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Decision Procedures for Elementary Sublanguages of Set Theory. V. Multilevel Syllogistic Extended by the General Union Operator

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1. INTRODUCTION

In this paper, which extends earlier work on decision proceures for various quantified and unquantified restricted sublanguages of set theory (see [FOS80, BFOS81, BF84, CFMS86]), we consider the language $\mathscr{L}$ built using the elementary Boolean connectives (conjunction, disjunction, implication, negation) from set-theoretic clauses of the forms

$$x = y \cup z, \qquad x = y \backslash z, \qquad x \in y, \qquad x = \varnothing, \qquad u = \mathrm{Un}(y). \tag{1}$$

In (1), the symbol $\mathrm{Un}(y)$ designates the union of all members of y, i.e., $\{x \mid (\exists z \in y)\, x \in z\}$. Note that relationships $x \subseteq y$, $x = y \cap z$, etc. (and obviously $x \notin y$, $x \neq y$, etc.), can easily be expressed in this language. The still more restricted language obtained by forbidding appearances of the operator Un is studied in [FOS80] and a (relatively simple) decision algorithm given for it. The case in which only one clause of the form $u = \mathrm{Un}(y)$ is allowed was treated in [BF84].

As in the previous papers in this sequence, the intended meaning of the language is that in which variables range over (possibly infinite) sets in the standard universe of "naive" set theory, and the various standard set-theoretic operator and predicate symbols appearing in (1) have their standard meanings; hence an *interpretation* M of a set of sentences P of the language $\mathscr{L}$ is a function which maps every variable x into a set Mx. If all the sentences of P are true under some interpretation of this kind, P is said to be *satisfiable* and each interpretation which satisfies P is called a *model* of P. Our aim is to exhibit an algorithm which decides the satisfiability of such sets P of sentences.

As the domain of the interpretation is fixed (the standard universe of von Neumann), we should speak of *standard interpretations* (resp. *standard models* of P) rather than interpretations (resp. models of P). But we will not belabor this technical point since this paper is concerned with computational rather than foun-

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dational or model-theoretic questions, so that all our discussions are carried out in ordinary "naive" set theory, no other domain of interpretation ever being intended. (Note in this connection that all our considerations are easily formalizable in ZFC (see [J]), and, in fact even in weaker set-theoretical systems, since the language with which we work includes only a very few constructs.)

The question we address is motivated by the large goal of implementing a proof-verifier which makes essential use of decision procedures of the kind developed in this paper and others in the same series (see also [CFOS85]). Such a verifier would include the following components (cf. [S78]), among others:

(a) An *inferential core*, comprising a collection of decision procedures for fragments of mathematical theories (e.g., predicate calculus, simple set-theoretic languages, elementary analysis, and geometry, etc.). These procedures would be managed by

(b) An *outer layer of administrative routines*. These routines would, e.g., maintain a growing library of proved theorems, keep track of demonstrations in progress, define the temporary set of hypotheses under which a proof is currently proceeding, etc.

(c) A family of *extension mechanisms*, to allow the system's user to define personalized families of auxiliary routines, and also to allow new decision procedures to be added to the inferential core.

2. Preliminaries

As in the preceding papers of this sequence, we can limit ourselves without loss of generality to considering simply conjunctions of clauses of the form (1) as well as clauses of the form $x \notin y$. In what follows, this assumption is made unless the contrary is explicitly indicated.

Suppose that a set P of simple clauses of the kind described above is given. Then a *place* α (for P) is a 0/1-valued function defined on the set of all variables in P such that $\alpha(x) = \alpha(y) \vee \alpha(z)$ (resp. $\alpha(x) = \alpha(y)$ & $\neg \alpha(z)$) if $x = y \cup z$ (resp. $x = y \setminus z$) appears in P, and such that $\alpha(x) \equiv 0$ if $x = \emptyset$ appears in P. Given a variable x, the place α is said to be a *place at* x (for P) if $\alpha(y) = 1$ whenever $x \in y$ appears in P and $\alpha(y) = 0$ when $x \notin y$ appears in P.

Any model M of the statements of P defines a set of places for P, and the structure of this set of places goes a long way toward describing the structure of the model M. More specifically, let p be any point appearing in the model; then the function α defined by $\alpha(x) = 1$ if $p \in Mx$, $\alpha(x) = 0$ if $p \notin Mx$ is clearly a place, and for each x, the place which contains Mx is clearly a place at x. Moreover, if we are given any model M and any place α, then we can consider the set

$$\sigma_\alpha = \{p \mid p \in Mx \leftrightarrow \alpha(x) = 1, \text{ for all variables } x\}, \tag{2}$$

which can be called the set of points (of the universal space of the model M) associated with the place α. It is convenient to consider only places α for which $\sigma_\alpha \neq \emptyset$ as places of the model M and to exclude the others. This will be done in what follows. With this understanding, the subsets σ_α are clearly disjoint and $\sigma_\alpha \subseteq Mx$ if and only if $\alpha(x) = 1$. Each set σ_α is either wholly contained in Mx or wholly disjoint from it, and $Mx = \bigcup_{\alpha(x)=1} \sigma_\alpha$. Note also that two variables x, y have the same representation in a model M if and only if $\alpha(x) = \alpha(y)$ for all places of the model. It will be convenient in what follows always to use lowercase Greek letters to designate places, and also to write $\alpha \subseteq x$ when $\sigma_\alpha \subseteq x$, i.e., when $\alpha(x) = 1$.

The set Π_1 of all possible places associated with the set P of clauses is clearly finite and easily calculated. We aim to state the condition that P should be satisfiable using only combinatorial conditions on the clauses of P and on the set of places which actually appear in a model M of P. This is clearly some subset Π of Π_1, which we suppose to have been chosen in advance. As noted just above, once Π is known we know exactly which variables are equal. We shall therefore suppose that (after Π is chosen) equal variables are identified in our set of clauses.

All the essential complications that need to be faced are connected with the presence in P of finitely many clauses of the form $u_i = \mathrm{Un}(y_i)$, which will be referred to as the *Uclauses* of P. The variables y_i appearing on the right of clauses of this form will be called *Uvariables*. Since $u = \mathrm{Un}(y)$ and $u' = \mathrm{Un}(y)$ implies $u = u'$, we can clearly suppose without loss of generality that each Uvariable y_i appears in just one Uclause.

The following definition takes a first step toward elucidating the logical weight of the Uclauses in P.

Definition 1. Given P and Π as above, the *Ugraph* G of P, Π is the graph whose set of nodes is Π, plus one additional node Ω, and whose edges are as follows:

(i) A directed edge connects α to Ω if and only if $\alpha(y_i) = 0$ for every Uvariable y_i. (Intuitively, this means that the Uclauses of P tell us nothing about the set $\mathrm{Un}(\sigma_\alpha)$).

(ii) Otherwise, a directed edge connects the place α to the place β if and only if $\beta(u_i) = 1$ for all clauses $u_i = \mathrm{Un}(y_i)$ such that $\alpha(y_i) = 1$. In this case, we write $\alpha \Rightarrow \beta$. (If there are no such β, then α is not the source node of any edge of G.) Intuitively, the nodes β such that $\alpha \Rightarrow \beta$ represent all the sets σ_β in which elements of $\mathrm{Un}(\sigma_\alpha)$ can appear. If there are no such β, $\mathrm{Un}(\sigma_\alpha)$ is necessarily null.

We shall call a node α of G *safe* if there is a directed path through G starting at α which reaches Ω. A node α will be called *null* if there is no β such that $\alpha \Rightarrow \beta$, and is said to be *trapped* if every sufficiently long path forward from α eventually reaches a null node. A node α which is neither safe nor trapped will be called *cyclic*; some path forward from such a node can always be extended indefinitely, but must then traverse certain other nodes repeatedly. Note that if α is safe, so is every β such that $\beta \Rightarrow \alpha$; hence if α is trapped or cyclic and $\alpha \Rightarrow \beta$, β is also trapped or cyclic.

