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Nonsingular CS-rings coincide with tight PP rings

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Abstract

It is shown that if M is a self-generator right R-module, then M is non-M-singular and CS iff M is M-tight and $\mathrm{End}(M_R)$ is a right PP ring. In particular, right nonsingular right CS-rings R are precisely right PP and right R-tight. As applications we show, among others, that for any domain R, R^2_R is right CS if and only if R is two-sided Ore domain and two-sided 2-hereditary, giving answer to an open question known previously in special cases. As another application, we show that for a von Neumann regular ring R, the matrix ring $M_n(R)$, $n > 1$, is right weakly selfinjective if and only if R is right selfinjective.

Keywords: CS-rings; Weakly selfinjective rings; PP rings; Tight rings; n-hereditary rings

1. Introduction

A submodule N of an R-module M is called closed in M if it has no proper essential extension in M. Closed submodules are precisely complement submodules. Clearly, every

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✠ K.I. Beidar passed away on March 9, 2004. We dedicate this paper to him in his memory.

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direct summand of M is closed in M. A module M is called a CS-module if every closed submodule of M is a direct summand of M. CS-modules are also known as extending modules (see [5]). A ring R is called a right CS-ring if R is CS as a right R-module (see Chatters and Hajarnavis [4]). The property of being a CS-module is not preserved under direct sums. It has been an open question for more than a decade to characterize domains R such that the finite direct sum R^n of copies of R is a right CS-module (equivalently, the $n \times n$ matrix ring $M_n(R)$ is a right CS-ring) where n is some fixed positive integer greater than 1. It is known that if R is a commutative integral domain, then $(R \times R)_R$ is CS if and only if R is a Prüfer domain [5, Corollary 12.10], and that if R is a local (noncommutative) domain, then $(R \times R)_R$ is CS if and only if R is a valuation domain [2, Lemma 3.6]. It is also known that if R is a semiprime Goldie ring, then R_R^n is CS for all $n > 0$ if and only if R is a two-sided semihereditary ring [5, Corollary 12.18]. Theorems 4.9 and 4.15 of this paper answer the above stated question for a class of rings which include integral domains.

Let M_R and N_R be R-modules. We denote by $E(M)$ the injective hull of M. M is called N-injective if for any submodule K of N and any R-homomorphism $\phi : K \to M$ there exists an R-homomorphism $\psi : N \to M$ such that $\psi|_K = \phi$. It was shown by Azumaya that M is N-injective if and only if for any R-homomorphism $f : N \to E(M)$, $f(N) \subset M$. More generally, M is said to be weakly N-injective if for any R-homomorphism $f : N \to E(M)$, there exists a submodule $X \subset E(M)$ such that $f(N) \subset X$ and $X \simeq M$ (see Jain–Lopez [10]). M is said to be N-tight if for any R-homomorphism $f : N \to E(M)$, $f(N)$ is embeddable in M (see Golan–Lopez [7]). M is called weakly injective (tight) if M is N-weakly injective (respectively N-tight) for all finitely generated modules N. A ring R which is weakly R-injective as a right R-module is called weakly selfinjective. Unlike injectivity the property that the ring $S = M_n(R)$, $n > 1$, is right weakly selfinjective need not imply that R is right weakly selfinjective. It is known that for a Boolean ring R if the $n \times n$ matrix ring $S = M_n(R)$, $n > 1$ is right weakly selfinjective then R is right selfinjective [13, Theorem 3.6].

In this paper we first prove that if M is non-M-singular and CS, then M is M-tight and End(M_R) is right PP. Furthermore, if M is a self-generator then the converse also holds. As a particular case, it follows that right nonsingular right CS-rings R are precisely right R-tight right PP-rings. This fact is indeed surprising: while the class of CS-modules is closed under direct summands but not under direct sums (finite or infinite), the class of tight (and also weakly injective) modules, in general, has the opposite properties with respect to direct summands and direct sums.

As applications of our main theorem, we prove, among others, that

(i) for any ring R having no infinite set of nonzero orthogonal idempotents, R_R^n is nonsingular CS right R-module if and only if R is Utumi and Baer if and only if ${}_R R^n$ is a nonsingular CS left R-module (Theorem 4.9);

(ii) for any reduced ring R, R^n is CS as a right R-module if and only if R is right n-hereditary and left classical quotient ring $Q_{\mathrm{cl}}^l(R)$ of R is same as the right maximal quotient ring $Q_{\max}^r(R)$ of R, if and only if R is right n-hereditary and right weakly injective, if and only if R^n is CS as a left R-module (Theorem 4.15); and

(iii) for a von Neumann regular ring R, the $n \times n$ matrix ring $S = M_n(R)$, $n > 1$, is right weakly S-injective if and only if S (and hence R) is right selfinjective (Theorem 4.4).

