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An application of Gegenbauer polynomials in queueing theory

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Abstract

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The symmetric coupled processor model is a queueing system in which a server divides his service capacity between two independent streams of customers, unless one queue is empty, in which case the full capacity is granted to the other queue. Customers demand an exponentially distributed service time with mean μ^{-1} and their arrivals are determined according to their stream by two independent Poisson processes each with rate λ. The symmetric coupled processor model can be represented by a continuous time Markov process $X(t) := (X_1(t), X_2(t))$, where $X_i(t)$ is the number of customers in the ith queue. Let $p_{m,n}(t) := \Pr\{X(t) = (m, n)\}$. If $\rho = 2\lambda/\mu < 1$, the equilibrium probabilities exist and are given by $p(m, n) = \lim_{t\to\infty} p_{m,n}(t)$. We prove that the equilibrium probability $p(n, n)$ can be written as $p(n, n) = (1-\rho)\rho^{2n}\sum_{k=0}^{\infty} a_k(n)\rho^k$, where the coefficients $a_k(n)$ are computed explicitly.

Keywords: Queueing theory; Gegenbauer polynomials; symmetric coupled processor

1. Introduction

In this paper a queueing system is studied where an exponential server divides his service capacity μ between two queues — each formed from Poisson arrivals with rate λ. The server attends both queues with equal rate, except when one of the queues is empty, in which case the server grants full capacity to the other queue. More specifically, if $X(t) = (X_1(t), X_2(t))$ denotes the number of customers in the two queues at time t, then $X(t)$ is a continuous time Markov chain with state space $\mathbb{N}^2$ (where $\mathbb{N} = \{0, 1, 2, \dots\}$) and generator $G = (g_{z,z'})$, $z, z' \in \mathbb{N}^2$, given by (we only specify the entries of G which are not equal to 0), for $z = (0, 0)$,

$$g_{z,z'} = \begin{cases} -2\lambda, & z' = z, \\ \lambda, & z' = (1, 0) \text{ or } z' = (0, 1), \end{cases}$$

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for $z=(m, 0)$, $m \geqslant 1$,

$$g_{z,z'} = \begin{cases} -(2\lambda+\mu), & z'=z, \\ \lambda, & z'=(m+1, 0) \text{ or } z'=(m, 1), \\ \mu, & z'=(m-1, 0), \end{cases}$$

for $z=(0, n)$, $n \geqslant 1$,

$$g_{z,z'} = \begin{cases} -(2\lambda+\mu), & z'=z, \\ \lambda, & z'=(0, n+1) \text{ or } z'=(1, n), \\ \mu, & z'=(0, n-1), \end{cases}$$

for $z=(m, n)$, $m \geqslant 1$ and $n \geqslant 1$,

$$g_{z,z'} = \begin{cases} -(2\lambda+\mu), & z'=z, \\ \lambda, & z'=(m, n+1) \text{ or } z'=(m+1, n), \\ \frac{1}{2}\mu, & z'=(m-1, n) \text{ or } z'=(m, n-1). \end{cases}$$

A fair number of papers has been written on the calculation of the stationary distribution $p(m, n)$ of the Markov chain X. We mention [2,3] where the model was studied as a Riemann–Hilbert problem, [5] where the generating function techniques were used to study the problem, and finally [1,4] where a power series expansion was used for $p(m, n)$ in the variable $\rho = 2\lambda/\mu$. More specifically, the authors of [4] showed that for ρ in a neighbourhood of the origin,

$$p(m, n) = \rho^{n+m}(1-\rho) \sum_{k=0}^{\infty} a_k(m, n)\rho^k, \tag{1}$$

where the coefficients $a_k(m, n)$ can be computed recursively. The main conjecture in [4] is that

$$|a_k(m, n)| \leqslant 1,$$

uniformly in $k \in \mathbb{N}$ and $(m, n) \in \mathbb{N}^2$.