It is very easy to see that complications greater than those encountered when no clauses $u_i = \mathrm{Un}(y_i)$ are present must be expected in the case before us. For example, the clauses $u = \mathrm{Un}(v)$, $v = \mathrm{Un}(u)$, $u \neq \varnothing$ can be satisfied, but only by an infinite model. Nevertheless, the arguments which follow will show that it is not hard to deal with these infinities. However, worse combinatorial difficulties are connected with the possible existence of trapped places. To see why this should be so, define the *height* of a trapped place τ as one more than the length of the longest path forward from τ to a null place. Suppose that there is a model for our set of clauses, which therefore associates a set Mx with every variable x and a set σ_α with every place α. If τ is of height 1, i.e., null, we have $\mathrm{Un}(\sigma_\tau) = \varnothing$, so $\sigma_\tau = \{\varnothing\}$; hence there can be only one such place, which must be a place at $\varnothing$. Define the height of any set s inductively as one more than the maximum height of any of its elements. Then it follows inductively that if τ is a trapped place the height of σ_τ is at most the height of τ. This restricts σ_τ to one of a finite collection of possible values, namely if H is the maximum height of any trapped place and F_H is the (finite) collection of all sets of height less than H, σ_τ must have some value in F_{H+1}. We will see in the next section that if there are no trapped places, restrictions of this kind, which prevent σ_α from being infinite and cause the combinatorial complications alluded to above, do not occur.

3. The Decision Algorithm in the Absence of Trapped Places

In this section we deduce some conditions which are necessary for P to be satisfiable, regardless of the presence or absence of trapped places. Moreover, we show that if trapped places are absent then these conditions are also sufficient for the satisfiability of P.

The conditions with which we work assert that the Ugraph G of P and Π has certain connectivity properties. Then imply that the sets σ_α, $\alpha \in \Pi$, can be initialized in a manner assuring that the initial interpretation $Mx = \bigcup_{\alpha(x)=1} \sigma_\alpha$ satisfies all equalities in P and allows a subsequent "stabilization" phase to force all remaining clauses of P of the type $(\in, \notin)$ to be satisfied without disrupting any other clause already modeled correctly.

To deduce our first condition we argue as follows. Suppose once more that a model of P exists. Form the union Σ of σ_α, α running over all trapped and cyclic places. Then since every β such that $\alpha \Rightarrow \beta$ must also be trapped or cyclic, it follows that $\mathrm{Un}(\Sigma) \subseteq \Sigma$. Take any element $p_1 \in \sigma_\alpha \subseteq \Sigma$. If $p_1 \neq \varnothing$, it has an element p_2 belonging to some β such that $\alpha \Rightarrow \beta$; if $p_2 \neq \varnothing$, we can repeat this argument to produce p_3, etc. This gives a sequence $\cdots p_3 \in p_2 \in p_1 \in \sigma_\alpha$, which by the set-theoretic axiom of well-foundedness cannot be infinite. It follows that there must be a path through G to a node α which is a place at $\varnothing$. This gives a first necessary condition for satisfiability:

Condition C1. Let the set P of clauses be satisfiable by a model whose set of places is Π, and define the Ugraph G corresponding to P, Π as above. Then, if there are any non-safe places in Π, there must exist a non-safe place γ which lies along a path through G from every non-safe node. Moreover, γ must be a place at $\varnothing$.

If condition C1 is satisfied, we can define a useful auxiliary map ψ of places to places as follows: given α, let $\psi(\alpha)$ be any node β which is one step closer to Ω (resp. γ) along a path of minimum length leading from α to Ω (resp. γ). If $\alpha \Rightarrow \Omega$, put $\psi(\alpha) = \Omega$. Moreover if γ is not null (which implies that no α is null) choose any α such that $\gamma \Rightarrow \alpha$, and put $\psi(\gamma) = \alpha$. The map ψ will be used later when we construct a model for P. Before this, however, we need to state additional satisfiability conditions.

Suppose once more that we have a model M for P, and derive the sets σ_α and the Ugraph G from this model as above. For any two sets s, t write $s \in^* t$ if there is a chain of intermediate elements s_i such that $s \in s_1 \in \cdots \in s_k \in t$. Since in set theory a circular sequence of membership relations $s_i \in^* s_j$ is impossible, any finite collection C of sets can be enumerated in such a way as to ensure that no set s of C can satisfy $s \in^* t$ for a set t coming earlier in sequence. In the following discussion it is supposed that the variables appearing in P are arranged in a sequence derived from such an enumeration of the sets Mx. For each variable x, consider the set Π_x of all places α such that $Mx \in^* \sigma_\alpha$. Then plainly we must have $\alpha(y) = 0$ for all y preceding x in sequence. Moreover, if $Mx \in^* \sigma_\alpha$ and $\sigma_\alpha \subseteq Mu_i = \mathrm{Un}(My_i)$ for some Uvariable y_i and clause $u_i = \mathrm{Un}(y_i)$, then there must exist a place $\beta \subseteq y_i$ such that $\beta \Rightarrow \alpha$, and such that $Mx \in^* \sigma_\beta$. For each α such that $Mx \in^* \sigma_\alpha$ for any variable x and for each Uvariable y_i such that $\alpha \subseteq u_i$, choose any $\beta \subseteq y_i$ such that $\beta \Rightarrow \alpha$ and $Mx \in^* \sigma_\beta$ and call it $\phi_x(\alpha, y_i)$. Finally, define $\phi(\alpha, y_i)$ for all Uvariables y_i such that $\alpha \subseteq u_i$ as any $\beta \subseteq y_i$ such that $\beta \Rightarrow \alpha$. This gives us a collection of maps ϕ, ϕ_x and a collection of sets Π_x of places, one for each variable x appearing in P, having the following properties:

(i) $\phi(\alpha, y_i)$ is defined for all places α and Uvariables y_i such that $\alpha \subseteq u_i$, where $u_i = \mathrm{Un}(y_i)$ is in P; and the value $\beta = \phi(\alpha, y_i)$ is a place such that $\beta \subseteq y_i$ and $\beta \Rightarrow \alpha$.

(ii) For each variable x, the place a_x at x defined by $\alpha_x(y) = 1$ iff $Mx \in My$ belongs to Π_x, and moreover if $\alpha \in \Pi_x$ and $\alpha \subseteq u_i$, then $\phi_x(\alpha, y_i)$ is defined and $\phi_x(\alpha, y_i) \in \Pi_x$, $\phi_x(\alpha, y_i) \Rightarrow \alpha$, $\phi_x(\alpha, y_i) \subseteq y_i$.

(iii) For each variable x, none of the places $\alpha \in \Pi_x$ satisfy $\alpha \subseteq y$ for any variable y which is either equal to x or comes before x in the enumeration of variables defined above.

In what follows, it will be convenient to call an enumeration of variables and maps ϕ and ϕ_x having properties (i)–(iii) a *good Uorder* (of variables) and *good Umaps* respectively; we will not bother to introduce a corresponding term for the sets Π_x of places, though of course such sets of places must be defined in connection with any purported good Umap ϕ_x.

The preceding discussion allows us to state a second condition necessary for satisfiability:

CONDITION C2. Let P, Π, G, etc., be as in condition C1 above. Then (if P is satisfiable) a place $\alpha_x \in \Pi$ such that $\alpha_x(y) = 1$ (resp. $\alpha_x(y) = 0$) if $x \in y$ (resp. $x \notin y$) occurs in P must be defined for each variable x appearing in P and there must exist sets $\Pi_x \subseteq \Pi$ for each variable x, a good Uorder of variables, and good Umaps ϕ and ϕ_x, which by definition will have the properties listed in (i)–(iii) above.

