3. Proof of theorem 1

In this section the inversion formulas (0.8) and (0.10) will be proved respectively. The formula (0.8) is equivalent to

\[
\lim_{\mu \to \infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)} y_2(x, \lambda) d\lambda \int_{0}^{\infty} f(t) y_1(t, \lambda) dt = \frac{1}{2} \{ f(x-0) + f(x+0) \}.
\]

To prove (3.1) we consider two cases:
I. \( f(t) \equiv 0 \) if \( 0 < t < x \),
II. \( f(t) \equiv 0 \) if \( t > x \).

Case I is contained in lemma 1, because we may specify

\[
\varphi(\lambda) = (2\pi)^{\frac{1}{2}} \frac{\lambda}{W_{12}(\lambda)} y_2(x, \lambda)
\]

in view of (1.31) and (1.32).

To treat case II we remark that the function \( K(x, t; \lambda) \) satisfying (0.2) as a function of \( x \) with initial conditions

\[
K(t, t; \lambda) = 0, \frac{\partial}{\partial x} K(x, t; \lambda)|_{x=t} = 1,
\]

is equal to

\[
K(x, t; \lambda) = \frac{y^*(x, \lambda)y(t, \lambda) - y(x, \lambda)y^*(t, \lambda)}{W(y, y^*; \lambda)}
\]

for any pair \( y \) and \( y^* \) of linearly independent solutions of (0.2), where \( W(y, y^*; \lambda) \) is the Wronskian of \( y \) and \( y^* \). This means that case II is equivalent to

\[
\int_{0}^{\infty} f(t) y_2(t, \lambda) dt = \frac{1}{2\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)} y_1(x, \lambda) d\lambda \int_{0}^{\infty} f(t) y_1(t, \lambda) dt \]

\[
+ \lim_{\mu \to \infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_{0}^{\infty} f(t) K(x, t; \lambda) dt = \frac{1}{2\pi i} f(x-0).
\]
In view of (1.23) and (1.32) we may apply lemma 2 with
\[ \varphi(\lambda) = \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \frac{\lambda}{W_{12}(\lambda)} y_2(x, \lambda) \]
to the first limit in (3.3), because (0.7) implies (2.18).

Hence it remains to show that
\[ \lim_{\mu \to \infty} \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} f(t) K(x, t; \lambda) \, dt = 0. \tag{3.4} \]

First consider the integral in the \( \lambda \)-plane
\[ \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} \lambda K(x, t; \lambda) \, d\lambda = \int_{-\mu}^{\mu} + \int_{-\mu}^{-\mu} \lambda_1 + i\mu \lambda_1 - i\mu \lambda_1 + i\mu \lambda_1 - i\mu \]
\[ = -I_1(t, \mu) + I_2(t, \mu) + I_3(t, \mu). \tag{3.5} \]

On the line \( \text{Re} \lambda = 0 \) the function \( K(x, t; \lambda) \) is an even function of \( \lambda \), because as a solution of the differential equation (0.2) with the initial conditions (3.2) it is an entire function of \( \lambda^2 \). Hence \( I_1(t, \mu) \) vanishes.

For \( I_2(t, \mu) \) and \( I_3(t, \mu) \) we use
\[ \lambda K(x, t; \lambda) = \frac{y_1(x, \lambda) y_2(t, \lambda) - y_2(x, \lambda) y_1(t, \lambda)}{W_{12}(\lambda)}. \tag{3.6} \]