In this paper we calculate (in Section 2) explicit expressions for the coefficients $a_k(n) = a_k(n, n)$, $k \geqslant 0$, $n \geqslant 1$, and establish the upper bound for $|a_k(m, n)|$ on the diagonal $n = m$. Moreover, we obtain excellent error bounds when $p(n, n)$ is approximated by the partial sums

$$\rho^{2n}(1-\rho) \sum_{k=0}^{M} a_k(n)\rho^k.$$

Finally, when $p(n, n)$, $n \geqslant 1$, is known, the other equilibrium probabilities can be calculated from the equilibrium equations.

2. The main result

Since $X_1(t) + X_2(t)$ is the row length at time t of an M/M/1 queue with traffic intensity ρ, it is immediate that $\rho < 1$ is a necessary and sufficient condition for ergodicity of the Markov chain $X(t)$. The equilibrium equations follow from $p^{t}G=0$ and are for $\rho<1$ given by

$$\begin{cases} \rho p(0,0)=p(0,1)+p(1,0), \\ 2(1+\rho)p(m,0)=\rho p(m-1,0)+p(m,1)+2p(m+1,0), \quad m\geqslant 1, \\ 2(1+\rho)p(0,n)=\rho p(0,n-1)+p(1,n)+2p(0,n+1), \quad n\geqslant 1, \\ 2(1+\rho)p(m,n)=\rho(p(m-1,n)+p(m,n-1)) \\ \qquad\qquad +p(m+1,n)+p(m,n+1), \quad m,n\geqslant 1. \end{cases} \tag{2}$$

We define

$$F(x,y):=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}p(n+r,n)x^{n}y^{r}. \tag{3}$$

The stationary probability of $2n+r$ customers present in an M/M/1 system with traffic intensity ρ is given by $\rho^{2n+r}(1-\rho)$; this implies that $p(0,0)=1-\rho$ and that $p(n+r,n)\leqslant\rho^{2n+r}(1-\rho)$. Hence $F(x,y)$ exists and is analytic for $(x,y)\in A$, where

$$A:=\{(x,y)\in\mathbb{C}^{2}\colon |x|<\rho^{-2},\ |y|<\rho^{-1}\}.$$

Let for $(x,y)\in A$,

$$h(x,y):=2(1+\rho)xy-(1+\rho x)(x+y^{2}). \tag{4}$$

Lemma 1. *On the set* $D=\{(x,y)\in A\colon h(x,y)=0\}$ *there holds*

$$(1+\rho x)(x-y^{2})F(x,0)=2(1-\rho)x(y-1)+2(x-y^{2})F(0,y). \tag{5}$$

Proof. The bivariate generating function $F(x,y)$ is analytic on A and it follows from (2) that

$$h(x,y)F(x,y)=\{(1+\rho)xy-x(1+\rho x)\}F(x,0)+(x-y^{2})F(0,y) \\ +(1-\rho)x(y-1). \tag{6}$$

Hence the right-hand side of (6) is equal to 0 on D. The equality $h(x,y)=0$ is equivalent to $(1+\rho)xy=\frac{1}{2}(1+\rho x)(x+y^{2})$ and insertion of this equality yields (5). □

Theorem 2. *For* $\rho<1$ *the equilibrium probability* $p(n,n)$*, on having n customers in each queue, is given by*

$$p(n,n)=(1-\rho)\rho^{n}(-1)^{n}\left(1+\frac{1}{\pi}\int_{0}^{\pi}C_{n}(\phi)U(\phi)\,\mathrm{d}\phi\right),\quad n\geqslant 1,$$

where

$$C_{n}(\phi):=-1+\frac{(-1)^{n}(\cos(n\phi)-\cos\{(n+1)\phi\})}{1-\cos\phi},$$

$$U(\phi):=\frac{\rho\cos\phi-1+\{(1+\rho)^{2}-\rho(1+\cos\phi)^{2}\}(1+\rho^{2}-2\rho\cos\phi)^{-1/2}}{\rho(1+\cos\phi)}.$$