Still one more necessary condition remains to be stated. To see what this is, let M, σ_α, α_x, etc., be as above. Then if $u_i = \mathrm{Un}(y_i)$ is a clause and $Mx \in My_i$, we must have $Mx \subseteq Mu_i$. Hence the following condition must obviously be satisfied:

CONDITION C3. If $u_i = \mathrm{Un}(y)$ is a Uclause of P and $\alpha_x \subseteq y_i$ & $\alpha \subseteq x$, then $\alpha \subseteq u_i$.

This completes the statement of all conditions for satisfiability, at least in the absence of trapped places. That is, we can now go on to show that if there are no trapped places in the Ugraph G of P, and if conditions C1–C3 are all satisfied, then a model for the clauses of P can be constructed. The construction of this model is easy once a sufficient supply of "auxiliary elements" is assured; accordingly, we will begin by assuming that such auxiliary elements with the needed properties have been constructed, and will show how these can be used to build a model M. After this, the narrower technical problem of constructing the auxiliary elements will be adressed.

The properties which the auxiliary elements must have are as follows:

(a) Suppose that condition C1 is satisfied, and let the set Π of places, the Ugraph G, and the map ψ, etc., be as in that condition. Then we assume that infinitely many distinct singleton sets A, called *auxiliary elements*, as well as various other sets B, not necessarily singletons, can be associated with each place $\alpha \in \Pi$. The elements B will be called *secondary elements*, and any auxiliary or secondary element associated with $\alpha \in \Pi$ will be said to be *resident at* α. Every secondary element B must satisfy $B \in^* A$, where A is some auxiliary element. (As above, the relationship $s \in^*$ is defined by the condition that there should exist a chain of sets $s_1, \ldots, s_k$ such that $s \in s_1 \in \cdots \in s_k \in t$.)

(b) No two auxiliary elements A, A' can satisfy $A \in^* A'$.

(c) If $\psi(\alpha) \neq \Omega$, every element of an auxiliary or secondary element A resident at the place α is a secondary element resident at the place $\psi(\alpha)$.

(d) The sets of auxiliary and secondary elements resident at distinct places α, β are always disjoint.

Suppose that infinitely many distinct auxiliary and secondary elements having all the properties (a), (b), (c), (d) are available. Then we can build a model M for the clauses of P as follows:

(1) Arrange the infinite sequence of auxiliary elements resident at each α in 1–1 association with the lattice points of the plane, i.e., divide them into infinitely many infinite "rows." The construction to be described will iterate through a sequence of steps, each of which may require countably many elements, and this arrangement simply ensures that the construction will never exhaust the available supply of auxiliary elements A resident at any α. In what follows, we will suppose that the A have the lexicographic order imposed by this arrangement and when A are required we will select them in this order.

(2) Initialize each of the sets σ_α by inserting all the secondary elements resident at α into σ_α. In addition, put three distinct and unique auxiliary elements into each σ_α. Let $\{A_1 \cdots A_{3n}\}$ be the set of all auxiliary elements used for this. Note that at the end of this step, all the σ_α are disjoint and every one of them contains at least 3 elements.

(3) By the *stabilization process* defined by the maps ϕ and ϕ_x appearing in condition C2 we designate the following operation. If p has been put into σ_α (either in initialization step (2) or in the first phase of the stabilization process itself), then for every Uclause $u_i = \mathrm{Un}(y_i)$ such that $\alpha \subseteq u_i$, choose a previously unused auxiliary element A resident at α, put the pair $\{p, A\}$ into the set σ_{β_i}, where $\beta_i = \phi(\alpha, y_i)$, and also put A into σ_α. Note that when it is generated, the pair $\{p, A\}$ must be distinct from all elements previously inserted into any of the σ_α, and so must the auxiliary element A. Indeed, A, which is a singleton, cannot be a previously formed pair; we will also see below that it cannot equal any of the sets Mx that we form, because such sets always contain at least three elements. For the same reason, $\{p, A\}$ can never equal a set Mx or a previously used auxiliary element, nor can it equal any secondary element B, since then there would exist an auxiliary A' such that $A \in^* A'$, which is impossible. Finally, $\{p, A\}$ can never equal any previously formed pair $\{q, A'\}$, since this could only happen if $p = A'$, $q = A$, but A follows A' in lexicographic order so that the pair $\{A, A'\}$ would never have been formed.

The stabilization process continues until such a pair $\{p, A\}$ has been formed for every p inserted into any one of the sets σ_α. The argument just given shows that the sets σ_α remain disjoint throughout the stabilization process. Moreover, whenever $\{p, A\}$ is inserted in $\beta = \phi(\alpha, y_i)$, we have $\alpha \subseteq u_i$ and p is already in σ_α. A is put in σ_α, but all the elements of A are secondary elements which will already have been put into $\sigma_{\psi(\alpha)}$ if $\psi(\alpha) \neq \Omega$, i.e., if $\alpha \subseteq y_i$ for any Uvariable y_i. Hence, since the condition

$$\mathrm{Un}\Big(\bigcup_{\alpha \subseteq y_i} \sigma_\alpha\Big) \subseteq \bigcup_{\beta \subseteq u_i} \sigma_\beta$$

holds initially for every Uclause $u_i = \mathrm{Un}(y_i)$, it holds throughout the stabilization process. Thus if Mx denotes the value $\bigcup_{\alpha \subseteq x} \sigma_\alpha$, $Mu_i \supseteq \mathrm{Un}(My_i)$ must hold when the stabilization process ceases to generate new pairs. But because of all the pairs $\{p, A\}$ inserted, we must also have $Mu_i \subseteq \mathrm{Un}(My_i)$, and therefore we must have

$Mu_i = \mathrm{Un}(My_i)$ for every Uclause $u_i = \mathrm{Un}(y_i)$. Moreover, since all the σ_α remain disjoint, all clauses of the form $x = y \cap z$, $x = y \backslash z$, and $x = \emptyset$ must also be modeled correctly. Thus it only remains to force all clauses $x \in y$ and $x \notin y$ to be modeled correctly.

For this, we simply work through the sequence of all variables, x, treating them in the (ascending) good Uorder mentioned in condition C2. When a variable x is processed, all the places $\alpha \subseteq x$ will have received values σ_α which will never change subsequently, so that we can define $Mx = \bigcup_{\alpha \subseteq x} \sigma_\alpha$. The variable $\emptyset$ can be bypassed, since $M\emptyset = \emptyset \in \alpha_\emptyset$ will always hold (see below). To process other variables x, Mx is inserted in the set σ_{α_x} (where α_x is the designated place at x (see above)), and the stabilization process is applied, this time using the map ϕ_x in place of the map ϕ. Note in this connection that

(i) Mx cannot be identical with any previously generated element. To see this, note that, for reasons already explained, Mx cannot be identical with any auxiliary or secondary element, or any pair $\{p, A\}$. Moreover, no two sets Mx, My can be equal, since at the start of our construction $Mx \cap \{A_1, ..., A_{3n}\} = \bigcup_{\alpha \subseteq x} \sigma_\alpha \cap \{A_1, ..., A_{3n}\}$, and this relationship is never disrupted by a subsequent insertion of any one of the elements of $\{A_1, ..., A_{3n}\}$ into any of the sets σ_α.

(ii) If $\alpha_x \subseteq y_i$ and $\alpha \subseteq x$, then $\alpha \subseteq u_i$ by condition C3. Hence if Mx is inserted into My_i all the elements of Mx must already belong to Mu_i, proving that the relationship $Mu_i \supseteq \mathrm{Un}(My_i)$ is not disrupted by insertion of Mx into σ_{α_x}. Thus application of the stabilization process restores all relationships $Mu_i = \mathrm{Un}(My_i)$.