From (1.23), (1.31) and (1.32) we deduce
\[ K(x, t; \lambda) = -i t^{\frac{1}{2}} e^{i(x-\lambda t)} \left( 1 + O\left( \frac{1}{\mu} \right) + O\left( \frac{1}{\mu^t} \right) \right) \]
\[ + \frac{1}{2} e^{i(x-\lambda t)} \left( 1 + O\left( \frac{1}{\mu} \right) + O\left( \frac{1}{\mu^t} \right) \right) + e^{-\lambda t \pm x \mu + i x \mu} \left( O\left( \frac{1}{\mu} \right) + O\left( \frac{1}{\mu^t} \right) \right) \tag{3.7} \]
as \( \mu t \to \infty, \mu \to \infty \) uniformly in \( t \) on \( 0 < t \leq x \), uniformly in \( \lambda \) on the line segments joining \(-i\mu \) with \( \lambda_1 - i\mu \) and \( i\mu \) with \( \lambda_1 + i\mu \); the upper or lower sign in the exponential has to be taken as \( \text{Im} \lambda \geq 0 \). Hence
\[ I_2(t, \mu) + I_3(t, \mu) = \frac{1}{\mu} \left[ \cos \left( \mu + i \lambda_1 \right)(t - x) - \cos \left( \mu - i \lambda_1 \right)(t - x) \right] \]
\[ + e^{i(x-\lambda t)} \left( O\left( \frac{1}{\mu} \right) + O\left( \frac{1}{\mu^t} \right) \right) \tag{3.8} \]
as \( \mu \to \infty, \mu t \to \infty \) uniformly in \( t \) on \( 0 < t \leq x \). From this, the lemma of Riemann-Lebesgue, (3.5) and (0.7) we obtain
\[ \lim_{\mu \to \infty} \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} f(t) dt \int_{-\mu}^{\mu} \lambda K(x, t; \lambda) \, d\lambda = 0. \tag{3.9} \]

From the formulas (1.14), (1.13) and (1.3) we obtain in case \( \nu \neq 0 \)
\[ y_1(t, \lambda) = O((\lambda t)^{\frac{1}{2} + \nu}) + O((\lambda t)^{\frac{1}{2} - \nu}) + O\left( \frac{1}{\lambda} \right) \tag{3.10} \]
as $\lambda t \to 0$, $\lambda \to \infty$ on $\Re \lambda \geq 0$, uniformly in $t$ on $t > 0$. If $\nu = 0$ the first two $O$-functions in (3.10) should be replaced by $O((\lambda t)^{\frac{1}{2}} \log (\lambda t))$. On account of (1.23), (1.31), (1.33), (3.10) and (1.32) we have

\begin{equation}
\lambda K(x, t; \lambda) = O((\lambda t)^{\frac{1}{2}}) + O((\lambda t)^{\frac{1}{2}}) + O(\lambda t) + O\left(\frac{1}{\lambda}\right)
\end{equation}

as $\lambda t \to 0$, uniformly in $t$ on $0 \leq t \leq x$, uniformly in $\lambda$ on the line segments joining $-i\mu$ with $\lambda_1 - i\mu$ and $i\mu$ with $\lambda_1 + i\mu$. Hence if $\nu \neq 0$:

\begin{equation}
I_2(t, \mu) + I_3(t, \mu) = O((\lambda t)^{\frac{1}{2}}) + O((\lambda t)^{\frac{1}{2}}) + O\left(\frac{1}{\lambda}\right)
\end{equation}

as $\lambda t \to 0$, $\mu \to \infty$ uniformly in $t$ on $0 \leq t \leq x$. In case $\nu = 0$ the terms $O((\lambda t)^{\frac{1}{2}}) + O((\lambda t)^{\frac{1}{2}})$ in (3.12) have to be replaced by $O((\lambda t)^{\frac{1}{2}} \log (\lambda t))$. From this, (3.5) and (0.7) we obtain

\begin{equation}
\lim_{\mu \to \infty} \int_0^{\frac{\pi}{\mu}} f(t) dt \int_{\lambda \to \lambda_1 - i\mu}^{\lambda_1 + i\mu} \lambda K(x, t; \lambda) d\lambda = 0
\end{equation}

for all $\nu$ satisfying (0.4). Combining this with (3.9) gives

\begin{equation}
\lim_{\mu \to \infty} \int_0^{\frac{\pi}{\mu}} f(t) dt \int_{\lambda \to \lambda_1 - i\mu}^{\lambda_1 + i\mu} \lambda K(x, t; \lambda) d\lambda = 0.
\end{equation}

Changing the order of integration in (3.14) is justified by (3.6), (1.14), (1.17) and (0.7); which shows (3.4). This completes the proof of formula (0.8).

