplus_{h:\ \mathrm{out}(h)=k} V_{\mathrm{in}(h)} \right), \tag{6.5}$$

and that a collection of subspaces $S_l \subset V_l$ such that $S_l = 0$ for $l \neq k$ is B-stable if and only if S_k is contained in the right-hand side of Eq. (6.5). Thus, the flag variety analogue of the diagram (6.3) (for $\mathbf{v} - \mathbf{v}' = c\mathbf{e}^k$) is

$$\mathcal{F}_{\mathbf{a}^{k,c},x} \xleftarrow{\pi_1} \mathcal{F}_{\mathbf{a},x}(k,c) \xrightarrow{\pi_2} \mathcal{F}_{\mathbf{a},x},$$

where $\mathbf{a} = \mathbf{a}(\mathbf{v},\mathbf{w}) = (a_1, \dots, a_n)$ and $\mathbf{a}^{k,c} = (a_1, \dots, a_{k-1}, a_k + c, a_{k+1} - c, a_{k+2}, \dots, a_n)$, and

$$\mathcal{F}_{\mathbf{a},x}(k,c) = \left\{ (F,S) \;\middle|\; F \in \mathcal{F}_{\mathbf{a},x},\ F_k \subset S \subset F_{k+1} \cap x^{-1}(F_{k-1}),\ \dim S/F_k = c \right\}.$$

In particular,

$$\mathcal{L}\left(\mathbf{v},\mathbf{w}; c\mathbf{e}^k\right) \cong \mathcal{F}_{\mathbf{a},x}(k,c).$$

Let $\mathcal{B}(\mathbf{a},x)$ denote the set of irreducible components of $\mathcal{F}_{\mathbf{a},x}$ and let $\mathcal{B}(x) = \bigsqcup_{\mathbf{a}} \mathcal{F}_{\mathbf{a},x}$. Let

$$\varepsilon_k(F) = \dim\left(F_{k+1} \cap x^{-1}(F_{k-1})\right) - \dim F_k,$$

and for $X \in \mathcal{B}(\mathbf{a},x)$ define $\varepsilon_k(X) = \varepsilon_k(F)$ for a generic flag $F \in X$. Then for $c \in \mathbb{Z}_{\geqslant 0}$ define

$$\mathcal{B}(\mathbf{a},x)_{k,c} = \left\{ X \in \mathcal{B}(\mathbf{a},x) \;\middle|\; \varepsilon_k(X) = c \right\}.$$

Then just as for quiver varieties, we have $\mathcal{B}(\mathbf{a}^{k,c},x)_{k,0} \cong \mathcal{B}(\mathbf{a},x)_{k,c}$ and we define $\tilde{f}_k$ and $\tilde{e}_k$ just as before. We also define

$$\mathrm{wt}(X) : \mathcal{B}(x) \to P, \quad \mathrm{wt}(X) = \sum_{k \in I} a_k \epsilon_k \quad \text{for } X \in \mathcal{B}(\mathbf{a},x),$$
$$\varphi_k(X) = \varepsilon_k(X) + \left\langle h_k, \mathrm{wt}(X) \right\rangle.$$

Then, by translating Proposition 6.4 into the language of flag varieties, we have the following theorem.

Theorem 6.8. $\mathcal{B}(x)$ *is a crystal and is isomorphic to the crystal of the highest weight* $U_q(\mathfrak{sl}_n)$*-module with highest weight* $w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1}$ *where* ω_i *are the fundamental weights of* $\mathfrak{sl}_n$ *and* w_i *is the number of* $(i \times i)$*-Jordan blocks in the Jordan normal form of* x.

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