Proof. Note from (3) that $F(x,0)=\sum_{n=0}^{\infty}p(n,n)x^{n}$. If we define

$$H(x):=\frac{(1+\rho x)F(x,0)}{1-\rho},\quad |x|<\rho^{-2},$$

and put $H(x)=\sum_{n=0}^{\infty} b_n x^n$, then $p(n, n)$ can be expressed in b_k, $0 \leqslant k \leqslant n$. So, our goal is to calculate the coefficients b_k, $k=0, 1, \ldots$. For given y, $|y| < \rho^{-1}$, the equation $h(x, y)=0$ has two solutions $x_1 = x_1(y)$ and $x_2 = x_2(y)$ given by

$$\begin{cases} \rho x_1 x_2 = y^2, \\ \rho(x_1 + x_2) = 2y(1+\rho) - \rho y^2 - 1. \end{cases} \tag{7}$$

Hence,

$$H(x_1) = \frac{2x_1(y-1)}{(x_1 - y^2)} + \frac{2}{1-\rho} F(0, y), \qquad H(x_2) = \frac{2x_2(y-1)}{(x_2 - y^2)} + \frac{2}{1-\rho} F(0, y),$$

and consequently

$$H(x_1) - H(x_2) = \frac{2\rho(y-1)(x_2 - x_1)}{(1-\rho x_1)(1-\rho x_2)}.$$

Elimination of y from the two equations given in (7) yields

$$H(x_1) - H(x_2) = \frac{\rho(1+\rho x_1)(1+\rho x_2)(x_2 - x_1)}{(1+\rho)(1-\rho x_1)(1-\rho x_2)} - \frac{2\rho(x_2 - x_1)}{(1-\rho x_1)(1-\rho x_2)}, \tag{8}$$

whenever

$$4\rho(1+\rho)^2 x_1 x_2 = (1+\rho x_1)^2 (1+\rho x_2)^2, \quad |x_i| < \rho^{-2}, \ i = 1, 2. \tag{9}$$

Note that for fixed $0 < x_1 < 1$, the solutions x_2^+ and x_2^- of (9) satisfy

$$x_2^{\pm} = \rho^{-1} e^{\pm i\phi}, \quad \text{with } \cos\phi = \frac{2(1+\rho)^2 x_1}{(1+\rho x_1)^2} - 1 \in (-1, 1).$$

Proceeding as in [5], we insert x_2^- and x_2^+ into (8) and subtract the equations to obtain

$$\begin{aligned} H(x_2^+) - H(x_2^-) &= \frac{\rho(1+\rho x_1)(x_2^- - x_2^+)\left[1 + \rho(x_2^- + x_2^+) - \rho^2 x_2^- x_2^+ - 2\rho x_1\right]}{(1+\rho)(1-\rho x_1)\left(1 - \rho(x_2^- + x_2^+) + \rho^2 x_2^- x_2^+\right)} \\ &\quad - \frac{2\rho(x_2^- - x_2^+)}{1 - \rho(x_2^- + x_2^+) + \rho^2 x_2^- x_2^+} \\ &= \frac{(1+\rho x_1)}{(1+\rho)(1-\rho x_1)} \frac{2i(\rho x_1 - \cos\phi)\sin\phi}{1-\cos\phi} + \frac{2i\sin\phi}{1-\cos\phi}. \end{aligned} \tag{10}$$

If we substitute $H(x)=\sum_{n=0}^{\infty} b_n x^n$, using that $x_2^{\pm} = \rho^{-1} e^{\pm i\phi}$ and the fact that

$$\int_0^{\pi} \sin(n\phi)\sin(m\phi)\, d\phi = \tfrac{1}{2}\pi\delta_{n,m},$$

we obtain

$$\frac{b_n}{\rho^n} = \frac{2}{\pi}\int_0^{\pi} \left\{ \frac{(1+\rho x_1)(\rho x_1 - \cos\phi)}{(1+\rho)(1-\rho x_1)} + 1 \right\} \frac{\sin\phi \sin(n\phi)}{1-\cos\phi}\, d\phi, \quad n \geqslant 1,$$

where in view of (9),

$$\rho x_1 = a - 1 - \sqrt{a(a-2)}, \quad a := \frac{(1+\rho)^2}{\rho(1+\cos\phi)}.$$

From the definition $H(x) = (1+\rho x)F(x, 0)/(1-\rho)$ we obtain

$$b_0 = (1-\rho)^{-1} p(0, 0) = 1,$$

$$b_n = (1-\rho)^{-1}(p(n, n) + \rho p(n-1, n-1)), \quad n \geqslant 1,$$

and so, by induction,

$$p(n, n) = (1-\rho)\rho^n(-1)^n\left\{1 + \sum_{k=1}^{n} (-1)^k \rho^{-k} b_k\right\}, \quad n \geqslant 1.$$

This proves Theorem 2. □

Next we formulate and prove the main result of the paper.