2. Definitions and notation

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right unital. A CS-module M is called continuous if a submodule N of M isomorphic to a direct summand of M is itself a direct summand of M. A module M is called nonsingular if for any essential right ideal E of R and any element m in M, $mE = 0$ implies $m = 0$. A right R-module M has finite uniform dimension if it does not contain any infinite direct sum of nonzero submodules. It is known that for a module M_R with finite uniform dimension, there exists an integer $n \geqslant 1$ such that every direct sum of submodules contains less than or equal to n terms. We denote the uniform dimension of M by $\text{u.dim}(M)$.

A right R-module N is said to be generated by a right R-module M if there is an epimorphism $M^{(A)} \to N \to 0$. N is said to be subgenerated by M if it is isomorphic to a submodule of an M-generated module. For a right R-module M, $\sigma[M]$ will denote the full subcategory of the category of right R-modules whose objects are all R-modules subgenerated by M [14, Section 15, p. 118].

For any two R-modules M and N, $M \subset_e N$ will denote that N is an essential extension of M. $\text{Tr}_M(N)$ will denote $\sum\{\text{Im}(f) \mid f \in \text{Hom}(N, M)\}$.

Let M be a right R-module. M is said to be self-generator if it generates all its submodules, equivalently, if $L = \text{Tr}_L(M)$ for every submodule L of M. A module $N \in \sigma[M]$ is called M-singular if there exists a module $K \in \sigma[M]$ with essential submodule L such that $N \cong K/L$. It is known that the class of M-singular modules is closed under submodules, homomorphic images and direct sums. Hence every module $N \in \sigma[M]$ contains a largest M-singular submodule, $Z_M(N)$ [5, p. 29]. N is called non-M-singular if $Z_M(N) = 0$.

A ring R is called right continuous if R_R is continuous. R is called a Baer ring if each right annihilator ideal (equivalently, left annihilator ideal) is a direct summand. R is called Utumi if its right maximal quotient ring coincides with its left maximal quotient ring. R is said to be right nonsingular if R_R is nonsingular. R is said to be right n-hereditary if each n-generated right ideal is projective. Right 1-hereditary rings are called right PP-rings. R is called right semihereditary if R is right n-hereditary for all $n \geqslant 1$. R is called directly finite if for $a, b \in R$, $ab = 1$ implies $ba = 1$. R is called (von Neumann) regular if for each $a \in R$ there exists $x \in R$ such that $axa = a$. If, in addition, x is unit then R is called unit regular. A regular ring R is called abelian regular if all its idempotents are central. We note that a regular ring is right and left nonsingular, right and left right semihereditary, and is right (left) CS if and only if it is right (left) continuous.

For a ring R, $Q^r_{\max}(R)$ ($Q^l_{\max}(R)$) will denote the right (left) maximal quotient ring of R; $Q^r_{\text{cl}}(R)$ ($Q^l_{\text{cl}}(R)$) will denote the right (left) classical ring of quotients of R. $Q_{\text{cl}}(R)$ and $Q_{\max}(R)$ will respectively denote the two-sided classical quotient ring and two-sided maximal quotient ring of R. For an element $a \in R$, $\text{r.ann}_R(a)$ will denote the right annihilator of a in R. CS-ring will mean both right and left CS and nonsingular ring will mean both right and left nonsingular. For all other notation and terminology the reader is referred to [5,8,11,12].

3. Main theorem on nonsingular CS modules and rings

Recall that $\widehat{M} = \mathrm{Tr}_{E(M)}(M)$ is the injective hull of M in $\sigma[M]$ (see [14, Section 17.9, p. 141]). In particular, $\widehat{M}$ is a quasi-injective R-module. We first prove a result for non-M-singular CS-modules.

Theorem 3.1. *Consider the following two conditions*:

(1) *M is non-M-singular and CS*;
(2) *M is M-tight and* $\mathrm{End}(M_R)$ *is right PP.*

Then (1) $\Rightarrow$ (2). *Moreover, if M is a self-generator, then* (2) $\Rightarrow$ (1).

Proof. Let $A = \mathrm{End}(M_R)$. We first prove (1) $\Rightarrow$ (2). Let $f : M \to E(M)$ be an R-homomorphism. Clearly $\mathrm{Im}(f) \subseteq \widehat{M}$. Moreover, $\widehat{M} \in \sigma[M]$ and M is an essential submodule of $\widehat{M}$. Since $Z_M(\widehat{M}) \cap M = Z_M(M) = 0$, it follows that $Z_M(\widehat{M}) = 0$. Clearly $\sigma[M] = \sigma[\widehat{M}]$ and $Z_{\widehat{M}}(\widehat{M}) = Z_M(\widehat{M}) = 0$. By [5, Section 4.1, p. 30], $\ker(f)$ is a closed submodule of M. Since M is CS, $\ker(f)$ is a direct summand of M. It follows that $\mathrm{Im}(f)$ is isomorphic to a direct summand of M. Thus M is M-tight.

