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Low and high frequency approximations to eigenvibrations of string with double contrasts

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ABSTRACT

We study eigenvibrations for inhomogeneous string consisting of two parts with strongly contrasting stiffness and mass density. In this work we treat a critical case for the *high frequency approximations*, namely the case when the order of mass density inhomogeneity is the same as the order of stiffness inhomogeneity, with heavier part being softer. The limit problem for high frequency approximations depends nonlinearly on the spectral parameter. The quantization of the spectral semiaxis is applied in order to get a close approximations of eigenvalues as well as eigenfunctions for the prime problem under perturbation.

1. Introduction and problem statement

Models with high contrasts are widely studied since their unusual properties give insight into the behaviour of new meta and nanomaterials, including those which already exist or are reachable nowadays via modern technologies. The corresponding mathematical problems often cause computational difficulties and require new methods of numerical approximation. A system under consideration possessing two components with double high contrasts, both in stiffness and mass density, expresses two distinguishing cases of the limit eigenvibration behaviour for each of low and high frequency levels. The description of such systems should not be restricted to the construction of classical number-by-number eigenfunction asymptotics, which are called *low frequency approximations*. They only ensure close approximations to several eigenfunctions corresponding to the bottom of the spectrum. For more precise eigenfunction description in the upper part of the spectrum the classical approach sets the requirement for ε to be negligibly small. Nevertheless, in actual physical models the parameter ε, denoting the ratio of inhomogeneity for a certain physical characteristic, is often *small but fixed*. Then describing actual vibrating systems, a problem of adequate approximation to eigenfunctions with large numbers arises. In order to solve the problem we propose a new asymptotics, being called *high frequency approximations*, and compare them with the classical ones. The high frequency approximations quite precisely describe eigenvibrations for which low frequency approximations are not precise enough.

Methods and results. Starting from an operator with a discrete spectrum a classical spectral analysis provides the discreteness of low frequency limits for eigenelements of the system with high contrasts. Nevertheless, the standard approach misses

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a certain important characteristic, because the completeness of eigenfunction system is lost in the limit. Accomplishing the investigation and filling up the gaps in the limit behaviour description we construct and justify high frequency approximations to the eigenfunctions. The quantization conditions play a vital part in the asymptotics providing an ε-network on the spectral axis in the range of approximation. Therefore even the leading terms in the spectral approximations change along with ε. Thus we obtain a quite precise approximations to eigenelements of the prime problem with a fixed small ε. Comparing to the previous study of the stiff problems [1], where the leading terms of high frequency approximations are independent of ε and the quantization provides a right choice of the correctors, in the present problem the quantization conditions, arising in particular from matching WKB and power series expansions, come along with the choice of the leading terms.

The preliminary results on the limit behaviour of the system under consideration have been discussed in [2]. The question of asymptotic description of low and high frequency eigenvibrations originates in work [3] arising again in [4,5,1,6] for problems with perturbations of the stiffness only. Elastic problems with perturbations of stiffness and mass density, either with other geometries or at different perturbation rates, have been studied in [6–12].

Problem statement. Let a soft and heavy part of the string, which occupies an interval $(a, 0)$, be complemented by a stiff and relatively light body part occupying $(0, b)$ with $a < 0 < b$. We consider a stiffness coefficient being $k(x)$ on $(a, 0)$ and $\varepsilon\varkappa(x)$ on $(0, b)$, and mass density being $\varepsilon r(x)$ on $(a, 0)$ and $\rho(x)$ on $(0, b)$, with all functions being positive and smooth in $[a, 0]$ and $[0, b]$ respectively. We assume that eigenvibrations of the string are described by the self-adjoint eigenvalue problem

$$(k(x)u_\varepsilon')' + \varepsilon\lambda^\varepsilon r(x)u_\varepsilon = 0, \quad x \in (a, 0),\ u_\varepsilon(a) = 0, \tag{1}$$

$$\varepsilon(\varkappa(x)u_\varepsilon')' + \lambda^\varepsilon \rho(x)u_\varepsilon = 0, \quad x \in (0, b),\ u_\varepsilon(b) = 0, \tag{2}$$

$$u_\varepsilon(-0) = u_\varepsilon(+0), \qquad k(0)u_\varepsilon'(-0) = \varepsilon\varkappa(0)u_\varepsilon'(+0). \tag{3}$$

We investigate the question how the eigenvibrations of the media, namely eigenvalues λ^ε and eigenfunctions u_ε, change if the parameter ε tends to 0. More precisely, we look for the good approximations of λ^ε and u_ε as $\varepsilon \to 0$.

2. Low frequency approximations

It is well known that for each fixed $\varepsilon > 0$ the spectrum of problem (1)–(3) is real and discrete, consisting of simple eigenvalues that form a sequence $0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_n^\varepsilon < \cdots \to \infty$ as $n \to \infty$. The corresponding eigenfunctions $\{u_{\varepsilon,n}\}_{n=1}^\infty$ form a basis in $L^2(a, b)$. Moreover, for each number n the eigenvalue branch λ_n^ε is a continuous function of $\varepsilon \in (0, 1)$ such that $\lambda_n^\varepsilon \le c_n\varepsilon$ with a positive constant c_n independent of ε, which follows from the mini–max principle since quadratic forms are continuously depending on ε [13].