Theorem 3. *For $\rho < 1$, the equilibrium probability $p(n, n)$, $n \geqslant 1$, can be written as*

$$p(n, n) = (1-\rho)\rho^{2n} \sum_{k=0}^{\infty} a_k(n)\rho^k,$$

where

$$a_{2k}(n) = \frac{(-1)^n n(n+1)}{(2k+n)(2k+n+1)} \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k+n}, \qquad k \geqslant 0,$$

$$a_{2k+1}(n) = \frac{(-1)^n n(n+1)}{(2k+n+1)(2k+n+2)} \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{k+n+1}, \quad k \geqslant 0.$$

Proof. The Gegenbauer polynomials $C_k^\lambda(x)$ are defined by

$$(1+\rho^2 - 2\rho x)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(x)\rho^k.$$

We rewrite $U(\phi)$, given in Theorem 2, to

$$U(\phi) = \frac{(1+\rho^2-2\rho\cos\phi)^{1/2} + \rho\cos\phi - 1 + \rho(1-\cos^2\phi)(1+\rho^2-2\rho\cos\phi)^{-1/2}}{\rho(1+\cos\phi)}.$$

Since $(1+\rho^2-2\rho\cos\phi)^{1/2} = 1 - \rho\cos\phi + O(\rho^2)$, as $\rho \to 0$ we can write

$$(1+\rho^2-2\rho\cos\phi)^{1/2} + \rho\cos\phi - 1 = \sum_{k=2}^{\infty} C_k^{-1/2}(\cos\phi)\rho^k. \tag{11}$$

Differentiating (11) twice with respect to ρ, we obtain

$$(1+\rho^2-2\rho\cos\phi)^{-3/2}(1-\cos^2\phi) = \sum_{k=2}^{\infty} k(k-1)C_k^{-1/2}(\cos\phi)\rho^{k-2}.$$

Hence we find

$$C_{k+2}^{-1/2}(\cos\phi)=\frac{(1-\cos^2\phi)C_k^{3/2}(\cos\phi)}{(k+1)(k+2)},\quad k\geqslant 0,$$

and consequently for $U(\phi)$,

$$U(\phi)=(1-\cos\phi)\left\{1+\sum_{k=1}^{\infty}\rho^k\left(\frac{C_{k-1}^{3/2}(\cos\phi)}{k(k+1)}+C_k^{1/2}(\cos\phi)\right)\right\}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{\pi}C_n(\phi)U(\phi)\,\mathrm{d}\phi\\
&\quad=\frac{1}{\pi}\int_0^{\pi}\left(-1+\cos\phi+(-1)^n(\cos(n\phi)-\cos\{(n+1)\phi\})\right)\mathrm{d}\phi\\
&\qquad+\frac{1}{\pi}\int_0^{\pi}-(1-\cos\phi)\sum_{k=1}^{\infty}\rho^k\left(\frac{C_{k-1}^{3/2}(\cos\phi)}{k(k+1)}+C_k^{1/2}(\cos\phi)\right)\mathrm{d}\phi\\
&\qquad+\frac{1}{\pi}\int_0^{\pi}(-1)^n(\cos(n\phi)-\cos\{(n+1)\phi\})\\
&\qquad\times\left(\sum_{k=1}^{\infty}\rho^k\left(\frac{C_{k-1}^{3/2}(\cos\phi)}{k(k+1)}+C_k^{1/2}(\cos\phi)\right)\right)\mathrm{d}\phi.
\end{aligned}$$

