Discrete Mathematics 36 (1981) 325–326
North-Holland Publishing Company

NOTE

ON COMPLEMENTS IN LATTICES OF FINITE LENGTH

Anders BJÖRNER

Institut Mittag-Leffler, Auravägen 17, S-18262 Djursholm, Sweden

Received 25 September 1979

Let L be a lattice of finite length. It is shown that if L contains no 3-element interval then L is relatively complemented. Also, if for every $x \in L$ the set $C_x = \{y \in L \mid y \text{ covers } x\}$ satisfies $\bigvee C_x = 1$, then L is complemented.

A relatively complemented lattice is defined to be a lattice in which every interval is complemented. In this note we show that a relatively complemented lattice of finite length is in fact characterized by the exclusion of only one interval: the 3-element chain. We also discuss some connections between the existence of complements in a lattice of finite length and the presence of intervals having the property that the greatest element is the join of the atoms.

For the general theory and terminology of lattices we refer to Birkhoff [1]. To avoid trivialities we only consider lattices and intervals of length at least two. By interval we mean closed interval

$$[x, y] = \{z \in L \mid x \leq z \leq y\}.$$

We call an interval $[x, y]$ of a lattice L of finite length $\vee$-*regular if* y is the join of the atoms of $[x, y]$. Dually, $[x, y]$ will be called $\wedge$-*regular* if x is the meet of the coatoms of $[x, y]$. An interval $[x, y]$ is called *upper* if $y = 1$ and *lower* if $x = 0$.

It is easy to see that in a complemented lattice of finite length the interval $[0, 1]$ is $\vee$-regular. The example of the hexagon lattice shows that $[0, 1]$ may be the only interval of a complemented lattice which is $\vee$-regular. In the other direction we have the following result, cf. [3].

Theorem 1. *Let L be a lattice of finite length such that all upper intervals are* $\vee$-*regular. Then L is complemented.*

Proof. The conclusion is clearly true if L has length two. We will use induction on the length of L. Assume that the theorem holds for all lattices of length less than n and that $l(L) = n$, $n \geq 3$. Take any element $x \in L - \{0, 1\}$. There must be an atom a such that $a \nleq x$, since if all atoms of L were below x then the assumption that

$[0, 1]$ is $\vee$-regular would imply $1 \leqslant x$. Choose such an atom a. All upper intervals of the sublattice $[a, 1]$ are $\vee$-regular and $l([a, 1]) < n$. Hence, by the induction assumption $[a, 1]$ is complemented. Since a is an atom and $a \nleqslant x$, we know that $a \wedge x = 0$. We also know that $a \vee x$ must have a complement in $[a, 1]$, say c. Now,

$$c \vee x = (c \vee a) \vee x = c \vee (a \vee x) = 1$$

and

$$c \wedge x = c \wedge ((a \vee x) \wedge x) = (c \wedge (a \vee x)) \wedge x = a \wedge x = 0.$$

Hence, c is a complement to x in L. □

It is well-known that the family of all upper intervals cannot be replaced by the family of all lower intervals in Theorem 1 (see e.g. Fig. 13b in [1, p. 88]). In fact, even if we require that all lower intervals are $\vee$-regular and all upper intervals are $\wedge$-regular it doesn't follow that a lattice is complemented, as Example 2 in [2] shows.

Theorem 2. *Let L be a lattice of finite length. Then the following conditions are equivalent:*

(1) *L is relatively complemented*
(2) *all intervals are $\vee$-regular*
(3) *all intervals are $\wedge$-regular*
(4) *L has no 3-element interval.*

Proof. $(1) \Rightarrow (4)$ is clear.

$(4) \Rightarrow (2)$. Assume that (4) holds in L. (4) means that all intervals of length two are $\vee$-regular. We proceed by induction on the length of intervals in L. Assume that all intervals of length less than n are known to be $\vee$-regular and that $[x, y]$ is of length n, $n \geqslant 3$. Let $x = x_1 < x_2 < \cdots < x_{n+1} = y$ be a maximal chain in $[x, y]$. Because of condition (4) there must be some $z \neq x_n$ such that $x_{n-1} < z < y$. Since y covers x_n and x_n covers x_{n-1} we get that $z \vee x_n = y$. Now, let A_y be the atoms of the interval $[x, y]$, A_z the atoms of $[x, z]$ and A_{x_n} the atoms of $[x, x_n]$. Clearly, $(A_z \cup A_{x_n}) \subseteq A_y$. By the induction assumption $z = \bigvee A_z$ and $x_n = \bigvee A_{x_n}$. Therefore $y \geqslant \bigvee A_y \geqslant \bigvee (A_z \cup A_{x_n}) = (\bigvee A_z) \vee (\bigvee A_{x_n}) = z \vee x_n = y$. So $[x, y]$ is $\vee$-regular.

$(2) \Rightarrow (1)$ by Theorem 1.

Since conditions (1) and (4) are self-dual, the equivalence of (3) with the other conditions also follows. □

References

[1] G. Birkhoff, Lattice Theory, 3rd ed., Amer. Math. Soc. Colloq. Publ. 25 (Amer. Math. Soc., Providence, RI, 1967).

[2] A. Björner and I. Rival, A note on fixed points in semimodular lattices, Discrete Math. 29 (1980) 245–250.

[3] Problem E 2700, Amer. Math. Monthly 86 (1979) 310–311.