The formula (0.10) is equivalent to

\begin{equation}
\lim_{\mu \to \infty} \int_0^{\frac{\pi}{\mu}} f(t) y_1(x, \lambda) d\lambda \int_0^{\infty} f(t) y_2(t, \lambda) dt
\end{equation}

\begin{equation}
= \frac{1}{2}(f(x-0) + f(x+0)).
\end{equation}

In order to prove (3.15) under the conditions of theorem 1, we consider again the two cases, where $f(t)$ vanishes identically at the left or at the right of $t = x$ respectively. Now case II is contained in lemma 2, if we observe that (0.9) implies (2.18). Hence it remains to show that (3.15) holds in case I or equivalently

\begin{equation}
\lim_{\mu \to \infty} \int_{\lambda \to \lambda_1 - i\mu}^{\lambda_1 + i\mu} \frac{\lambda}{W_{12} (\lambda)} y_1(x, \lambda) d\lambda \int_0^{\infty} f(t) y_2(t, \lambda) dt
\end{equation}

\begin{equation}
- \lim_{\mu \to \infty} \int_{\lambda \to \lambda_1 - i\mu}^{\lambda_1 + i\mu} \lambda d\lambda \int_0^{\infty} f(t) K(x, t; \lambda) dt = \frac{1}{2} \pi i f(x + 0).
\end{equation}

Functions which satisfy (0.9) will certainly satisfy (2.1) and this means that lemma 1 can be applied to the first term in (3.16). So we have to show

\begin{equation}
\lim_{\mu \to \infty} \int_{\lambda \to \lambda_1 - i\mu}^{\lambda_1 + i\mu} \lambda d\lambda \int_0^{\infty} f(t) K(x, t; \lambda) dt = 0.
\end{equation}
The rest of the proof is very similar to the treatment of (3.4). We assume (3.5), where \( I_1(t, \mu) \) vanishes. To treat \( I_2(t, \mu) \) and \( I_3(t, \mu) \) we use (3.6) again, but now (3.7) takes the form

\[
\lambda K(x, t; \lambda) = -\frac{1}{2} e^{\lambda x - 0} \left( 1 + O \left( \frac{1}{\mu} \right) \right) - \frac{1}{2} e^{\lambda (t - x)} \left( 1 + O \left( \frac{1}{\mu} \right) \right)
\]

\[
\cdots + e^{-\lambda x + 0} \pm (n + \frac{1}{2}) \eta O \left( \frac{1}{\mu} \right)
\]

as \( \mu \to \infty \), uniformly in \( t \) on \( t \geq x \), uniformly in \( \lambda \) on the line segments joining \( -i\mu \) with \( \lambda_1 - i\mu \) and \( i\mu \) with \( \lambda_1 + i\mu \); the upper or lower sign in the exponential has to be taken as \( \text{Im} \lambda \lesssim 0 \). This can be seen from (1.14), (1.17) and (1.32). Hence

\[
I_2(t, \mu) + I_3(t, \mu) = \frac{1}{x - t} \left[ \cos \left( \mu + i\lambda_1 \right)(t - x) - \cos \left( \mu - i\lambda_1 \right)(t - x) \right]
\]

\[
\cdots + e^{i\mu x - 0} O \left( \frac{1}{\mu} \right)
\]

as \( \mu \to \infty \), uniformly in \( t \) on \( t \geq x \). Using the lemma of Riemann-Lebesgue and (0.9) it follows that

\[
\lim_{\mu \to \infty} \int_{\lambda_1 = \pm i\mu} \int_{\lambda_1 = \pm i\mu} \lambda K(x, t; \lambda) \, d\lambda = 0.
\]

Changing the order of integration in (3.20) is justified by (3.6), (1.14), (1.17) and (0.9); this shows (3.17). This completes the proof of formula (0.10).