Hence,

$$1+\frac{1}{\pi}\int_0^{\pi}C_n(\phi)U(\phi)\,\mathrm{d}\phi=S_n(\phi)-S_0(\phi),$$

where

$$\begin{aligned}
S_n(\phi)=(-1)^n\sum_{k=1}^{\infty}\rho^k\int_0^{\pi}&(\cos(n\phi)-\cos\{(n+1)\phi\})\\
&\times\left(\frac{C_{k-1}^{3/2}(\cos\phi)}{k(k+1)}+C_k^{1/2}(\cos\phi)\right)\mathrm{d}\phi.
\end{aligned}$$

In order to compute $S_n(\phi)-S_0(\phi)$, we use (cf. [6, (4.9.19) and (4.7.23)]),

$$C_{2n}^{\lambda}(\cos\phi)=2\sum_{k=0}^{n}\binom{k+\lambda-1}{k}\binom{2n-k+\lambda-1}{2n-k}\cos\{(2n-2k)\phi\}-\binom{n+\lambda-1}{n}^2, \tag{12}$$

$$C_{2n+1}^{\lambda}(\cos\phi)=2\sum_{k=0}^{n}\binom{k+\lambda-1}{k}\binom{2n-k+\lambda}{2n-k+1}\cos\{(2n-2k+1)\phi\}, \tag{13}$$

and the fact that

$$\int_0^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \frac{1}{2}\pi, & n = m \neq 0, \\ \pi, & n = m = 0. \end{cases}$$

This yields

$$\frac{1}{\pi}\int_0^{\pi}(\cos(n\phi) - \cos\{(n+1)\phi\})C_{2k}^{\lambda}(\cos\phi)\,\mathrm{d}\phi$$
$$= \begin{cases} 0, & 2k < n, \\ \binom{k-m+\lambda-1}{k-m}\binom{k+m+\lambda-1}{k+m}, & 2k \geqslant n = 2m, \\ -\binom{k-m+\lambda-2}{k-m-1}\binom{k+m+\lambda}{k+m+1}, & 2k \geqslant n = 2m+1, \end{cases}$$

and

$$\frac{1}{\pi}\int_0^{\pi}(\cos(n\phi) - \cos\{(n+1)\phi\})C_{2k+1}^{\lambda}(\cos\phi)\,\mathrm{d}\phi$$
$$= \begin{cases} 0, & 2k+1 < n, \\ -\binom{k-m+\lambda-1}{k-m}\binom{k+m+\lambda}{k+m+1}, & 2k+1 \geqslant n = 2m, \\ \binom{k-m+\lambda-1}{k-m}\binom{k+m+\lambda}{k+m+1}, & 2k+1 \geqslant n = 2m+1. \end{cases}$$

A straightforward calculation yields in the case $n = 2m$,

$$b(2m; 2k) := \frac{1}{\pi}\int_0^{\pi}(\cos(2m\phi) - \cos\{(2m+1)\phi\})\left(\frac{C_{2k-1}^{3/2}(\cos\phi)}{2k(2k+1)} + C_{2k}^{1/2}(\cos\phi)\right)\mathrm{d}\phi$$
$$= -\frac{1}{2k(2k+1)}\binom{k-m-\frac{1}{2}}{k-m-1}\binom{k+m+\frac{1}{2}}{k+m} + \binom{k-m-\frac{1}{2}}{k-m}\binom{k+m-\frac{1}{2}}{k+m}$$
$$= \begin{cases} \dfrac{\Gamma(k-m+\frac{1}{2})\Gamma(k+m+\frac{1}{2})}{\Gamma(k-m+1)\Gamma(k+m+1)\pi}\,\dfrac{2m(2m+1)}{2k(2k+1)}, & k \geqslant m, \\ 0, & k < m, \end{cases}$$

$$b(2m; 2k+1) := \frac{1}{\pi}\int_0^{\pi}(\cos(2m\phi) - \cos\{(2m+1)\phi\})$$
$$\times\left(\frac{C_{2k}^{3/2}(\cos\phi)}{(2k+1)(2k+2)} + C_{2k+1}^{1/2}(\cos\phi)\right)\mathrm{d}\phi$$
$$= \begin{cases} -\dfrac{\Gamma(k+m+\frac{3}{2})\Gamma(k-m+\frac{1}{2})}{\Gamma(k-m+1)\Gamma(k+m+2)\pi}\,\dfrac{2m(2m+1)}{(2k+1)(2k+2)}, & k \geqslant m, \\ 0, & k < m. \end{cases}$$