(iii) By condition C2, no α which is included either in x (i.e., $\alpha \subseteq x$) or in a variable y which comes before x in the good Uorder of variables can be part of a chain α_i of places satisfying $\alpha_x = \alpha_1$, $\alpha_{i+1} = \phi_x(\alpha_i, y_{j_i})$. However, it is only such places that are affected either by insertion of Mx into α_x or by the subsequent stabilization process. It follows that no relationship $My = \bigcup_{\alpha \subseteq y} \sigma_\alpha$ is disrupted by the said insertion or stabilization operations. This guarantees that literals of type $y \in z$ or $y \notin z$ are correctly modeled. Therefore at the end of the series of steps described M will be a model for all the clauses P.

We therefore will have proved that conditions C1, C2, and C3 are necessary and sufficient for satisfiability of P (at least in the situation in which there are no trapped places) as soon as we show how to construct a family of auxiliary and secondary elements having all the properties (a), (b), and (c) listed above. For this, we can proceed as follows.

Begin with all places α such that $\psi(\alpha) = \Omega$. Assign disjoint infinite sets of integers $n \geqslant 3$ to these places, and for each integer n assigned to α build the singleton $\{n\}$. Define half these singletons to be *auxiliary* elements resident at α, and the other half of these singletons to be *secondary* elements resident at α.

Next suppose that there are cyclic places α, but continue to suppose that there are no trapped places. Then, as has been shown earlier, there is a place $\gamma = \alpha_\emptyset$ at $\emptyset$ and a path through the Ugraph G (see condition (1)) to γ from any other cyclic node. Hence by definition of the map ψ there is some cycle $\gamma_1,..., \gamma_{m+1}$ of length m at least 2, such that $\gamma_1 = \gamma_{m+1} = \gamma$, and $\gamma_{i+1} = \psi(\gamma_i)$, $i = 1,..., m$. (Note that this cycle is allowed to contain repetitions.) Define the set $\varnothing_n$ for all $n \geqslant 0$ by $\varnothing_0 = \varnothing$, $\varnothing_{i+1} = \{\varnothing_i\}$, and let all the sets $\varnothing_n$ of this form with $n \equiv 1 - j \pmod m$ be secondary elements resident at γ_j. (Since all these elements are inserted into σ_{γ_j} initially, we have $\varnothing \in \sigma_{\gamma_1} = \sigma_{\alpha_\varnothing}$ as noted above.) Then form all pairs $\{\varnothing_n, \varnothing_{n+m}\}$ and let all such pairs with $n \equiv m - j \pmod m$ be additional secondary elements resident at γ_j. Finally, form all singletons $\pi_n = \{\{\varnothing_n, \varnothing_{n+m}\}\}$ and let all those with $n \equiv m - 1 - j$ $\pmod m$ be resident at γ_j. Take the infinite set of the singletons of this last form resident at γ_j and divide this set, in any convenient way, into disjoint parts, both infinite; define the singletons belonging to one of these parts to be *auxiliary* elements resident at γ_j, while the singletons of the other part are defined to be *secondary* elements resident at γ_j.

Next define further singletons $\pi_{n,j}$ by $\pi_{n,1} = \pi_n$, $\pi_{n,j+1} = \{\pi_{n,j}\}$. It is easy to see that $\pi_{n,j} \in^* \pi_{l,k}$ if and only if $n = l$ and $j < k$. Indeed, $\pi_{n,j} \in^* \pi_{l,k}$ implies that $\{\varnothing_n, \varnothing_{n+m}\} \in^* \pi_{l,k}$, and then clearly $\{\varnothing_n, \varnothing_{n+m}\} \in^* \{\{\varnothing_l, \varnothing_{l+m}\}\}$, so either $\{\varnothing_n, \varnothing_{n+m}\} = \{\varnothing_l, \varnothing_{l+m}\}$, implying $n = l$, or $\{\varnothing_n, \varnothing_{n+m}\} \in^* \varnothing_{l+m}$, which is impossible. But once we know that $\pi_{n,j} \in^* \pi_{l,k}$ implies $n = l$, it follows trivially that it must also imply $k > j$.

We have associated infinitely many auxiliary and secondary elements of the form $\{n\}$, where n is an integer $\geqslant 3$, with each place α such that $\psi(\alpha) = \Omega$. Much as previously, define $\pi^*_{n,1}$, by $\pi^*_{n,1} = \{n\}$, $\pi^*_{n,j+1} = \{\pi^*_{n,j}\}$. Then $\pi_{n,j} \in^* \pi^*_{l,k}$ would imply that $\{\varnothing_n, \varnothing_{n+m}\} \in^* \pi^*_{l,k}$, and hence $\{\varnothing_n, \varnothing_{n+m}\} \in^* l$, which is impossible since all the elements of an integer are themselves integers. For the same reason, $\pi^*_{n,j} \in^* \pi_{l,k}$ is impossible, and $\pi^*_{n,j} \varepsilon^* \pi^*_{l,k}$ implies that $n = l$ and $j < k$.

At this point we have associated infinitely many auxiliary and secondary elements $\pi_{n,1}$ with each place γ' of the cycle $\gamma_1 \cdots \gamma_{m+1}$, and with each γ such that $\psi(\gamma) = \Omega$, and it only remains to extend this association to the remaining cyclic and safe places. For this, a simple iterative construction can be used. Regard a place as having been *treated* if secondary and auxiliary elements $\pi_{n,j}$ or $\pi^*_{n,j}$ have already been associated with it. If any untreated places remain, choose some α which has already been treated, but for which there remain untreated $\beta_1,..., \beta_k$ such that $\psi(\beta_1) = \cdots = \psi(\beta_k) = \alpha$. Divide the infinitely many *secondary* elements $\pi_{n,j}$ or $\pi^*_{n,j}$ resident at α into k subsequences, all infinite, and define the elements $\pi_{n,j+1}$ (or $\pi^*_{n,j+1}$) such that $\pi_{n,j}$ (or $\pi^*_{n,j}$) belongs to the ith of these subsequences to be resident at β_i, $i = 1,..., k$. Divide the infinite set of resident items thereby associated with each of the β_i into two infinite subsequences, and define the elements of one of these subsequences to be *auxiliary* elements resident at β_i, while the elements of the other subsequence are defined to be *secondary* elements resident at β_i. Continue in this way as long as any untreated places remain. Finally, in order to ensure that every secondary element β is a member of some auxiliary element, we adopt the technical convention of forming $\{\beta\}$ as an auxiliary element without specific residence whenever β is a singleton secondary element for which $\{\beta\}$ is not otherwise introduced.

It is clear that the collection of auxiliary and secondary elements A constructed in this way satisfies all the conditions (a), (b), (c), (d) stated previously. This completes our treatment of the case in which no trapped places exist, i.e., shows that if the Ugraph G appearing in condition C1 has no trapped nodes, then conditions C1, C2, C3 are necessary and sufficient for the satisfiability of P by a model having Π as its set of places. The case in which trapped places can exist is considered in the next section.

4. The Decision Algorithm When Trapped Places Are Present

The construction of a model of P in the presence of trapped places is a bit subtler than that applicable in the case considered in the previous section. The main differences stem from the fact that in this case the role of the single place γ must be played by a finite set of places, called $\gamma_1, \gamma_2, ..., \gamma_k$ in the discussion which follows; moreover, sets associated with trapped places can only range over a finite family of finite sets known a priori. This last limitation makes the stabilization phase more complicated.