4. Some applications

Example A. The simplest application of the preceding theory may be obtained, when we consider the differential equation

\[
\frac{d^2 y}{dx^2} - 2y = 0, \quad x > 0.
\]

Here we may take \( e(x, \lambda) = e^{-\lambda x} \), while we find \( e(x, \lambda) = \cosh \lambda x \) or \( e(x, \lambda) = -\frac{1}{\lambda} \sinh \lambda x \), depending on the initial conditions (0.11) or (0.12) respectively. Applying theorem 1 in these two cases and adding the results, we obtain a complex inversion formula for transforms related to (4.1).

Theorem 2. Let the conditions on \( x, \lambda_1 \) and \( f(t) \) of theorem 1 be satisfied. If (0.7) holds, then

\[
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{\lambda_1 = \pm i\mu} \lambda K(x, t; \lambda) \, d\lambda = \frac{1}{2} \{ f(x - 0) + f(x + 0) \}.
\]
If (0.9) holds, then

\[
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{\lambda_i - i\mu}^{\lambda_i + i\mu} e^{-kz} \lambda \int_0^\infty f(t)e^{t\lambda} dt = \frac{1}{2} \{ f(x-0) + f(x+0) \}.
\]

The inversion formula (4.2) for the one-sided Laplace transform can be found in Widder [16, Ch. II, Theorem 7.3] under somewhat more general conditions on \(x\) and \(\lambda_i\).

Example B. Taking \(q(x) = 0\) in (0.2) we obtain the differential equation (1.1) for the modified Bessel functions. In one of the foregoing sections their properties have been shown and the following theorem is immediate.

**Theorem 3.** Let the conditions on \(x\), \(\lambda_1\) and \(f(t)\) of theorem 1 be satisfied and let \(-\frac{1}{2} \leq \Re v \leq \frac{1}{2} \).

If (0.7) holds, then

\[
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{\lambda_i - i\mu}^{\lambda_i + i\mu} (\lambda x)^{\frac{1}{2}} I_v(\lambda x) d\lambda \int_0^\infty f(t)(\lambda t)^{\frac{1}{2}} K_v(\lambda t) dt = \frac{1}{2} \{ f(x-0) + f(x+0) \}.
\]

If (0.9) holds, then

\[
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{\lambda_i - i\mu}^{\lambda_i + i\mu} (\lambda x)^{\frac{1}{2}} K_v(\lambda x) d\lambda \int_0^\infty f(t)(\lambda t)^{\frac{1}{2}} I_v(\lambda t) dt = \frac{1}{2} \{ f(x-0) + f(x+0) \}.
\]

The formula (4.4) is essentially due to Meyer [11, Satz 1], while (4.5) seems to be new, although Koh and Zemanian [9] have proved a formula similar to (4.5) for certain generalized functions.

Example C. We now turn to integral transforms with a hypergeometric function as kernel. The theory of this type of transform acting on \(\Omega_2(0, \infty)\) functions has been given by Titchmarsh [14, section 4.18], while we will treat this transform under \(\Omega_1\) conditions. Therefore we consider the hypergeometric differential equation (see [14]):

\[
X(X + 1) \frac{d^2 Y}{dX^2} + \{ y + (x + 1)X \} \frac{dY}{dX} - (2x^2 - \frac{1}{4}x^2) Y = 0, \quad X > 0,
\]

which has solutions

\[
Y_1 = X^{-a} F \left( a, 1 + a - c; 1 + a - b; -\frac{1}{X} \right), \quad Y_2 = F(a, b; c; -X)
\]

where \(a = \lambda + \frac{1}{2}x\), \(b = -\lambda + \frac{1}{2}x\) and \(c = y\). Putting

\[
X = \sinh^2 \frac{1}{2}x, \quad Y = r(x)y,
\]
then the equation (4.6) becomes

\[
\frac{d^2 y}{dx^2} - \left\{ \lambda^2 + \frac{(\gamma - 1)^2 - \frac{1}{4}}{x^2} + q(x) \right\} y = 0
\]