From these formulas it is immediate that $S_0(\phi)=0$, and that a factor ρ^n can be put in front of the series $S_n(\phi)$. By using $\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ and $\binom{2j}{j}=(-1)^j 2^{2j}\binom{-\frac{1}{2}}{j}$, we get

$$b(2m;2k)=2^{-4k}\binom{2k-2m}{k-m}\binom{2k+2m}{k+m}\frac{2m(2m+1)}{2k(2k+1)}$$
$$=\binom{-\frac{1}{2}}{k-m}\binom{-\frac{1}{2}}{k+m}\frac{2m(2m+1)}{2k(2k+1)}$$

and

$$a_{2j}(2m)=b(2m;2j+2m)=\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{j+2m}\frac{2m(2m+1)}{(2j+2m)(2j+2m+1)},\tag{14}$$

and similarly

$$b(2m;2k+1)=-\binom{-\frac{1}{2}}{k-m}\binom{-\frac{1}{2}}{k+m+1}\frac{2m(2m+1)}{(2k+1)(2k+2)}$$

and

$$a_{2j+1}(2m)=b(2m;2j+1+2m)$$
$$=\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{j+2m+1}\frac{2m(2m+1)}{(2j+2m+1)(2j+2m+2)}.\tag{15}$$

In the case $n=2m-1$ we get

$$b(2m-1;2k):=\frac{1}{\pi}\int_0^{\pi}(\cos\{(2m-1)\phi\}-\cos(2m\phi))$$
$$\times\left(\frac{C_{2k-1}^{3/2}(\cos\phi)}{2k(2k+1)}+C_{2k}^{1/2}(\cos\phi)\right)\mathrm{d}\phi$$
$$=-2^{-4k}\binom{2k-2m}{k-m}\binom{2k+2m}{k+m}\frac{2m(2m-1)}{2k(2k+1)},$$

$$b(2m-1;2k+1):=\frac{1}{\pi}\int_0^{\pi}(\cos\{(2m-1)\phi\}-\cos(2m\phi))$$
$$\times\left(\frac{C_{2k}^{3/2}(\cos\phi)}{(2k+1)(2k+2)}+C_{2k+1}^{1/2}(\cos\phi)\right)\mathrm{d}\phi$$
$$=2^{-4k-2}\binom{2k-2m+2}{k-m+1}\binom{2k+2m}{k+m}\frac{2m(2m-1)}{(2k+1)(2k+2)}.$$

Hence,

$$a_{2j}(2m-1)=b(2m-1;2j+2m-1)$$
$$=-\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{j+2m-1}\frac{2m(2m-1)}{(2j+2m-1)(2j+2m)},\tag{16}$$

$$a_{2j+1}(2m-1) = b(2m-1;\, 2j+2m) = -\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{j+2m}\frac{2m(2m-1)}{(2j+2m)(2j+2m+1)}. \tag{17}$$

Combination of (14)–(17) gives the desired result. □

3. Error estimates

It follows from Theorem 3 that the equilibrium probability $p(n, n)$ can be approximated by

$$p_{n,n}(M) := (1-\rho)\rho^{2n} \sum_{k=0}^{M} a_k(n)\rho^k, \quad M \in \mathbb{N}.$$

To estimate the difference $d_n(M) = |\, p(n, n) - p_{n,n}(M)\,|$ we need the following lemma.