Studying the asymptotic behaviour as $\varepsilon \to 0$ of each eigenvalue branch λ_n^ε with fixed number n and corresponding eigenfunctions $u_{\varepsilon,n}$, we immediately have the convergence $\varepsilon^{-1}\lambda_n^\varepsilon \to \lambda_n$ and $u_{\varepsilon,n} \to U_n$, where $U_n = 0$ in $(a, 0)$ and U_n in $(0, b)$ coincides with an eigenfunction u_n^+ of the limit problem (6) for the eigenvalue λ_n.

We look for the approximations of eigenvalues and eigenfunctions in the form

$$\lambda_n^\varepsilon \sim \varepsilon\mu_n + \varepsilon^2\nu_n + \cdots, \qquad u_{\varepsilon,n} \sim u_n(x) + \varepsilon w_n(x) + \cdots, \quad x \in (a, b). \tag{4}$$

Constructing standardly the asymptotic expansions we first define the leading terms, which satisfy the problem

$$\text{for } x \in (a, 0): \quad (k(x)u_n')' = 0, \quad u_n(a) = 0, \quad u_n'(-0) = 0. \tag{5}$$

Hence $u_n \equiv 0$ on $(a, 0)$ and therefore

$$\text{for } x \in (0, b): \quad (\varkappa(x)u_n')' = -\mu_n\rho(x)u_n, \quad u_n(+0) = u_n(b) = 0. \tag{6}$$

Since we are looking for the eigenfunction approximations, which are supposed to be different from zero, the limit μ_n has to be an eigenvalue with corresponding eigenfunction u_n of problem (6).

Let us fix an eigenvalue μ_n of (6), and corresponding eigenfunction u_n such that $\int_0^b \rho u_n^2 \mathrm{d}x = 1$. Then the next terms of (4) satisfy the problem

$$\text{for } x \in (a, 0): \quad (k(x)w_n')' = 0, \quad w_n(a) = 0, \quad (kw_n')(-0) = (\varkappa u_n')(+0). \tag{7}$$

Therefore on $(a, 0)$ we have $w_n = (\varkappa u_n')(+0)\int_a^x k^{-1}(t)\mathrm{d}t$, and

$$\text{for } x \in (0, b): \quad (\varkappa(x)w_n')' + \mu_n\rho(x)w_n = -\nu_n\rho(x)u_n, \tag{8}$$

$$w_n(b) = 0, \quad w_n(+0) = v_n(-0). \tag{9}$$

The solvability of (8) and (9) along with (7) and the normalization of u_n implies

$$\nu_n = -(kw_nw_n')(-0) = -\int_a^0 k(w_n')^2\mathrm{d}x. \tag{10}$$

Then on the interval $(0, b)$ we fix a unique w_n such that $\int_0^b \rho u_n w_n \mathrm{d}x = 0$.

Justification of low frequency approximations. We use the same letter f both for a function defined on the interval (a, b) and a vector (f_-, f_+), where f_-, f_+ are the restrictions of f to $(a, 0)$ and $(0, b)$ respectively. Let $\mathcal{L}$ be the Hilbert space $L^2_r(a, 0) \times L^2_\rho(0, b)$ with the scalar product $(u, v)_{\mathcal{L}} = \int_a^0 r u_- v_- \mathrm{d}x + \int_0^b \rho u_+ v_+ \mathrm{d}x$ and norm $\|u\| = (u, u)^{1/2}_{\mathcal{L}}$, where $u = (u_-, u_+)$. Let us introduce the matrix operator $\mathcal{A}_\varepsilon$ in $\mathcal{L}$

$$\mathcal{A}_\varepsilon = \begin{pmatrix} -\dfrac{1}{\varepsilon r}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(k\dfrac{\mathrm{d}}{\mathrm{d}x}\right) & 0 \\ 0 & -\dfrac{\varepsilon}{\rho}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\varkappa\dfrac{\mathrm{d}}{\mathrm{d}x}\right) \end{pmatrix}.$$

with the domain

$$\mathcal{D}(\mathcal{A}_\varepsilon) = \big\{u \in \mathcal{L}\colon u_- \in W^1_2(a, 0),\ u_-(a) = 0,\ u_+ \in W^1_2(0, b),\ u_+(b) = 0,$$
$$u_-(0) = u_+(0),\ k(0)u'_-(0) = \varepsilon\varkappa(0)u'_+(0)\big\}.$$

The $\mathcal{A}_\varepsilon$ is a self-adjoint operator with a compact resolvent. The spectrum $\sigma(\mathcal{A}_\varepsilon)$ is the set of all eigenvalues of (1)–(3).

Let B be a self-adjoint operator in Hilbert space H with a domain $\mathcal{D}(B)$. Recall that a pair $(\mu, u) \in \mathbb{R} \times \mathcal{D}(B)$ with $\|u\|_H = 1$ is a quasimode of the operator B with an accuracy up to $\sigma > 0$ if $\|(B - \mu I)u\|_H \le \sigma$.

Lemma 1. *Suppose that the spectrum of B is discrete. If (μ, u) is a quasimode of B with accuracy to σ, then interval $[\mu-\sigma, \mu+\sigma]$ contains an eigenvalue of B. Furthermore, if segment $[\mu - \tau, \mu + \tau]$, $\tau > 0$, contains one and only one eigenvalue λ of B, then $\|u - v\|_H \le 2\tau^{-1}\sigma$, where v is an eigenfunction of B for the eigenvalue λ, $\|v\|_H = 1$ [14].*

Theorem 2. *For each $n \in \mathbb{N}$ there exists $C_n > 0$ such that the $C_n\varepsilon^2$-vicinity of $\varepsilon\mu_n$ contains exactly one eigenvalue λ^ε_n of problem (1)–(3):*

$$|\lambda^\varepsilon_n - \varepsilon\mu_n| \le C_n\varepsilon^2. \tag{11}$$

The corresponding normalized eigenfunction $u_{\varepsilon,n}$ satisfies the estimate $\|u_{\varepsilon,n} - u_n - \varepsilon w_n\|_{L_2(a,b)} \le \tilde{C}_n\varepsilon^2$, with a certain $\tilde{C}_n > 0$ independent of ε.