where

\[
q(x) = \frac{2(x - 1)(2\gamma - x - 1) \cosh x + 2x^2 - 4\gamma x \cdot (1 - 2\gamma)^2 \cdot (\gamma - 1)^2 - \frac{1}{4}}{4 \sinh^2 x}
\]

and

\[
r(x) = \left( \frac{\cosh x}{e^x + 1} \right)^{1 + \frac{1}{2} - \gamma} \sinh^{-\frac{1}{2} - \gamma} x.
\]

The function \(q(x)\) thus obtained is continuous for positive \(x\) and \(q(x) \in L_1(0, \infty)\). The solution \(y_1\) corresponding to \(Y_1\) has an asymptotic behaviour

\[
y_1(x, \lambda) \sim A e^{-\lambda x} \text{ as } x \to \infty,
\]

while the function \(y_2\), corresponding to \(Y_2\) behaves as

\[
y_2(x, \lambda) \sim B x^{-\gamma} \text{ as } x \downarrow 0,
\]

where \(A\) and \(B\) are parameters, not depending on \(x\). Thus if \(\frac{1}{2} \leq \text{Re } \gamma \leq \frac{3}{2}\) the solutions \(y_1(x, \lambda)\) and \(y_2(x, \lambda)\) of (4.7) satisfy the conditions (0.5) and (0.6) respectively. Using some well-known properties of the hypergeometric function (cf. Erdélyi [3, 2.10 (2)]) we find for the Wronskian of \(y_1\) and \(y_2\):

\[
W_x[y_1(x, \lambda), y_2(x, \lambda)] = \frac{1}{\{r(x)\}^2} W_x[\sinh^{-2a} \frac{1}{2} x F(a, 1 + a - c; 1 + a - b; - \sinh^{-2} x),
\]

\[
F(a, b; c; - \sinh^2 x)]
\]

\[
= \frac{1}{\{r(x)\}^2} W_x [X^{-a} F\left(a, 1 + a - c; 1 + a - b; - \frac{1}{X}\right), F(a, b; c; - X)]
\]

\[
= \frac{1}{\{r(x)\}^2} \left[ X(X + 1) \right]^{\frac{1}{2}} W_x \left[ X^{-a} F\left(a, 1 + a - c; 1 + a - b; - \frac{1}{X}\right),
\right]
\]

\[
X^{-a} \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(c - a) \Gamma(b)} F\left(a, 1 + a - c; 1 + a - b; - \frac{1}{X}\right) + X^{-b} \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(c - b) \Gamma(a)}
\]

\[
\cdot F\left(b, 1 + b - c; 1 + b - a; - \frac{1}{X}\right)
\]

\[
= \frac{1}{\{r(x)\}^2} \left[ X(X + 1) \right]^{\frac{1}{2}} \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(c - b) \Gamma(a)}
\]

\[
W_x \left[ X^{-a} F\left(a, 1 + a - c; 1 + a - b; - \frac{1}{X}\right), X^{-b} F\left(b, 1 + b - c; 1 + b - a; - \frac{1}{X}\right)\right]
\]

\[
\sim \frac{1}{\{r(x)\}^2} \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(c - b) \Gamma(a)} (a - b) X^{-a - b} \sim 2^{a + b} \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} (a - b)
\]
as $X \to \infty$. Since $W_x[y_1(x, \lambda), y_2(x, \lambda)]$ is independent of $x$, we obtain
\[
W_x[y_1(x, \lambda), y_2(x, \lambda)] = 2^{a+b} \frac{I(c)I(a-b)}{I(a)I(c-b)}(a-b) = 2^{a+1} \frac{I(\gamma)I(2\lambda)}{I(\lambda + \frac{1}{2} \lambda)I(\lambda - \frac{1}{2} \lambda + \gamma)}.
\]
We are now ready to formulate our theorem.