Define the height of a trapped place τ and the height of a set s as in Section 2; let H be the maximum height of any trapped place τ, and suppose that there exists a model M with places Π, sets σ_α, etc., all as in our preceding discussion. For each h, let F_h designate the finite family of all sets of height $<h$; note (for implicit use in what follows) that the union of subsets of F_h is itself a subset of F_h. As shown earlier, $\sigma_\tau \subseteq F_H$ for all trapped τ. Put $\sigma'_\alpha = \sigma_\alpha \cap F_{H+1}$ for each α. Call a variable x *trapped* if $\alpha \subseteq x$ implies that α is trapped. Plainly if τ is trapped then $\sigma_\tau = \sigma'_\tau \subseteq F_H$, hence if x is trapped $Mx = \bigcup_{\alpha \subseteq x} \sigma_\alpha = \bigcup_{\alpha \subseteq x} \sigma'_\alpha \subseteq F_H$, that is, $Mx \in F_{H+1}$ and then $Mx \in \sigma_{\alpha_x} \cap F_{H+1} = \sigma'_{\alpha_x}$. Let x be a variable such that α_x is trapped. Since $\alpha \subseteq x$ implies $\alpha_x \Rightarrow \alpha$, it follows that x is trapped and $\bigcup_{\alpha \subseteq x} \sigma'_\alpha \in \sigma'_{\alpha_x}$. Let $u_i = \mathrm{Un}(y_i)$ be a Uclause. Let α be a trapped place and suppose that $\alpha \subseteq y_i$. Then $\mathrm{Un}(\sigma'_\alpha) = \mathrm{Un}(\sigma_\alpha) \subseteq (\bigcup_{\beta \in b} \sigma_\beta) \cap F_{H+1} = \bigcup_{\beta \in b} \sigma'_\beta$, where b is the set of places β such that $\alpha \Rightarrow \beta$. Note that these β's are all trapped and $\beta \subseteq u_i$.

For $\alpha, \beta \in \Pi \cup \{\Omega\}$ we put $\alpha \sim \beta$ if $\alpha \Rightarrow^* \beta$ and $\beta \Rightarrow^* \alpha$, where $\gamma \Rightarrow^* \delta$ means that there is a directed path, possibly null (i.e., $\gamma = \delta$), from γ to δ.

Clearly $\sim$ is an equivalence relation. Moreover the partition of Π induced by $\sim$ refines the crude partition of the set of all places into safe, trapped, and cyclic places.

By $[\alpha]$ we will denote the equivalence class (relative to the relation $\sim$) containing α. (These are the strongly connected components of the Ugraph G. See [AHU, p. 189].) Define the auxiliary directed graph $\bar{G}$ induced by the Ugraph G as the graph whose nodes are the equivalence classes of $\sim$ and whose edges are the following:

$[\alpha] \Rightarrow_{\bar{G}} [\beta]$ is an edge of $\bar{G}$ if $[\alpha] \neq [\beta]$ and there are $\alpha' \in [\alpha]$, $\beta' \in [\beta]$ such that $\alpha' \Rightarrow_G \beta'$ is an edge of the Ugraph G.

It is obvious that $\bar{G}$ has no self-loops, and that $\bar{G}$ is acyclic. Next suppose that there are cyclic places. Consider the subgraph $L = \{[\alpha] \mid \alpha \text{ is cyclic}\}$ induced by $\bar{G}$.

Obviously L is acyclic too, and hence since L is finite there exist elements of L with no outgoing edge to any other element of L, i.e., the set $M = \{[\alpha] \in L \mid [\alpha]$ has no outgoing edge in $L\}$ is non-null. Let $[\alpha] \in M$. Then if $[\alpha] \Rightarrow_{\bar{G}} [\beta]$, β must be trapped. Indeed, by the definition of M, β cannot be cyclic; and clearly β cannot be safe because otherwise α would also be safe. Moreover there must be at least one trapped place β such that $[\alpha] \Rightarrow_{\bar{G}} [\beta]$, because as observed earlier there is a path from every non-safe place to the place at $\varnothing$, and the place at $\varnothing$ is not an element of $[\alpha]$ since it is not cyclic. Let $[\alpha_1]$, $[\alpha_2]$,..., $[\alpha_k]$ be the elements of M. For each $[\alpha_i]$, $i = 1,..., k$, consder the set $S_i = \bigcup_{\beta \in [\alpha_i]} \sigma_\beta$. Let γ_i be an element of S_i having minimal height. Without loss of generality we can assume that $\gamma_i \in \sigma_{\alpha_i}$. Clearly $\gamma_i \subseteq \bigcup_{\beta \in b} \sigma_\beta$, where b is the set of places β such that $\alpha_i \Rightarrow_G \beta$. Moreover, by the minimality of the height of γ_i, no element of γ_i belongs to S_i; from which it follows that every such element lies outside the union $\bigcup \sigma_\alpha$ extended over all cyclic places α. Thus $\gamma_i \subseteq \bigcup_{\beta \in b'} \sigma_\beta$, where b' is the set of places in b which are trapped. Hence $\gamma_i \subseteq \bigcup_{\beta \in b'} \sigma'_\beta$, which implies that $\gamma_i \in \sigma_{\alpha_i} \cap F_{H+1} = \sigma'_{\alpha_i}$. Various other useful properties of the elements γ_i now follow easily. First, for every $i, j \in \{1, 2,..., k\}$, $\gamma_i \in^* \gamma_j$ is false. Indeed if there exist i, j such that $\gamma_i \in^* \gamma_j$, then $i \neq j$ and for some $s_1,..., s_t$ we must have $\gamma_i \in s_1 \in \cdots \in s_t \in \gamma_j$. That implies the existence of places $\beta_1,..., \beta_t$ such that $\alpha_j \Rightarrow_G \beta_t \Rightarrow_G \cdots \Rightarrow_G \beta_1 \Rightarrow_G \alpha_i$, and then plainly all the β_i's are cyclic places. From this it follows at once that $[\alpha_j] \Rightarrow_L^* [\alpha_i]$ which, by the definition of the $[\alpha_i]$'s, is a contradiction. Moreover for each $i \in \{1, 2,..., k\}$ and each trapped place τ, $\gamma_i \in^* \sigma'_\tau$ is false. Indeed if this were not the case, it would follow as above that $\alpha_\tau \Rightarrow_G \beta_t \Rightarrow_G \cdots \Rightarrow_G \beta_1 \Rightarrow_G \alpha_i$, which is impossible, since α_τ is trapped while α_i is cyclic. Another important property of the classes $[\alpha_i]$ is that every element in such a class lies in a cycle. To see this take any element α of $[\alpha_i]$. Then there must be a path from α to a cycle of places. But no edges along this path can exit $[\alpha_i]$, since if any did it would have to terminate at a trapped place, which is clearly impossible. It follows that $[\alpha_i]$ must contain at least one cycle; but then since all the elements of $[\alpha_i]$ are equivalent, it follows that every element of $[\alpha_i]$ lies on a cycle.

By the definition of $\alpha_1,..., \alpha_k$, there is a path from every cyclic place to at least one of the α_i's. This allows us to define various maps which will be useful in the following. Specifically, let $u_i = \mathrm{Un}(y_i)$ be a Uclause. For each place $\alpha \subseteq u_i$, there must exist a $\beta \subseteq y_i$ such that $\beta \Rightarrow_G \alpha$. If any such β is non-trapped, choose such a β and call it $\phi(\alpha, y_i)$; otherwise let $\phi(\alpha, y_i)$ be any trapped β such that $\beta \Rightarrow_G \alpha$ and $\beta \subseteq y_i$. If $\phi(\alpha, y_i)$ is trapped then α is trapped too, and $\sigma'_\alpha = \sigma_\alpha \subseteq \bigcup_{\beta \in b} \mathrm{Un}(\sigma_\beta) = \bigcup_{\beta \in b} \mathrm{Un}(\sigma'_\beta)$, where b is the set of all places β such that $\beta \subseteq y_i$ and $\beta \Rightarrow_G \alpha$. As in the previous case in which no trapped places exist, we can define a good Uorder of the variables occurring in P and a good Umap $\phi_x(\alpha, y_i)$ having properties (ii), (iii) listed just before condition C2 as well as the following property:

(iv) If α_x is non-trapped, then all places in Π_x are non-trapped.