Proof. We introduce a corrector $\phi_n(x) = a^{-1}w'_n(+0)x(x-a)$ on $(a, 0)$ and $\phi_n(x) = 0$ on $(0, b)$ such that $U^\varepsilon_n = u_n+\varepsilon(w_n+\phi_n)$ belongs to $\mathcal{D}(\mathcal{A}_\varepsilon)$. Let $\Lambda^\varepsilon_n = \varepsilon\mu_n + \varepsilon^2\nu_n$ and $\tilde{U}^\varepsilon_n = \tau^\varepsilon_n U^\varepsilon_n$ with $\tau^\varepsilon_n = \|U^\varepsilon_n\|^{-1}_{\mathcal{L}}$. By the construction

$$\|\mathcal{A}_\varepsilon\tilde{U}^\varepsilon_n - \Lambda^\varepsilon_n\tilde{U}^\varepsilon_n\|^2_{\mathcal{L}} \le K_1\varepsilon^4(\tau^\varepsilon_n)^2|\mu_n + \varepsilon\nu_n|^2(u'_n(+0)^2 + w'_n(+0)^2) + K_2\varepsilon^6(\tau^\varepsilon_n)^2\nu^2_n\|w_n\|^2_{L^2_\rho(0,b)}, \tag{12}$$

with positive constants K_j independent of ε and n, and also

$$|\tau^\varepsilon_n| \le (1 - \varepsilon\|w_n + \phi_n\|_{\mathcal{L}})^{-1} \le 1 + \hat{C}_n\varepsilon \tag{13}$$

for ε small enough. Therefore a pair Λ^ε_n and $\tilde{U}^\varepsilon_n$ is a quasimode of $\mathcal{A}_\varepsilon$ with the accuracy up to $C_n\varepsilon^2$. By Lemma 1, in $C_n\varepsilon^2$-vicinity of Λ^ε_n there exists a certain eigenvalue λ^ε_j of (1)–(3). Additionally, it can be easily shown that the eigenvalues converge saving multiplicity, $\varepsilon^{-1}\lambda^\varepsilon_n \to \mu_n$. Since the limit problem has only simple eigenvalues, in a certain $\hat{C}_n\varepsilon$-vicinity of μ_n there is no other eigenvalues of (1)–(3) that provides (11). Applying again Lemma 1 finishes the proof. □

Note that low frequency vibrations vanish in $(a, 0)$ as $\varepsilon \to 0$. This naturally raises the question on the possibility of constructing other *nontrivial on* $(a, 0)$ approximations of eigenvibrations addressed next.

3. High frequency approximations

Considering sufficiently large eigenvalues $\lambda^\varepsilon_n \sim \varepsilon^{-1}(\omega + \varepsilon\omega_1)^2$ with $\omega > 0$, we look for the asymptotic expansions of eigenfunctions $u_{\varepsilon,n}(x) \sim Y(\varepsilon, x)$ with

$$Y(\varepsilon, x) = \begin{cases} v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x), & x \in (a, 0), \\ (c_0(x) + \varepsilon c_1(x)) \sin\gamma_\varepsilon S(x) + \varepsilon c_2(x)\cos\gamma_\varepsilon S(x), & x \in (0, b), \end{cases} \tag{14}$$

where v_0 is different from zero and $\gamma_\varepsilon = \frac{\omega}{\varepsilon} + \omega_1$. The expansion in form (14) consists of power series on the interval $(a, 0)$ and two-term short-wave (WKB) approximation [15] on $(0, b)$ since Eq. (2) contains a small parameter near the highest derivative. Substituting these expressions into equation and boundary condition (1) gives

$$(kv'_0)' + \omega^2 r v_0 = 0, \quad v_0(a) = 0, \tag{15}$$

$$(kv'_1)' + \omega^2 r v_1 = -2\omega\omega_1 r v_0, \quad v_1(a) = 0, \tag{16}$$

$$(kv'_2)' + \omega^2 r v_2 = -\omega_1^2 r v_0 - 2\omega\omega_1 r v_1, \quad v_2(a) = 0. \tag{17}$$

Next, we substitute $Y(\varepsilon, x)$ into (2):

$$\varepsilon\gamma_\varepsilon^2\left(-\varkappa S'^2+\rho\right)Y(\varepsilon,\cdot)+\varepsilon\gamma_\varepsilon\Big(2\varkappa S'c_0'+(\varkappa S')'c_0\Big)\cos\gamma_\varepsilon S+\varepsilon\omega\Big(2\varkappa S'c_1'+(\varkappa S')'c_1\Big)\cos\gamma_\varepsilon S$$
$$-\varepsilon\Big(2\omega\varkappa S'c_2'+\omega(\varkappa S')'c_2-(\varkappa c_0')'\Big)\cos\gamma_\varepsilon S=O(\varepsilon^2). \tag{18}$$

Equating the expressions in the large parentheses to zero we minimize the discrepancy in (18). The *eikonal* equation $\varkappa S'^2=\rho$ has a solution

$$S(x)=\int_x^b \varkappa^{-1/2}(\tau)\rho^{1/2}(\tau)\mathrm{d}\tau,\quad x\in(0,b).$$

