A complete solution to a conjecture on chromatic uniqueness of complete tripartite graphs

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Abstract

Let \( P(G, \lambda) \) be the chromatic polynomial of a graph \( G \). A graph \( G \) is chromatically unique if for any graph \( H, P(H, \lambda) = P(G, \lambda) \) implies \( H \cong G \). Koh, Teo and Chia conjectured that for any integers \( n \) and \( k \) with \( n \geq k + 2 \geq 4 \), the complete tripartite graph \( K(n - k, n, n) \) is chromatically unique. Let \( K(n, m, r) - S \) denote the graph obtained by deleting all edges in \( S \) from the complete tripartite \( K(n, m, r) \). In this paper, we establish that for any positive integer \( n \geq m \geq r \geq 2 \), the chromatic equivalence class of \( K(n, m, r) \) is contained in the family \( \{ K(x, y, z) - S | 1 \leq x \leq y \leq z, m \leq z \leq n, x + y + z = n + m + r, S \subseteq E(K(x, y, z)) \textrm{ and } |S| = xy + xz + yz - nm - nr - mr \} \). By applying these results, we confirm this conjecture and show that \( K(n - k, n - 1, n) \) is chromatically unique if \( n \geq 2k \) and \( k \geq 2 \).

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1. Introduction

All graphs considered here are finite and simple. For notations and terminology not explained here, we refer to [1].

For a graph \( G \), let \( V(G), E(G), p(G) \) and \( q(G) \) be the vertex-set, edge-set, the number of vertices and the number of edges of \( G \), respectively. By \( \overline{G} \) we denote the complement

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of $G$. Then we let $O_n = \overline{K}_n$, where $K_n$ denotes the complete graph with $n$ vertices. By $K(n_1, n_2, n_3)$ we denote the complete tripartite graph with three parts of $n_1$, $n_2$ and $n_3$ vertices. Let $S$ be a set of $s$ edges of $G$. We denote by $G - S$ the graph obtained by deleting all edges in $S$ from $G$. Let $N_3(G)$ denote the number of triangles in $G$.

For a graph $G$, let $P(G, \lambda)$ be the chromatic polynomial of $G$. A partition $\{A_1, A_2, \ldots, A_r\}$ of $V(G)$, where $r$ is a positive integer, is called an $r$-independent partition of a graph $G$ if every $A_i$ is a nonempty independent in $G$. Let $z(G, r)$ denote the number of $r$-independent partitions of $G$. Then we have $P(G, \lambda) = \sum_{r \geq 1} z(G, r)(\lambda)^r$, where $(\lambda)^r = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1)$ (see [7]).

Two graphs $G$ and $H$ are chromatically equivalent, simply denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. Let $[G] = \{H | H \sim G\}$. A graph $G$ is chromatically unique (or simply $\chi$-unique) if $[G] = \{G\}$.

Let $G$ be a graph with $p$ vertices. Then the polynomial $\sigma(G, x) = \sum_{r=1}^{p} z(G, r)x^r$ is called the $\sigma$-polynomial of $G$, see [2]. Clearly, $P(G, \lambda) = P(H, x)$ if and only if $\sigma(G, x) = \sigma(H, x)$.

For disjoint graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$; $G + H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy | x \in V(G) \text{ and } y \in V(H)\} \cup E(G) \cup E(H)$.

**Lemma 1** (Brenti [2]). Let $G$ and $H$ be two disjoint graphs. Then

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, n_3, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

It has been shown [3–6] that the following complete tripartite graphs are $\chi$-unique graphs: $K(p_1, p_2, p_3)$ for $|p_i - p_j| \leq 1$ and $p_i \geq 2$, $i = 1, 2, 3$; $K(n, n, n+k)$ for $n \geq 2$ and $0 \leq k \leq 3$; $K(n-k, n, n)$ for $n \geq k + 2$ and $0 \leq k \leq 3$; $K(n-k, n, n+k)$ for $n \geq 5$ and $0 \leq k \leq 2$. Recently, Zou et al. in [8,9] gave the following $\chi$-unique graphs: $K(n-k, n, n)$ for $n > k + k^2/3$; $K(n, n, n+k)$ for $n > (k+k^2)/3$; $K(n-k, n, n+k)$ for $n > k^2 + 2\sqrt{3}/3k$; $K(n-4, n, n)$ for $n \geq 6$. Chia et al., and Koh and Teo in [4,5] proposed the following conjecture:

**Conjecture.** For any integers $n$ and $k$ with $n \geq k + 2 \geq 4$, $K(n-k, n, n)$ is $\chi$-unique.

In this paper, we investigate the chromaticity of $K(r, m, n)$ for $n \geq m \geq r \geq 2$. At first we obtain that $[K(r, m, n)] \subseteq \{K(x, y, z) - S | 1 \leq x \leq y \leq z, m \leq z \leq n, x+y+z=n+m+r, S \subseteq E(K(x, y, z)) \text{ and } |S| = xy + xz + yz - nm - nr - mr \}$ for $n \geq m \geq r \geq 2$. Then we give a positive answer to the above conjecture and show that $K(n-k, n-1, n)$ is $\chi$-unique for $n \geq 2k$ and $k \geq 2$. 