To show that if a model exists condition (iv) can always be satisfied along with the other conditions (ii), (iii) we reason as follows. Let α_x be non-trapped and $\alpha \in \Pi_x$. By the definition of Π_x (see the paragraph preceding the statement of conditions (i), (ii), and (iii)) we have $Mx \in^* \sigma_\alpha$. Hence there are elements $s_1, \ldots, s_t$ such that $Mx \in s_1 \in \cdots \in s_t \in \sigma_\alpha$. Suppose that σ_α is a trapped place. Let $\alpha_x, \beta_1, \ldots, \beta_t$ be places such that $Mx \in \sigma_{\alpha_x}$, $s_j \in \sigma_{\beta_j}$. None of these places can be safe, and the preceding chain of memberships implies (inductively) that $\alpha = \beta_t \Rightarrow_G \cdots \Rightarrow_G \beta_1 \Rightarrow_G \alpha_x$, so that all the places in the sequence must be trapped, contradicting our assumption that α_x is not trapped.

The preceding discussion shows that if there exists a model M of P with places Π and Ugraph G involving trapped places τ, the following combinatorial conditions must be satisfied:

Condition C1′. Let H be the maximum height, in the Ugraph G, of any trapped place τ, and let F_{H+1} be as above. Then there must be a map $\alpha \to \sigma'_\alpha$ of places to disjoint subsets of F_{H+1} such that $\sigma'_\alpha \neq \emptyset$ whenever α is trapped, and there must exist a map $x \to a_x$ of the set of all the variables occurring in P to the set of all places, such that:

(i) If $x \in y$ (resp. $x \notin y$) occurs in P then $\alpha_x(y) = 1$ (resp. $\alpha_x(y) = 0$).

(ii) If x is trapped (that is, all places α such that $\alpha \subseteq x$ are trapped), then $\bigcup_{\alpha \subseteq x} \sigma'_\alpha \in \sigma'_{\alpha_x}$.

(iii) If $u_i = \mathrm{Un}(y_i)$ is a Uclause and α is a trapped place for which $\alpha \subseteq y_i$, then $\mathrm{Un}(\sigma'_\alpha) \subseteq \bigcup_{\beta \in b} \sigma'_\beta$, where b is the set of places β such that $\alpha \Rightarrow_G \beta$.

(iv) If there are any cyclic places, then there exists a set $\{\alpha_1, \ldots, \alpha_k\}$ of such places, each lying in some cycle of the Ugraph G, and for each $i = 1, \ldots, k$ an element $\gamma_i \in \sigma'_{\alpha_i}$ such that $\gamma_i \in^* \gamma_j$ is false for every $i, j = 1, \ldots, k$, and such that $\gamma_i \in^* \sigma'_\beta$ is also false for every $i = 1, \ldots, k$ and every trapped place β. Moreover for each α_i the set b_i of trapped places β such that $\alpha_i \Rightarrow_G \beta$, is non-empty and $\gamma_i \subseteq \bigcup_{\beta \in b_i} \sigma'_\beta$. Finally, there must exist a path through the Ugraph G forward from every cyclic place α to some α_i, $i = 1, 2, \ldots, k$.

Condition C2′. There must exist maps $\phi(\alpha, y_i)$ and $\phi_x(\alpha, y_i)$ defined for places α, variables x, and Uclauses $u_i = \mathrm{Un}(y_i)$, both having values (when defined) which are places $\beta \subseteq y_i$ such that $\beta \Rightarrow_G \alpha$. Moreover, there must exist a good Uorder of the variables and a set of places Π_x, for each variable x, such that if α_x is non-trapped then all places in Π_x are non-trapped, and ϕ_x must be a good Umap in the sense of the previously stated Condition C2 in the preceding section, and must be defined for every $\alpha \in \Pi_x$ and every Uclause $u_i = \mathrm{Un}(y_i)$ such that $\alpha \subseteq u_i$. In addition, $\phi(\alpha, y_i)$ must be defined for each $\alpha \subseteq u_i$ and must be non-trapped if there is any non-trapped $\beta \subseteq y_i$ such that $\beta \Rightarrow_G \alpha$. Moreover if $\phi(\alpha, y_i)$ is trapped, then we must have $\sigma'_\alpha \subseteq \bigcup_{\beta \in b} \mathrm{Un}(\sigma'_\beta)$, where b is the set of all places $\beta \subseteq y_i$ such that $\beta \Rightarrow_G \alpha$.

Condition C3′. This is identical to condition C3 stated in Section 3.

We shall now complete our analysis by showing that the three conditions C1′, C2′, C3′ are not only necessary but also sufficient for satisfiability of the clauses P by a model having the set of places Π. Suppose therefore that these conditions are satisfied. To construct a model M for the clauses P, we will use much the same method as in the easier case in which no trapped places exist, but with the difference that no elements are ever added to a set σ_α if the place α is trapped; thus for trapped places we will always have $\sigma_\alpha = \sigma'_\alpha$. Our first step is to construct a sufficient supply of auxiliary and secondary elements resident at each non-trapped place. We begin by considering the case in which cyclic places do occur. (However, if there are no cyclic places the proof is much the same; in this case the reader has only to ignore what is said about cyclic places in the following paragraphs.) Define an auxiliary map ψ from non-trapped places to non-trapped places as follows. Clause (iv) of condition C1′ implies the existence of cyclic places $\alpha_1,\ldots,\alpha_k$, all of them lying in some simple cycle, which we will designate by $\alpha_i = \beta_{i,1} \Rightarrow_G \beta_{i,2} \Rightarrow_G \cdots \Rightarrow \beta_{i,m_i+1} = \alpha_i$, $i = 1,\ldots, k$, $m_i \geqslant 1$. For each of the places $\beta_{i,j}$ we put $\psi(\beta_{i,j}) = \beta_{i,j+1}$, $1 \leqslant j \leqslant m_i$, with the understanding that j and $j+1$ are taken modulo m_i. (Note that no two of the cycles $\beta_{i,1},\ldots, \beta_{i,m_i}$ intersect.) For all remaining cyclic places α we put $\psi(\alpha) = \beta$, where β lies one step closer than α along some shortest path through the Ugraph G to an element in one of these cycles. If α is safe, we put $\psi(\alpha) = \Omega$ if $\alpha \Rightarrow \Omega$; otherwise we put $\psi(\alpha) = \beta$, where β lies one step closer to Ω than does α (again, along some shortest path through G to one of the cycles above). Condition (iv) of C1′ ensures that ψ is well defined for all cyclic places. Moreover it is obvious that if α is cyclic than $\psi(\alpha)$ is cyclic too. For each α_i, $i = 1, 2,\ldots, k$, define sets $\gamma_i^{(n)}$ by $\gamma_i^{(0)} = \gamma_i$, $\gamma_i^{(n+1)} = \{\gamma^{(n)}\}$, where $\gamma_i \in \sigma'_{\alpha_i}$ is the element appearing in (iv) of C1′. Define all the $\gamma_i^{(n)}$ with $n \equiv 1 - j \pmod{m_i}$ to be secondary elements resident at $\beta_{i,j}$; also, form all pairs $\{\gamma_i^{(n)}, \gamma_i^{(n+m_i)}\}$, and let all the pairs of this form with $n \equiv m_i - j \pmod{m_i}$ be secondary elements resident at $\beta_{i,j}$ also. Next define singletons $\pi_i^n = \{\{\gamma_i^{(n)}, \gamma_i^{(n+m_i)}\}\}$, and take each such singleton with $n \equiv m_i - 1 - j \pmod{m_i}$ to be resident at $\beta_{i,j}$. Divide the infinite set of these singletons resident at $\beta_{i,j}$ in any convenient way into two disjoint infinite parts; define singletons belonging to one of these parts to be *auxiliary* elements resident at $\beta_{i,j}$, and define the singletons belonging to the other of these parts to be *secondary* elements resident at $\beta_{i,j}$.