Consequently, the *transport* equation $2\varkappa S'c'+(\varkappa S')'c=0$ admits a solution $c(x)=\varkappa^{-1/4}(x)\rho^{-1/4}(x)$ up to a constant multiplier. Therefore $c_0(x)=\beta_0c(x)$ and $c_1(x)=\beta_1c(x)$. Introducing h as a unique solution of the problem

$$2\varkappa S'h'+(\varkappa S')'h=(\varkappa c')'\quad \text{for } x<b,\ h(b)=0,$$

we set $c_2(x)=\beta_0\omega^{-1}h(x)$ providing the boundary condition $Y(\varepsilon,b)=0$ is satisfied. By construction $Y(\varepsilon,\cdot)$ formally solves Eq. (1) up to the terms of order ε^3 and Eq. (2) up to the terms of order ε^2.

We now apply interface conditions (3) in order to define parameters ω, ω_1, β_0 and β_1. Before that, regularizing the ε-dependence of $Y(\varepsilon,+0)$ we apply the restriction

$$\left(\frac{\omega}{\varepsilon}+\omega_1\right)S(0)=\delta+\pi l,\quad \delta\in(-\pi/2,\pi/2],\ l\in\mathbb{Z}. \tag{19}$$

Satisfying the interface conditions up to the terms of order ε^2, we set

$$\begin{cases}v_0(0)=(-1)^l\beta_0c(0)\sin\delta\\ k(0)v_0'(0)=(-1)^l\beta_0\omega S'(0)\varkappa(0)c(0)\cos\delta,\end{cases} \tag{20}$$

$$\begin{cases}v_1(0)=(-1)^l(\beta_1c(0)\sin\delta+g_1)\\ k(0)v_1'(0)=(-1)^l(\beta_1\omega S'(0)\varkappa(0)c(0)\cos\delta+\beta_0\varkappa(0)g_2),\end{cases} \tag{21}$$

where $g_1=\beta_0\omega^{-1}h(0)\cos\delta$, $g_2=\omega_1S'(0)c(0)\cos\delta+(c'(0)-S'(0)h(0))\sin\delta$.

Combining (15) and (20) we obtain that v_0 is a solution to the problem

$$\begin{aligned}&(kv')'+\omega^2rv=0\quad\text{in }(a,0),\\ &v(a)=0,\qquad k(0)v'(0)\sin\delta-\omega\varkappa(0)S'(0)v(0)\cos\delta=0.\end{aligned} \tag{22}$$

Proposition 3. *For every $\omega>0$ there exists a unique $\delta(\omega)\in(-\pi/2,\pi/2]$ such that problem (22) has a nontrivial solution v.*

Proof. If ω^2 is an eigenvalue of the problem $(kv')'+\omega^2rv=0$, $v(a)=0$, $v(0)=0$, we put $\delta(\omega)=0$. Otherwise we consider the eigenvalue problem

$$(kv')'+\omega^2rv=0\quad\text{in }(a,0),\qquad v(a)=0,\qquad k(0)v'(0)+\mu v(0)=0 \tag{23}$$

with respect to the spectral parameter μ. For each ω under consideration the problem has a unique eigenvalue $\mu(\omega)$, which is due to the fact that the spectral parameter is missed in equation. Therefore $\delta(\omega)$ can be found as a unique root in $(-\pi/2,\pi/2]$ of the equation

$$\omega\varkappa(0)|S'(0)|\operatorname{cotan}\delta=\mu(\omega). \tag{24}$$

Recall that $S'(0)<0$. □

Fixing an arbitrary $\omega>0$ we also fix $v_0=v(\omega,x)$ being corresponding eigenfunction of nonlinear pencil (22) with $\delta=\delta(\omega)$ defined by Proposition 3. Let additionally v_0 be unity normalized in $L_r^2(a,0)$. Consequently, (20) provides

$$\beta_0=\begin{cases}\dfrac{(-1)^lv_0(0)}{c(0)\sin\delta(\omega)} & \text{if }\delta(\omega)\neq0,\\[2ex] \dfrac{(-1)^lk(0)v_0'(0)}{\omega\varkappa(0)S'(0)c(0)} & \text{if }\delta(\omega)=0.\end{cases}$$

We conclude from condition (22) that the function $\beta_0(\omega)$ is continuous at every point ω_* for which $\delta(\omega_*)=0$. From (21) and (24) we obtain

$$\begin{aligned}&(kv_1')'+\omega^2rv_1=-2\omega\omega_1rv_0\quad\text{in }(a,0),\\ &v_1(a)=0,\qquad k(0)v_1'(0)+\mu(\omega)v_1(0)=\varkappa(0)f,\end{aligned} \tag{25}$$

where $f=g_2\sin\delta(\omega)-\omega S'(0)g_1\cos\delta(\omega)$ and $\delta(\omega)\neq0$. The problem admits a solution if and only if $v_0(0)\varkappa(0)f=-2\omega\omega_1$, since $\mu(\omega)$ is an eigenvalue of (23). This solvability condition can be derived multiplying the equation by v_0 and integrating twice by parts. It may be written in the form

$$\omega_1 = \frac{\left(h(0)S'(0) - c'(0)\sin^2\delta(\omega)\right)v^2(\omega, 0)}{(2\omega + \varkappa(0)S'(0)\cos\delta(\omega))\,c(0)\sin\delta(\omega)} \quad \text{if } \delta(\omega) \neq 0.$$

Thus we get ω_1 as a function of ω. Additionally, we obtain

$$\omega_1 = -k(0)v_0'(0)c_2(0)(2\omega)^{-1} \quad \text{if } \delta(\omega) = 0.$$

We now can find v_1, which is ambiguously determined. Subordinating it to the condition $\int_a^0 rv_0v_1dx = 0$ we fix it uniquely. Then β_1 is given by (21). We fix an arbitrary v_2 being solution of (17).