**Theorem 4.** Let the conditions on $x$, $\lambda_1$ and $f(t)$ of theorem 1 be satisfied and let $\frac{1}{2} \leq \text{Re} \gamma \leq \frac{3}{2}$. If (0.7) holds, then
\[
\lim_{\mu \to \infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\Gamma(\lambda+\frac{1}{2} \lambda) \Gamma(\lambda+\gamma-\frac{1}{2} \lambda)}{\Gamma(2\lambda)} y_2(x, \lambda) d\lambda \int_0^\infty f(t) y_1(t, \lambda) dt = 2^{a} \frac{\Gamma(\gamma)}{\Gamma(2\lambda)} \{f(x-0)+f(x+0)}.
\]
If (0.9) holds, then
\[
\lim_{\mu \to \infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\Gamma(\lambda+\frac{1}{2} \lambda) \Gamma(\lambda+\gamma-\frac{1}{2} \lambda)}{\Gamma(2\lambda)} y_1(x, \lambda) d\lambda \int_0^\infty f(t) y_2(t, \lambda) dt = 2^{a} \frac{\Gamma(\gamma)}{\Gamma(2\lambda)} \{f(x-0)+f(x+0)}.
\]

**Example D.** Our last example deals with the generalized differential equation of Legendre
\[
(4.10) \quad (1-u^2) \frac{d^2w}{du^2} - 2u \frac{dw}{du} + \left\{k(k+1) - \frac{m^2}{2(1-u)} - \frac{n^2}{2(1+u)} \right\} w = 0, \quad u > 1,
\]
of which linearly independent solutions $P_{k,m,n}(u)$ and $Q_{k,m,n}(u)$ have been defined by Kuipers and Meulenbeld [6]. After the substitution $u = \cosh x$ and some transformations (4.10) becomes
\[
(4.11) \quad \frac{d^2y}{dx^2} - \left\{(k+\frac{1}{2})^2 + \frac{m^2 - \frac{1}{4}}{x^2} + q(x) \right\} y = 0,
\]
where
\[
q(x) = \frac{2(m^2-n^2)}{4 \sinh^2 x} \cosh x + \frac{2(m^2+n^2)}{x^2} + q(x) - \frac{m^2 - \frac{1}{4}}{x^2},
\]
and
\[
y(x) = (\sinh x)^{\frac{1}{2}} w(\cosh x).
\]
Hence the function $q(x)$ is continuous for positive values of $x$ and $q(x) \in L(0, \infty)$. The equation (4.11) is satisfied by $(\sinh x)^t P_{k,m,n}(\cosh x)$ and $(\sinh x)^t Q_{k,m,n}(\cosh x)$. From Kuipers and Meulenbeld [8, (1) and (7)] it can be seen that
\[
e^{-it} \sinh x)^{\frac{1}{2}} Q_{k,m,n}(\cosh x) \sim Ae^{-(k+t)x} \text{ as } x \to \infty,
\]
and
\[
(\sinh x)^{\frac{1}{2}} P_{k,m,n}(\cosh x) \sim Bx^{k-m} \text{ as } x \downarrow 0,
\]
where $A$ and $B$ parameters, not depending on $x$; cf. (0.5) and (0.6).
In order to find the Wronskian of the solutions we refer to Meulenberg [10, (8)]:

\[
W_\pi[ P_k^{m,n}(z), Q_k^{m,n}(z) ] =
\frac{\Gamma \left( k + \frac{m+n}{2} + 1 \right) \Gamma \left( k - \frac{m-n}{2} + 1 \right)}{\Gamma \left( k - \frac{m+n}{2} + 1 \right) \Gamma \left( k - \frac{m-n}{2} + 1 \right)} \frac{1}{1 - z^2}
\]