Lemma 4. *For each $n \geqslant 1$, the sequence $(a_k(n))_{k=0}^{\infty}$ is of alternating sign and its absolute value is strictly decreasing.*

Proof. It is immediate from the explicit form in Theorem 3 that $a_{2k}(n)$ and $a_{2k+1}(n)$ have opposite signs. Further,

$$|a_{2k}(n)| = \frac{(2k+n+2)(2k+2n+2)}{(2k+n)(2k+2n+1)}|a_{2k+1}(n)| > |a_{2k+1}(n)|$$

and

$$|a_{2k+1}(n)| = \frac{(2k+2)(2k+n+3)}{(2k+1)(2k+n+1)}|a_{2k+2}(n)| > |a_{2k+2}(n)|. \qquad \square$$

Corollary 5. *For $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we have*

$$|a_k(n)| \leqslant 1.$$

Proof.

$$|a_k(n)| \leqslant |a_0(n)| = \left|\binom{-\frac{1}{2}}{n}\right| \leqslant 1. \qquad \square$$

Table 1
Computation of $p_{1,1}$, $p_{2,2}$, $p_{3,3}$ and $p_{4,4}$ for increasing values of ρ

ρ	$p_{1,1}$	$p_{2,2}$	$p_{3,3}$	$p_{4,4}$
0.1	$0.4390 \cdot 10^{-2}$ [3] {3}	$0.3238 \cdot 10^{-4}$ [3] {3}	$0.2669 \cdot 10^{-6}$ [3] {3}	$0.2318 \cdot 10^{-8}$ [5] {3}
0.3	$0.2929 \cdot 10^{-1}$ [5] {5}	$0.1880 \cdot 10^{-2}$ [5] {4}	$0.1364 \cdot 10^{-3}$ [5] {5}	$0.1049 \cdot 10^{-4}$ [5] {5}
0.5	$0.5546 \cdot 10^{-1}$ [7] {6}	$0.9557 \cdot 10^{-2}$ [10] {8}	$0.1880 \cdot 10^{-2}$ [10] {9}	$0.3946 \cdot 10^{-3}$ [9] {8}
0.7	$0.6229 \cdot 10^{-1}$ [13] {11}	$0.2028 \cdot 10^{-1}$ [13] {11}	$0.7605 \cdot 10^{-2}$ [15] {13}	$0.3062 \cdot 10^{-2}$ [14] {12}
0.9	$0.3277 \cdot 10^{-1}$ [16] {14}	$0.1693 \cdot 10^{-1}$ [24] {20}	$0.1015 \cdot 10^{-1}$ [32] {25}	$0.6564 \cdot 10^{-2}$ [34] {30}

Theorem 6. *The difference* $d_n(M) = |p(n, n) - p_{n,n}(M)|$ *is at most*

$$d_n(M) \leqslant (1-\rho)\rho^{2n+M+1}|a_{M+1}(n)|. \tag{18}$$

Proof. The proof of Theorem 6 follows immediately from Theorem 3 and Lemma 4. □

In Table 1 we present the equilibrium probabilities $p(n, n)$ with four correct significant digits. The number in brackets denotes the theoretical value of M necessary to obtain the prescribed accuracy. The number in braces denotes the minimum M to attain the four correct digits in practice.

Starting from the probabilities $p(n, n)$, $n = 1,\ldots, K+1$, other equilibrium probabilities can be obtained from the equilibrium equations (2) as follows. First, we obtain $p(n+1, n)$ for $n = 1,\ldots, K-1$ from the recursive equation

$$p(n+1, n) = (1+\rho)p(n, n) - \rho p(n, n-1)$$

and the initial value $p(1, 0) = \frac{1}{2}\rho(1-\rho)$. Next we calculate for $r = 2,\ldots, 2K$, the probabilities $p(n+r, n)$, for $n = 1,\ldots, K - [\frac{1}{2}(r+1)]$, from the recursion

$$p(n+r, n) = 2(1+\rho)p(n+r-1, n) - \rho(p(n+r-2, n) + p(n+r-1, n-1)) \\ - p(n+r-1, n+1),$$

with initial value

$$p(r, 0) = (1+\rho)p(r-1, 0) - \tfrac{1}{2}\rho p(r-2, 0) - \tfrac{1}{2}p(r-1, 1).$$

In order to check the accuracy of the above procedure on can use that for m is even,

$$p(\tfrac{1}{2}m, \tfrac{1}{2}m) + 2\sum_{j=m/2+1}^{m} p(j, m-j) = \rho^m(1-\rho),$$

while for m odd,

$$2\sum_{j=(m+1)/2}^{m} p(j, m-j) = \rho^m(1-\rho).$$

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