Let us return to condition (19). Now it may be considered as the countable set of equations for ω:

$$\left(\frac{\omega}{\varepsilon} + \omega_1(\omega)\right)S(0) - \delta(\omega) = \pi l, \quad l \in \mathbb{Z}. \tag{26}$$

Since $\omega_1(\omega)$ can have a vertical asymptote in the interval $I = [0, \frac{1}{2}\varkappa(0)|S'(0)|)$, Eq. (26) can have roots in I. More subtle analysis shows that for each l there always exists a unique root of (26) in the set $[\frac{1}{2}\varkappa(0)|S'(0)|, \infty)$ because $\omega_1(\omega) \to 0$ as $\omega \to +\infty$ and quantity $\delta(\omega)$ is bounded. We consider the roots that increase along with l.

Definition 4. We say that $\omega(l)$ is an *admissible limit frequency* for given $\varepsilon > 0$ and $l \in \mathbb{Z}$ if it is the largest root of (26).

Let us establish connection between the exact eigenfrequencies $\sqrt{\lambda_l^\varepsilon}$ and admissible limit frequencies $\omega(l)$ in the case of constant coefficients. Indeed, for $k = \varkappa = r = \varrho = 1$ we have $\sqrt{\lambda_n^\varepsilon} = \sqrt{\varepsilon}\pi n(b - \varepsilon a)^{-1}$, $n \in \mathbb{N}$. Counting the admissible frequencies in this case we note that $S(x) = b - x$ and $v_0(x) = C_0 \sin\omega(x - a)$ providing, via the proof of Proposition 3, $\delta(\omega) = 0$ if $\omega = \pi n a^{-1}$ for natural n and cotan $\delta(\omega) = \omega^{-1}\mu(\omega)$ for all other ω. Moreover, (23) yields $\mu(\omega) = -v_0'(0)v_0(0)^{-1} = \omega\text{cotan }(\omega a)$ gaining $\delta(\omega) = \arctan(\tan(\omega a)) \in (-\pi/2, \pi/2)$ or $\delta = \pi/2$. Observe that $c_0(x) = \beta_0$, $h(x) = 0$ and $c_2(x) = 0$ providing $\omega_1 = 0$. Then (26) becomes $\omega b\varepsilon^{-1} = \arctan(\tan(\omega a)) + \pi l$. Therefore, $\omega b\varepsilon^{-1} \in (-\pi/2 + \pi l, \pi/2 + \pi l]$ and $\tan(\omega b\varepsilon^{-1}) = \tan(\omega a)$ providing $\omega = \varepsilon\pi k(l)(b - \varepsilon a)^{-1}$ for $k(l) \in \mathbb{N} \cap K_l^\varepsilon$ with $K_l^\varepsilon = (z_\varepsilon(l - 1/2), z_\varepsilon(l + 1/2)]$, where $z_\varepsilon = (1 + \varepsilon a(b - \varepsilon a)^{-1})^{-1}$. Since $z_\varepsilon > 1$ and therefore the length $|K_l^\varepsilon|$ is also larger then 1, we have at least one natural $k(l) \in K_l^\varepsilon$. Picking up the maximal value $k_l^{\max}(\varepsilon) \in K_l^\varepsilon \cap \mathbb{N}$ we fix the admissible frequency $\omega(l) = \varepsilon\pi k_l^{\max}(\varepsilon)(b - \varepsilon a)^{-1}$. Note that $k_l^{\max}(\varepsilon) = l$ for the range of numbers $l < \frac{b+\varepsilon a}{2\varepsilon|a|}$. Therefore, $\sqrt{\lambda_l^\varepsilon} = \frac{\omega(l)}{\sqrt{\varepsilon}}$ for $l < \frac{b+\varepsilon a}{2\varepsilon|a|}$.

Having exact correspondence for the range of eigenfrequencies and admissible frequencies in the case of constant coefficients, in general case we further use the set of admissible frequencies as the first approximation for the eigenfrequencies. Let Φ_ε denote the set of all admissible limit frequencies. The subset Φ_ε of $\mathbb{R}_+$ is thick enough, the distance between neighboring roots is comparable with ε. In some sense (26) could be regarded as a kind of WKB quantization condition. The positive spectral ray $\omega > 0$ is covered by the ε-net Φ_ε, for each point of which we can construct the asymptotics (14). For each admissible frequency $\omega \in \Phi_\varepsilon$ we will denote by $Y_\omega(\varepsilon, x)$ the corresponding asymptotic solution (14).