For convenience, in this paper we denote simply $\sigma(G,x)$ by $\sigma(G)$ and $G \cong H$ by $G=H$.

2. Some lemmas

Lemma 2 (Zou [8]). Let $G = K(n_1, n_2, n_3)$. Then

(i) $\alpha(G, 3) = 1$ and $\alpha(G, 4) = \sum_{i=1}^{3} 2^{ni-1} - 3$.

(ii) If $H \subseteq [G]$, there is a complete tripartite graph $F = K(m_1, m_2, m_3)$ such that $H = F - S$ and

$m_1 + m_2 + m_3 = n_1 + n_2 + n_3$, where $S$ is a set of $s$ edges of $F$ and $s = q(F) - q(G)$.

Lemma 3 (Zou [8]). Let $G = K(n_1, n_2, n_3)$ with $n_3 \geq n_2 \geq n_1 \geq 2$ and let $H = G - S$ for a set $S$ of $s$ edges of $G$. If $n_1 \geq s + 1$, then $s \leq \alpha(H, 4) - \alpha(G, 4) \leq 2^s - 1$.

Lemma 4 (Koh and Teo [5]). Let $G$ and $H$ be two graphs with $G \sim H$. Then $|V(G)| = |V(H)|, |E(G)| = |E(H)|, N_3(G) = N_3(H)$ and $\alpha(G, r) = \alpha(H, r)$ for $r = 1, 2, 3, \ldots, p(G)$.

Lemma 5 (Koh and Teo [5]). Let $n \geq m \geq 2$. Then $K(n,m)$ is $\chi$-unique.

3. Main results

Theorem 1. For any integers $n \geq m \geq r \geq 2$, we have $[K(r,m,n)] \subseteq \{K(x, y, z) - S | 1 \leq x \leq y \leq z, m \leq z \leq n, x + y + z = n + m + r, S \subseteq E(K(x, y, z)) \text{ and } |S| = xz + yz + xz - nm - nr - mr \}$. In particular, if $z = n$, $K(r, m, n) = K(x, y, z)$.

Proof. Let $G = K(r, m, n)$ and $H \in [G]$. From Lemmas 2(ii) and 4, we know that there exists a graph $F = K(x, y, z)$ and $S \subseteq E(F)$ such that $H = F - S$ and $|S| = s$. We may assume that $1 \leq x \leq y \leq z$. Clearly, $s = q(F) - q(G) = xyz - nmr - nm - nr - mr$ and $x + y + z = n + m + r$.

By Lemma 4, $N_3(G) = N_3(H)$. Hence, we shall consider the number of triangles in $G$ and $H$. Without loss of generality, let $S = \{e_1, e_2, \ldots, e_z\} \subseteq E(F)$. Denote by $N_3(e_i)$ the number of triangles containing the edge $e_i$ in $F$. It is not hard to see that $N_3(e_i) \leq z$. Then

\[ N_3(H) \geq N_3(F) - sz, \tag{1} \]

and the equality holds only if $N_3(e_i) = z$ for all $e_i \in S$.

Let $\beta = N_3(F) - N_3(G)$. It is obvious that $N_3(F) = xyz, N_3(G) = nmr$ and $\beta = xyz - nmr$. So, we have

\[ N_3(G) = N_3(F) - \beta. \tag{2} \]

Since $N_3(G) = N_3(H)$, from (1) and (2) it follows that

\[ \beta \leq sz. \tag{3} \]
Let $f(z) = \beta - sz$. Recalling that $x + y = n + m + r - z$, $\beta = xyz - nmr$ and $s = xy + xz + yz - nm - nr - mr$, we have

$$f(z) = xyz - nmr - [xy + (x + y)z - nm - nr - mr]z$$

$$= (z - n)(z - m)(z - r).$$

(4)

From the fact that $x + y + z = n + m + r$ and $x \leq y \leq z$, we have $z \geq (n + m + r)/3 \geq r$. Note that if $z = r$, then $n = m = r$. From (4), it is not hard to see that inequality (3) holds if and only if $m \leq z \leq n$. This implies $|G| \leq \{K(x, y, z) - S|1 \leq x \leq y \leq z, m \leq z \leq n, |S| = xy + xz + yz - nm - nr - mr, x + y + z = n + m + r\}.$

From now on we assume that $z = n$ and distinguish the following cases:

Case 1: $m < y \leq n$. Clearly $x + y = m + r$ and $x < r$. Hence $s = xy + xn + yn - nm - nr - mr = xy - mr$. One can show that $s < 0$ for $x < r$ and $y > m$. This contradicts $s \geq 0$.