Next define further singletons $\pi_i^{j,n}$ by $\pi_i^{j,0} = \pi_i^j$, $\pi_i^{j,n+1} = \{\pi_i^{j,n}\}$. Using the fact that $\gamma_i \in^* \gamma_l$ is false for every $i, l = 1,\ldots, k$ it is easy to see that we have $\pi_i^{j,n} \in^* \pi_l^{p,m}$ iff $i = l$, $j = p$, and $n < m$. The definitions stated in the preceding paragraph associate infinitely many secondary and auxiliary elements of the form $\pi_i^{j,0}$ with each place β belonging to any cycle α_i, $\psi(\alpha_i)$, $\psi^2(\alpha_i)$,... with $i \in \{1, 2,\ldots, k\}$, but we need to extend this association to the remaining cyclic places and to treat the safe places. For this, much the same simple construction as before is available. We use the fact that if α is cyclic (resp. safe) then $\psi(\alpha)$ is cyclic (resp. safe or Ω), and that repeated application of the map ψ must eventually bring any place α to one of the places with which auxiliary and secondary places have already been associated. More specifically, regard a cyclic place as having been *treated* if secondary and auxiliary elements $\pi_i^{j,n}$ have already been associated with it. If any untreated cyclic places remain, choose some α which has already been treated but for which there remain untreated $\beta_1,..., \beta_l$ such that $\psi(\beta_1) = \cdots = \psi(\beta_l) = \alpha$ (by the observation made just above, such an α must exist). Divide the infinitely many *secondary* elements $\pi_i^{j,n}$ resident at α into l subsequences, all infinite, and let the elements $\pi_i^{j,n+1}$ such that $\pi_i^{j,n}$ belongs to the pth of these subsequences be resident at β_p, $p = 1,..., l$. Divide the infinite set of resident items associated in this way with each of the β_p into two infinite subsequences, and define the elements of one of these subsequences to be auxiliary elements resident at β_p, and the elements of the other subsequence to be secondary elements resident at β_p. Continue in this way as long as there remain any untreated cyclic places.

To handle the safe elements begin with the finite set N of places α such that $\psi(\alpha) = \Omega$. Divide the infinite set of singletons $\{n\}$, where n is an integer and $n \geq H + 1$, into an appropriate number of infinite subsets, and define the elements of each of these subsequences to be resident at a corresponding place α in N. Divide the singletons thereby assigned to α into two infinite subsequences, and define the elements of one of these subsequences to be secondary elements resident at α; the elements of the other subsequence are defined to be auxiliary elements resident at α. Then use the map ψ in the same iterative fashion as in the preceding paragraph, until resident auxiliary and secondary elements have been assigned to all safe places. (Again, we adopt the technical convention of regarding $\{A\}$ as an auxilary element without specific residence whenever A is a singleton secondary element for which $\{A\}$ is not otherwise introduced.)

Much as in the simple case, free of trapped places, treated earlier, the construction just outlined associates infinitely many resident auxiliary elements A and secondary elements B with each non-trapped place α. These are easily seen to have the following properties:

(a) Every secondary element B satisfies $B \in^* A$, where A is some auxiliary element (not necessarily resident at the same place).

(b) No two auxiliary elements A, A' can satisfy $A \in^* A'$.

(c) If $\psi(\alpha) \neq \Omega$, every element of an auxiliary or secondary element resident at a non-trapped place α is either a secondary element resident at $\psi(\alpha)$, or an element of σ'_β for some trapped place β such that $\alpha \Rightarrow \beta$, the second possibility only arising for elements of secondary items.

(d) No auxiliary or secondary element A resident at a non-trapped place α satisfies $A \in^* \sigma'_\beta$ for any trapped place β.

(e) The sets of auxiliary and secondary elements resident at distinct non-trapped places are disjoint.

Once having associated infinitely many distinct auxiliary and secondary places with each non-trapped place α in a manner satisfying conditions (a)–(e), we can build a model for the clauses of P as follows:

(1) Arrange the infinite sequence of auxiliary elements resident at each non-

trapped α in 1–1 association with the lattice points of the plane, thereby giving them a lexicographic order. As in the simpler case considered previously, this ensures that the iterative construction described in the next few paragraphs will never exhaust the supply of auxiliary elements A resident at any α.

(2) Initialize each of the sets σ_α, for trapped α only, by inserting all the elements of σ'_α into σ_α. If α is not trapped, insert all the secondary elements resident at α into σ_α. In addition, if α is non-trapped, put three distinct and unique auxiliary elements resident at α into σ_α of height at least $H+1$. Let $\{A_1,..., A_{3n}\}$ be the set of all auxiliary elements used for this. Note that at the end of this step, all the σ_α are disjoint, and if α is non-trapped σ_α contains at least three elements.

(3) By the *ϕ–ψ-stabilization process* defined by the map ϕ appearing in condition C2′ and the map ψ defined previously we designate the following operation:

If p has been put into σ_α, then for all Uclauses $u_i=\mathrm{Un}(y_i)$ such that $\alpha\subseteq u_i$ which are such that the place $\beta_i=\phi(\alpha, y_i)$ is non-trapped, proceed as follows. Choose a previously unused auxiliary element A_i resident at $\psi(\beta_i)$ (observe that since β_i is non-trapped and $\beta_i \Rightarrow_G \psi(\beta_i)$, $\psi(\beta_i)$ is a non-trapped place and thus has associated auxiliary and secondary elements), put the pair $\{p, A_i\}$ into the set σ_{β_i}, and also put A_i into $\sigma_{\psi(\beta_i)}$.

Note that when it is generated, the pair $\{p, A_i\}$ must be distinct from all elements previously inserted into any of the σ_α, and so must the auxiliary element A_i. Indeed, the singleton A_i cannot be a previously formed pair, nor can it equal any element of any σ'_τ, τ trapped, or any secondary element resident at any non-trapped place or any previously used auxiliary element A. Moreover, $\{p, A_i\}$ can never equal any previously used auxiliary element, nor can it equal any secondary element B, since then there would exist an auxiliary A such that $A_i \in^* A$, which is impossible. Finally, for the same reason as in the simpler case considered previously, in which there exist no trapped places, $\{p, A\}$ can never equal any previously formed pair $\{q, A'\}$.

It follows that the sets σ_α remain disjoint throughout the ϕ–ψ-stabilization process, which continues until a pair $\{p, A\}$ has been formed for any p inserted into any set σ_α such that there is a Uclause $u_i=\mathrm{Un}(y_i)$ for which $\alpha\subseteq u_i$ and $\phi(\alpha, y_i)$ is non-trapped. Moreover, before the ϕ–ψ-stabilization process begins, we have $\mathrm{Un}(\sigma_\alpha)\subseteq \bigcup_{\beta\subseteq u_i}\sigma_\beta$ for each Uclause $u_i=\mathrm{Un}(y_i)$ and $\alpha\subseteq y_i$. Indeed, for α trapped $\mathrm{Un}(\sigma'_\alpha)\subseteq\bigcup_{\beta\in b}\sigma'_\beta$ by (iii) of condition C1′, where b is the set of places $\beta\subseteq u_i$ such that $\alpha\Rightarrow_G\beta$. Moreover by condition (c) just above, every element of a secondary element p inserted into σ_α is either a secondary element inserted into $\psi(\beta)\subseteq u_i$ or an element of σ'_β for some trapped β such that $\beta\subseteq u_i$ and $\alpha\Rightarrow_G\beta$. On the other hand the ϕ–ψ-stabilization process does not disturb this condition, since a pair $\{p, A\}$ is only inserted into σ_β, where $\beta\subseteq y_i$, when p is already in some σ_α with $\alpha\subseteq u_i$; moreover A is then inserted into $\sigma_{\psi(\beta)}$, which must also satisfy $\psi(\beta)\subseteq u_i$. (Note also that by (c) above, when A is inserted into $\sigma_{\psi(\beta)}$, all the elements of A are already present: $\bigcup_{\beta\subseteq u_i}\sigma_\beta$.)