4. Justification of high frequency approximations

The function $Y_\omega(\varepsilon, x)$ can be used to construct a quasimode of the operator $\mathcal{A}_\varepsilon$. Clearly, $Y_\omega \in \mathcal{L}$, but $Y_\omega \notin \mathcal{D}(\mathcal{A}_\varepsilon)$ because of discontinuity at $x = 0$. Let us introduce functions $\zeta_0, \zeta_1 \in C^2(a, 0)$ such that $\zeta_0(a) = 0$, $\zeta_0(0) = 1$, $\zeta_0'(0) = 0$ and $\zeta_1(a) = 0$, $\zeta_1(0) = 0$, $\zeta_1'(0) = 1$. Both functions are extended by zero into $(0, b)$. Introducing

$$\tau_0(\varepsilon) = (Y_\omega(\varepsilon, +0) - Y_\omega(\varepsilon, -0))\,\varepsilon^{-2}, \qquad \tau_1(\varepsilon) = \left(\varepsilon Y_\omega'(\varepsilon, +0) - Y_\omega'(\varepsilon, -0)\right)\varepsilon^{-2},$$

which are bounded in ε by construction, we obtain that the function

$$\tilde{Y}_\omega(\varepsilon, \cdot) = Y_\omega(\varepsilon, \cdot) + \varepsilon^2(\tau_0(\varepsilon)\zeta_0 + \tau_1(\varepsilon)\zeta_1)$$

belongs to $\mathcal{D}(\mathcal{A}_\varepsilon)$. Setting $\Upsilon_\omega(\varepsilon, \cdot) = \|\tilde{Y}_\omega\|_{\mathcal{L}}^{-1} \cdot \tilde{Y}_\omega(\varepsilon, \cdot)$ we prove the following estimate $\|(\mathcal{A}_\varepsilon - \varepsilon\gamma_\varepsilon^2 I)\Upsilon_\omega(\varepsilon, \cdot)\|_{\mathcal{L}} \leq C\varepsilon^2$ by recalling $\varepsilon^{-1}(\omega + \varepsilon\omega_1)^2 = \varepsilon\gamma_\varepsilon^2$.

Proposition 5. *The pair $(\varepsilon\gamma_\varepsilon^2, \Upsilon_\omega(\varepsilon, \cdot))$ is a quasimode of $\mathcal{A}_\varepsilon$ with accuracy to $O(\varepsilon^2)$ for every admissible frequency $\omega \in \Phi_\varepsilon$.*

Proposition 6. *For the range of numbers $n \leq \theta\varepsilon^{\sigma-1/2}$ with arbitrary $\theta > 0$ and $0 < \sigma < 1/2$, the eigenvalues satisfy the estimate $|\lambda_n^\varepsilon - \varepsilon\mu_n| \leq K_*\varepsilon^{1+2\sigma}$.*

Proof. In order to improve (11) we calibrate (12). Eigenfrequencies $\eta_n = \mu_n^{1/2}$ and normalized eigenfunctions u_n of problem (6) can be represented as [15]

$$\eta_n = \frac{\pi n}{S(0)} + \frac{\pi S(0)}{n} + O\left(\frac{1}{n^3}\right) \quad \text{as } n \to +\infty, \tag{27}$$

$$u_n = \sqrt{2/S(0)}(\varkappa\rho)^{-1/4}(\sin\eta_n S)(1 + O(\eta_n^{-1})) \quad \text{on } (0, b), \tag{28}$$

where (28) is uniform on $[0, b]$ and admits differentiation in x. Then we have the approximation of the right-hand side in (8) and (9)

$$v_n(-0) = \eta_n\beta_1\beta_2(1 + O(\eta_n^{-1})), \qquad v_n = -\eta_n^2\beta_1^2\beta_2(1 + O(\eta_n^{-1})), \quad n \to \infty, \tag{29}$$

with $\beta_1 = \sqrt{2/S(0)}(\varkappa\rho)^{-1/4}(0)$ and $\beta_2 = \int_a^0 k(t)^{-1}dt$. Since there exists the fundamental set of solutions corresponding (8) in the form [15]

$$y_n = (1 + O(\eta_n^{-1}))(\varkappa\rho)^{-1/4}\sin\eta_n S \quad \text{and} \quad g_n = (1 + O(\eta_n^{-1}))(\varkappa\rho)^{-1/4}\cos\eta_n S,$$

w_n admits representation $w_n = p_n(x)y_n(x) + q_n(x)g_n(x)$ for certain functions p_n and q_n. Exploring this structure of solution in problem (8) and (9) we obtain

$$w_n(x) = (1 + O(\eta_n^{-1}))\eta_n K(\varkappa\rho)^{-1/4}(x)\cos\eta_n S(x), \quad n \to \infty, \tag{30}$$

with constant $K = \beta_1\beta_2(\varkappa\rho)^{1/4}(0)S(0)^{-1}$. Then (30) and (27) provide

$$\|w_n\|_{L^2_\rho(0,b)} \le K_1 n, \qquad |w_n'(+0)| \le K_2 n. \tag{31}$$

Finally, counting (27)–(31) in (12) we obtain

$$\|\mathcal{A}_\varepsilon\tilde{U}_n^\varepsilon - \Lambda_n^\varepsilon\tilde{U}_n^\varepsilon\|_{\mathcal{L}}^2 \le K_3|\tau_n^\varepsilon|\varepsilon^2 n^2\sqrt{1 + \varepsilon^6 n^6} \le K_4\varepsilon^{1+2\sigma}, \tag{32}$$

for $n \le \theta\varepsilon^{\sigma-1/2}$ and $|\tau_n^\varepsilon| \le 1 - K_5\varepsilon^{\sigma+1/2}$, which follows from (13). Then the application of Lemma 1 finishes the proof. □

Theorem 7. *Let $\theta \ge 1$, $0 < \sigma < 1/2$, $0 < \gamma < 1/2 - \sigma$. If $\omega = \omega(n)$ is an admissible limit frequency from the number range $n \in [\theta^{-1}\varepsilon^{-\gamma}, \theta\varepsilon^{\sigma-1/2}]$ and $\delta(\omega) \ne \pi/2$ then the eigenvalue λ_n^ε and eigenfunction $y_{\varepsilon,n}$ satisfy the estimates*

$$|\lambda_n^\varepsilon - \varepsilon^{-1}(\omega + \varepsilon\omega_1(\omega))^2| \le \alpha_1\varepsilon^2, \qquad \|y_{\varepsilon,n} - \Upsilon_\omega(\varepsilon,\cdot)\|_{L_2(a,b)} \le \alpha_2\varepsilon^{1+\gamma},$$

with positive constants α_1, α_2 being independent of ε.