(z not lying on the cut \((-\infty, 1]\))

and to Kuppers and Meulenberg [7, (6)]:

\[
e^{\pi i m} Q_k^{-m,-n}(z) = 2^{m-n} \frac{\Gamma \left( k - \frac{m+n}{2} + 1 \right) \Gamma \left( k + \frac{m-n}{2} + 1 \right)}{\Gamma \left( k + \frac{m+n}{2} + 1 \right) \Gamma \left( k + \frac{m-n}{2} + 1 \right)} e^{-\pi i n} P_k^{m,n}(z).
\]

Hence we conclude

\[
W_\pi \left[ (\sinh x)^{\frac{1}{2}} e^{\pi i m} Q_k^{-m,-n}(\cosh x), (\sinh x)^{\frac{1}{2}} P_k^{m,n}(\cosh x) \right] = 1.
\]

This enables us to formulate the next theorem.

**Theorem 5.** Let \(x\) and \(k_1\) be real numbers with \(x > 0\) and \(k_1 > -\frac{1}{2}\). Let \(f(t)\) be a function, defined for positive values of \(t\) and of bounded variation in a neighbourhood of \(t = x\). Let \(-\frac{1}{2} \leq \text{Re} \, m \leq \frac{1}{2}\).

If

\[
e^{-(k_1+i)t} f(t) \in \mathcal{L}_1(0, \infty),
\]

then

\[
\left\{ \begin{aligned}
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1)(\sinh x)^{\frac{1}{2}} P_k^{m,n}(\cosh x) \, dk \\
\int_0^\infty f(t) e^{\pi i m} (\sinh t)^{\frac{1}{2}} Q_k^{-m,-n}(\cosh t) \, dt = \frac{1}{2} \{ f(x - 0) + f(x + 0) \}.
\end{aligned} \right.
\]

If

\[
e^{(k_1+i)t} f(t) \in \mathcal{L}_1(0, \infty),
\]

then

\[
\left\{ \begin{aligned}
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1) e^{\pi i m} (\sinh x)^{\frac{1}{2}} Q_k^{-m,-n}(\cosh x) \, dk \\
\int_0^\infty f(t) (\sinh t)^{\frac{1}{2}} P_k^{m,n}(\cosh t) \, dt = \frac{1}{2} \{ f(x - 0) + f(x + 0) \}.
\end{aligned} \right.
\]
Restating this result in another way, we obtain a theorem which has been found by Braaksma and Meulenbeld [1, Theorem 1] under somewhat different conditions 1).

**Theorem 5a.** Let $x$ and $k_1$ be real numbers with $x > 1$ and $k_1 > -\frac{1}{2}$. Let $f(t)$ be a function, defined for $t > 1$ and of bounded variation in a neighbourhood of $t = x$. Let $-\frac{1}{2} \leq \text{Re } m \leq \frac{1}{2}$.

If

\begin{equation}
(t - 1)^{-1} f(t) \in \mathcal{L}_1(1, a), t^{-k_1-1} f(t) \in \mathcal{L}_1(a, \infty) \text{ with } a > 1,
\end{equation}

then

\begin{equation}
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{k_i-\mu}^{k_i+\mu} (2k + 1) P_{k}^{m,n}(x) \, dk \int_{1}^{\infty} f(t) e^{-\imath m} Q_{k}^{m,-n}(t) \, dt = \frac{1}{2} \{f(x - 0) + f(x + 0)\}.
\end{equation}

If

\begin{equation}
(t - 1)^{-1} f(t) \in \mathcal{L}_1(1, a), t^{k_1} f(t) \in \mathcal{L}_1(a, \infty) \text{ with } a > 1,
\end{equation}

then

\begin{equation}
\lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{k_i-\mu}^{k_i+\mu} (2k + 1) e^{-\imath m} Q_{k}^{m,-n}(x) \, dk \int_{1}^{\infty} f(t) P_{k}^{m,n}(t) \, dt = \frac{1}{2} \{f(x - 0) + f(x + 0)\}.
\end{equation}

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**REFERENCES**


1) In their statement of the formula (4.19) the second condition in (4.18) should be added to ensure convergence of the integral with respect to $t$. 

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