Case 2: $y = m$. Then $x = r$ and $F = K(r, m, n)$. So $s = 0$ and $H = G$.

Case 3: $x \leq y < m$. Let $X_1, X_2, X_3$ be the unique 3-independent partition of $K(x, y, n)$ such that $|X_1| = x, |X_2| = y$ and $|X_3| = n$. By $f(z) = f(n) = 0$, we have that $\beta = sn$. From (1) and (2), we have $N_3(G) = N_3(H) = N_3(F) - sn$ and $N_3(e_i) = n$ for all $e_i \in S$. Thus for every edge $e_i$ in $S$, an end-vertex of $e_i$ belongs to $X_1$, whereas the other end-vertex belongs to $X_2$. Hence $\tilde{H}$ contains $K_n$ as its component. Set $\tilde{H} = \tilde{H}_1 \cup K_n$. Then $H = H_1 + O_n$. From Lemma 1 and $\sigma(H) = \sigma(K(r, m, n))$, we have

$$\sigma(H_1)\sigma(O_n) = \sigma(O_r + O_m)\sigma(O_n).$$

So

$$\sigma(H_1) = \sigma(O_r + O_m),$$

which implies that $P(H_1, \chi) = P(K_{r,m}, \chi)$. Hence, from Lemma 5 and the condition of the theorem, we have $H_1 = K_{r,m}$. So $y = m$, which contradicts $y < m$. This completes the proof.

From Theorem 1, we know that $z = n$ if $n = m$. Therefore, a positive answer to the conjecture is described in the following theorem.

**Theorem 2.** For any integers $n$ and $k$ with $n \geq k + 2 \geq 4$, $K(n - k, n, n)$ is $\chi$-unique.

Further we can give another set of $\chi$-unique complete tripartite graphs.

**Theorem 3.** For any integers $n$ and $k$ with $n \geq 2k \geq 4$, $K(n - k, n - 1, n)$ is $\chi$-unique.

**Proof.** Let $G = K(n - k, n - 1, n)$ and let $H \in [G]$. Then by Theorem 1, $H \in \{K(x, y, z) - S|1 \leq x \leq y \leq z, n - 1 \leq z \leq n, |S| = xy + xz + yz - 3n^2 + 2nk + 2n - k, x + y + z = 3n - k - 1\}.$

Further $H = G$ if $z = n$. For $z = n - 1$, we distinguish the following cases.

Case 1: $y = z = n - 1$. Then $H = K(n - k + 1, n - 1, n - 1) - S$. Let $F = K(n - k + 1, n - 1, n - 1)$ and $|S| = s$. Obviously, $s = q(F) - q(G) = k - 1$. Let $\theta(H) = \chi(H, 4) - \chi(F, 4)$.
From Lemmas 2 and 3,
\[ \chi(G, 4) = 2^{n-k-1} + 2^{n-2} + 2^{n-1} - 3, \]
\[ \chi(H, 4) = 2^{n-k} + 2^{n-1} - 3 + \theta(H). \]

By Lemma 3 and the condition of the theorem, one can see that
\[ s \leq \theta(H) \leq 2^{z} - 1. \]
Since \( k \geq 2 \), from (5) and (6) it follows that
\[ \chi(G, 4) - \chi(H, 4) \geq 2^{n-3} - \theta(H). \]

Remembering the condition of the theorem and \( s = k - 1 \), we have immediately that
\[ \chi(G, 4) - \chi(H, 4) \geq 2^{k-1} - 2^{k-1} + 1 \geq 1. \]
This contradicts that \( \chi(G, 4) = \chi(H, 4) \).

**Case 2:** \( z = n - 1 \) and \( x \leq y \leq n - 2 \). By completely analogous arguments with Case 3 in the proof of Theorem 1, we can obtain that \( H = H_{1} + O_{n-1} \) and \( P(H_{1}, \chi) = P(K_{r,n}, \chi) \). Hence we have \( y = n \), which contradicts \( y \leq n - 2 \). \( \square \)

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**References**