Proof. Let $\omega_n^\varepsilon = \sqrt{\lambda_n^\varepsilon}$. Proposition 6 and (27) for the given number range yield

$$\omega_n^\varepsilon = \varepsilon^{1/2}\pi\left(\frac{n}{S(0)} + \frac{S(0)}{n}\right) + O\left(\varepsilon^{\frac{1}{2}+2\sigma+\gamma}\right) \quad \text{and} \quad \lambda_n^\varepsilon = \frac{\varepsilon\pi^2 n^2}{S^2(0)} + O(\varepsilon). \tag{33}$$

We now estimate the distance between neighboring eigenvalues of $\mathcal{A}_\varepsilon$. From (33) we have $\lambda_{n+1}^\varepsilon - \lambda_n^\varepsilon = \varepsilon\pi^2(2n+1)S^{-2}(0) + O(\varepsilon)$. If $n \ge \theta^{-1}\varepsilon^{-\gamma}$ then

$$|\lambda_{n+1}^\varepsilon - \lambda_n^\varepsilon| \ge 2\pi^2 S^{-2}(0)n\varepsilon + O(\varepsilon) \ge \theta_0\varepsilon^{1-\gamma}, \tag{34}$$

with a constant θ_0 being positive and independent of n. By the similar argument,

$$|\omega_{n+1}^\varepsilon - \omega_n^\varepsilon| \ge \sqrt{\varepsilon}\left(\frac{\pi}{S(0)} - c\left(n^{-2} + \varepsilon^{2\sigma+\gamma}\right)\right) \ge \sqrt{\varepsilon}\left(\frac{\pi}{S(0)} - \theta_1\varepsilon^\gamma\right). \tag{35}$$

Let for a certain number l the admissible frequency $\omega_* = \omega(l)$ minimize the difference $|\sqrt{\varepsilon}\omega_n^\varepsilon - (\omega_* + \varepsilon\omega_1(\omega_*))|$, which equals $|\varepsilon\pi(n - l - \frac{\delta}{\pi}) + O(\varepsilon^{1+\gamma})|$ by (26) and (33). Note that if $|\delta| < \frac{\pi}{2}$ the latter is minimized only for $l = n$ (for sufficiently small ε) and then we have

$$\left|\sqrt{\varepsilon}\omega_n^\varepsilon - (\omega_* + \varepsilon\omega_1(\omega_*))\right| \le |\delta(\omega_*)|S^{-1}(0)\varepsilon + \theta_2\varepsilon^{1+\gamma}. \tag{36}$$

Suppose that frequency ω_{n+1}^ε (the same for ω_{n-1}^ε) also satisfies the last inequality. Then we obtain the estimate

$$|\omega_{n+1}^\varepsilon - \omega_n^\varepsilon| \le 2|\delta(\omega)|S^{-1}(0)\sqrt{\varepsilon} + 2\theta_2\varepsilon^{1+\gamma}$$

that contradicts (35) for $\varepsilon < \varepsilon_0$ because $2|\delta(\omega)| < \pi$. The number ε_0 can be found from the equation $S(0)\varepsilon^\gamma(2\theta_2\varepsilon^{1/2} + \theta_1) = \pi - 2|\delta(\omega)|$. Hence, ω_n^ε is a unique eigenfrequency that satisfies (36) with $\omega(n)$ being the root of (26).

In view of Lemma 1 and Proposition 5, we can improve inequality (36) to $|\lambda_n^\varepsilon - \varepsilon^{-1}(\omega + \varepsilon\omega_1(\omega))^2| \le \alpha_1\varepsilon^2$. Repeated application of Lemma 1 enables us to write $\|u_{\varepsilon,n} - \Upsilon_\omega(\varepsilon,\cdot)\|_{\mathcal{L}} \le \alpha_2\varepsilon^{1+\gamma}$, because the spectral gap is of order $\varepsilon^{1-\gamma}$ due to (34). □

Note that the case $\delta = \pi/2$ is not a typical situation. Indeed, in this case the admissible frequency ω coincides with the eigenfrequency of problem (22) with Neumann condition $v'(0) = 0$. In this case we cannot establish the vicinity of ω that would contain one and only one eigenfrequency $\sqrt{\lambda_n^\varepsilon}$. The arguments of the last proof show only that there exist not more