Thus, for each Uclause $u_i = \mathrm{Un}(y_i)$ and $\alpha \subseteq y_i$, we continue to have $\mathrm{Un}(\sigma_\alpha) \subseteq \bigcup_{\beta \subseteq u_i} \sigma_\beta$ at the end of the ϕ–ψ-stabilization process. However, we also have $\sigma_\beta \subseteq \mathrm{Un}(\sigma_{\phi(\beta, y_i)})$ if $\beta \subseteq u_i$ and $\phi(\beta, y_i)$ is non-trapped. Moreover, if $\phi(\beta, y_i)$ is trapped then β is trapped too and it follows by condition C2′ that $\sigma_\beta = \sigma'_\beta \subseteq \bigcup_{\gamma \in b} \mathrm{Un}(\sigma'_\gamma)$, where b is the set of places $\gamma \subseteq y_i$ such that $\gamma \Rightarrow_G \beta$ (all these places are trapped). But in this case $\bigcup_{\gamma \in b} \mathrm{Un}(\sigma'_\gamma) = \bigcup_{\gamma \in b} \mathrm{Un}(\sigma_\gamma)$, and hence $\sigma_\beta \subseteq \bigcup_{\gamma \in b} \mathrm{Un}(\sigma_\gamma)$ in every case; i.e., at the end of the ϕ–ψ-stabilization process all Uclauses are correctly modeled. Moreover, since the non-empty sets σ_β remain disjoint throughout the ϕ–ψ-stabilization process, all clauses $x = y \cup z$, $x = y \backslash z$, and $x = \emptyset$ are correctly modeled also. In addition, since the value Mx assigned to a variable x is always understood to be $\bigcup_{\alpha \subseteq x} \sigma_\alpha$, it follows from (i) and (ii) of condition C1′ that the clauses $x \in y$ and $x \notin y$ containing a given variable x are correctly modeled whenever the place α_x is trapped (indeed if α_x is trapped, by condition C3′ the variable x is also trapped).

(4) It only remains to extend the model M so as to force clauses $x \in y$ and $x \notin y$ to be correctly modeled even if α_x is not trapped. This can be done by applying exactly the method described previously for the case in which no trapped places exist. That is, we arrange all the variables appearing in P in the (ascending) good Uorder mentioned in condition C2′. To process a variable x, we insert Mx into the set σ_{α_x} if Mx is not already in σ_{α_x}. (Note, in particular, that variables x such that α_x is trapped require no processing.) After each such insertion, we restabilize to ensure the validity of all Uclauses, using the ϕ_x-stabilization process, just as in the absence of trapped places, rather than the ϕ–ψ-stabilization process used in step (3). Note that if α_x is non-trapped then by condition C2′ all the places β for which σ_β is affected by the ϕ_x-stabilization process are non-trapped. Moreover, as was pointed out in (iii) just preceding the definition of the auxiliary and secondary elements in the case in which no trapped places exist, neither insertion of the Mx's into σ_{α_x} nor the subsequent stabilization operations disrupt any relationship $My = \bigcup_{\alpha \subseteq y} \sigma_\alpha$, where y either precedes x in the good Uorder of variables or is equal to x.

As in the simpler case considered previously, to justify the remarks made in the preceding paragraph we must establish that no set Mx inserted into a set σ_{α_x} at the start of a phase of the construction described in the preceding paragraph is equal to a previously constructed auxiliary or secondary element A, a pair $\{p, A\}$, an element of a set σ'_β with β trapped, or a previously constructed model My. This can be shown as follows. Suppose, first, that x is non-trapped, so that Mx contains at least three elements, all of height at least $H + 1$. Thus, Mx clearly cannot equal any auxiliary element A or pair $\{p, A\}$, nor can it equal any secondary element B since every such element is $B \in^* A$ for some auxiliary A, and thus we would have $A' \in^* A$, where $A' \in Mx$. Moreover, $Mx = My$ cannot hold if the variables x and y are distinct because at every stage of our construction the sets σ_α remain pairwise disjoint. Finally, $Mx \notin \bigcup_{\beta\ \mathrm{trapped}} \sigma'_\beta$, because, as observed above, Mx has elements of height at least $H + 1$. This shows that if x is non-trapped, neither insertion of Mx into σ_{α_x} nor the subsequent stabilization process disrupts the disjointness of sets σ_α. Next consider the case in which the variable x is trapped, but in which the place α_x is non-trapped (since otherwise we would not have to insert Mx in σ_{α_x}, because by (ii) of condition C1′, Mx would already be there). Since x is trapped, Mx has height at most H. Hence Mx is different from any auxiliary or secondary element resident at any safe place, since these elements have height greater than H. For the same reason Mx is different from any pair $\{p, A\}$ with A resident at a safe place. On the other hand, Mx cannot equal any auxiliary A or pair $\{p, A\}$ with A resident at a cyclic place, nor can it equal any secondary element B resident at a cyclic place and different from the γ_i's, because for each such pair or element C, we have $\gamma_j \in^* C$ for some $j \in \{1, 2, \ldots, k\}$, whereas by (iv) of C1′, it follows from $Mx \subseteq \bigcup_{\beta \text{ trapped}} \sigma'_\beta$ that $\gamma_j \in^* Mx$ is false for all γ_j. Moreover, by (ii) of C1′, we have $Mx \in \sigma'_{\alpha_x}$, and hence since α_x is non-trapped it follows by the disjointness of the σ'_α that $Mx \notin \bigcup_{\beta \text{ trapped}} \sigma'_\beta$. For the same reason, Mx can neither equal any γ_i which does not belong to σ'_{α_x} nor equal an element $\gamma_i \in \sigma'_{\alpha_x}$ since we suppose that Mx is not in σ_{α_x} before processing of the variable x (whereas all γ_i are inserted into the σ_α to which they belong during the initialization phase). Finally Mx cannot equal any My with y distinct from x since at every stage of our construction the sets σ_α remain pairwise disjoint.

As in the absence of trapped places, the insertion of Mx into σ_{α_x} does not upset any relationship $\text{Un}(My_i) \subseteq Mu_i$, since whenever Mx is inserted into My_i we have $\alpha_x \subseteq y_i$, and then $\alpha \subseteq x$ implies $\alpha \subseteq u_i$ by condition C3′, so $\text{Un}(My_i) \subseteq Mu_i$ remains valid.

Taken all in all it follows that, just as in the simpler case considered previously (i.e., in the absence of trapped places), all the clauses of P will be modeled correctly at the end of the series of steps described. This shows that P is satisfiable by a model having Π as its set of places if and only if conditions C1′, C2′, and C3′ are satisfied. Note finally that, even though the wording of the preceding occasionally assumes that cyclic places are present, no real use is made of the existence of cyclic places; i.e., simply by ignoring what is said about such places we can still build a model of P.

This completes the proof of decidability of multilevel syllogistic extended by the general union operator in all possible cases.

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