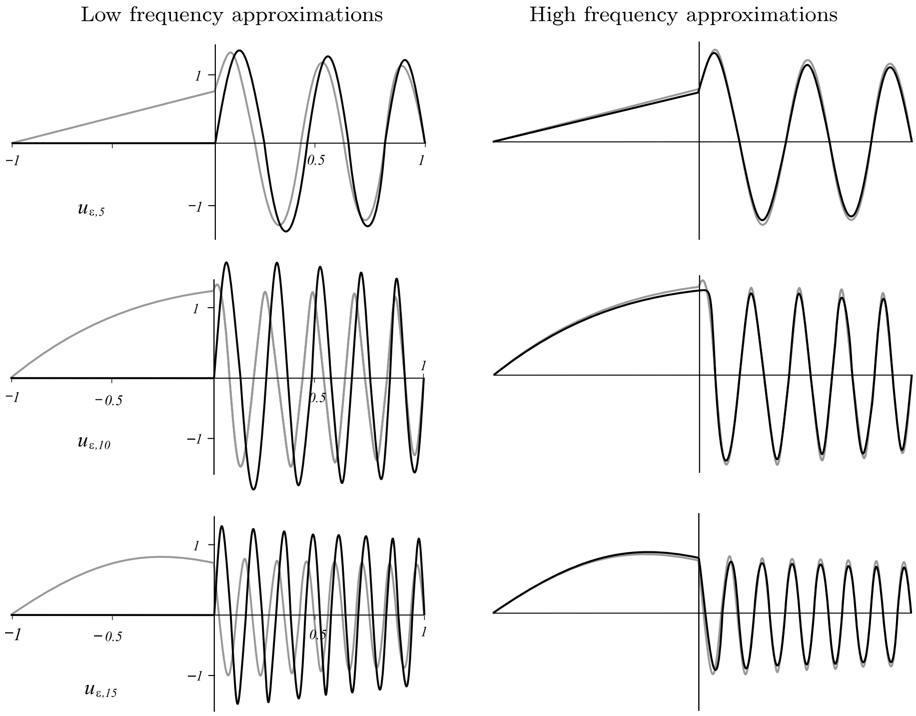

Fig. 1. A comparison of the low and high frequency approximations (black plots) with the eigenfunctions (grey plots) for $u_{\varepsilon,5}$, $u_{\varepsilon,10}$ and $u_{\varepsilon,15}$ (from top to bottom).

then two eigenfrequencies satisfying (36). If ω_n^ε and ω_{n+1}^ε satisfy (36) then $\Upsilon_\omega(\varepsilon,\cdot)$ is not yet a good approximation to any of $u_{\varepsilon,n}$ or $u_{\varepsilon,n+1}$. The situation could be improved by the next terms of asymptotics but that is beyond the scope of this paper.

Numerical example. Let us consider the coefficients $k = 1$ and $r = 1 + x^2$ on the interval $(-1, 0)$, $\varkappa = 1$ and $\rho = 1 + x$ on $(0, 1)$. For the value of small parameter $\varepsilon = 0.05$, in Fig. 1 we have plotted the eigenfunctions of (1)–(3) and the leading terms of the low and high frequency approximations given by (4) and (14). Let us emphasize that the purpose of high frequency approach is in a good approximation of eigenfunctions. In the example only the first few eigenfunctions could be well approximated by the low frequency approach. Already $u_{\varepsilon,5}$ is quite far away from its low frequency limit (see Fig. 1), and that cannot be improved by the next terms of low frequency asymptotics, because the absolute error is large enough. Note that the eigenvalue λ_5^ε is still far from zero and thus $\sqrt{\lambda_5^\varepsilon}$ cannot be treated as a low frequency. In the right-hand side plots we observe that the high frequency approximations work well for the range of numbers between 5 and 15. We have to mention that the proof of Theorem 7 is done by asymptotic methods, so it would be challenge to tell in particular examples the exact range of numbers n, for which high frequency approximations are valid in the case of fixed ε.

We refer to the values of $\sqrt{\lambda_n^\varepsilon}$ that are calculated with high accuracy as to "exact". The numerical values under discussion are represented in the Table

n	5	10	15
Exact $\sqrt{\lambda_n^\varepsilon}$	2.76675	5.52678	8.27450
Low freq. approximation $\sqrt{\varepsilon\mu_n}$	2.88055	5.76252	8.64418
ω	0.6270	1.260	1.860
ω_1	−0.22224	−0.53779	−0.02669
δ	−0.63509	−1.3217	−0.03770
High freq. approximation $\frac{\omega}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\omega_1$	2.75433	5.51464	8.3122

As for numerical example we present the low and high frequency approximations to eigenfunctions by the leading terms of the expansions only. Thus the low frequency approximations $\sqrt{\varepsilon\mu_n}$ to eigenfrequencies $\sqrt{\lambda_n^\varepsilon}$ are given in the Table and are accomplished by visualization of eigenfunctions u_n of problem (6) in the left columns of Fig. 1 (u_n is extended by zero to $(-1,0)$). In order to find the admissible frequency ω we need also δ and ω_1. In the vicinity of expected ω we create network over ω and for each of this ω we find $\delta(\omega)$ satisfying (22) (up to 10^{-7}) and then find $\omega_1(\omega)$ such that (25) has a solution.

Finally, we find the admissible frequency $\omega = \omega(n)$ giving the best approach to (26) over tabulated ω. The high frequency approximations to the eigenfunctions we depict from $u_{\varepsilon,l} \sim v(\omega, x)$ for $x \in (-1, 0)$ and $u_{\varepsilon,l} \sim c_0(x)\sin(\frac{\omega}{\varepsilon} + \omega_1)S(x)$ for $x \in (0, 1)$ with $v(\omega, x)$ from (22), $c_0(x) = \frac{v(\omega,0)}{\sin\delta(\omega)\sqrt[4]{1+x}}$ and $S(x) = \frac{2}{3}(2\sqrt{2} - \sqrt{(1+x)^3})$. All depicted eigenfunctions $u_{\varepsilon,n}$ are normalized in $L^2(-1, 1)$.

Note that the method that is applied for the approximations in one-dimensional case is also applicable in a multi-dimensional situation. Nevertheless, the justification of it requires another technique, which is beyond the scope of this paper.

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