Now let $g \in A$. By [5, Section 4.1, p. 30], $\ker(g)$ is a closed submodule of M. Thus there exists $e \in A$ such that $e = e^2$ and $\ker(g) = eM$. Now

$$\mathrm{r.ann}_A(g) = \bigl\{h \in A \mid \mathrm{Im}(h) \subseteq \ker(g)\bigr\} = \{h \in A \mid eh = h\} = eA. \tag{1}$$

Consequently, A is right PP.

Next assume that M is a self-generator and that the condition (2) holds. We first prove that *M is non-M-singular*. Suppose that $Z_M(M) \neq 0$. Since $Z_M(M) = \mathrm{Tr}_{Z_M(M)}(M)$, it follows from [5, Proposition 4.3.3, p. 31] that there exists a nonzero homomorphism $g : M \to Z_M(M)$ with $\ker(g)$ essential in M. Since A is right PP, there exists an idempotent $e \in A$ such that $\mathrm{r.ann}_A(g) = eA$. Using (1), we get $\ker(g) = \mathrm{Tr}_{\ker(g)}(M) = eM$. Hence $\ker(g)$ is a direct summand of M. Since $\ker(g)$ is essential in M, it follows that $e = 1$. Thus $g = 0$, a contradiction. Therefore $Z_M(M) = 0$.

Now let K be a closed submodule of M. We show that K is a direct summand of M. Let L be a closure of K in $\widehat{M}$. Then $L \cap M = K$. Since $\widehat{M}$ is a quasi-injective module, L is a direct summand of $\widehat{M}$. Thus $L = v\widehat{M}$ for some $v = v^2 \in \mathrm{End}(\widehat{M}_R)$. Obviously, $K = L \cap M = \ker(1 - v) \cap M$ is the kernel of the map $M \to (1 - v)M \subseteq \widehat{M}$. Since M is M-tight, there exists an embedding of $(1 - v)M$ into M. That is, there exists an endomorphism $g : M \to M$ with $\ker(g) = K$. Since A is right PP, $\mathrm{r.ann}_A(g) = eA$ for some $e = e^2 \in A$. Once again using (1), we get

$$K = \mathrm{Tr}_K(M) = \sum \bigl\{\mathrm{Im}(h) \mid \mathrm{Im}(h) \subseteq K\bigr\} = eM.$$

Thus K is a direct summand of M and hence M is CS. □

Since R_R is a self-generator, we have the following theorem for right nonsingular right CS-rings. The theorem is of independent interest and will be used throughout Section 4.

Theorem 3.2. *A ring R is a right nonsingular and right CS-ring if and only if R is a right R-tight right PP-ring.*

Recall that a ring R is called directly finite if for $a, b \in R$, $ab = 1$ implies $ba = 1$.

Lemma 3.3. *Let M be a non-M-singular module such that* $\mathrm{End}(\widehat{M}_R)$ *is directly finite. Then M is M-tight if and only if M is weakly M-injective.*

Proof. It is enough to show that if M is M-tight, then it is weakly M-injective. Therefore, let M be M-tight and let $f : M \to E(M)$ be an R-homomorphism. Clearly $K = \mathrm{Im}(f) \subseteq \widehat{M}$. Since M is M-tight, there exists a submodule L of M with $L \cong K$. Let L' and K' be closures (in $\widehat{M}$) of L and K, respectively. It is well known that L' and M' are M-injective hulls of L and K, respectively. Therefore the isomorphism $L \to K$ can be extended to an isomorphism $L' \to K'$, and both L' and K' are direct summands of $\widehat{M}$. Let L'' and K'' be the complements in $\widehat{M}$ of L' and K', respectively. Then $\widehat{M} = K' \oplus K'' = L' \oplus L''$.

Since M is non-M-singular, $\mathrm{End}(\widehat{M}_R)$ is von Neumann regular [5, Section 4.9(c)]. By [5, p. 35], $\mathrm{End}(\widehat{M}_R)$ is right self-injective. Since, by assumption, $\mathrm{End}(\widehat{M}_R)$ is directly finite, it follows from [9, Theorem 9.17] that $\mathrm{End}(\widehat{M}_R)$ is unit-regular. Therefore, by [9, Theorem 4.1], $L'' \cong K''$. Thus the isomorphism $L \to K$ can be extended to an automorphism g of $\widehat{M}$. Consequently, $\mathrm{Im}(f) = K = g(L) \subseteq g(M) \cong M$. □

Corollary 3.4. *Let M be a right R-module such that* $\mathrm{End}(\widehat{M}_R)$ *is directly finite. Suppose that M is self-generator. Then M is non-M-singular and CS if and only if M is weakly M-injective and* $\mathrm{End}(M_R)$ *is right PP.*

Corollary 3.5. *Let R be a right nonsingular ring such that $Q^r_{\max}(R)$ is unit regular* (*equivalently, directly finite*)*. Then R is right CS if and only if R is a right weakly selfinjective and right PP-ring.*

4. Applications

In this section we give applications of Theorem 3.2 and Corollary 3.5. We first state a well-known result.

Lemma 4.1 [5, Lemma 12.8]. *R^n_R is a CS-module if and only if the $n \times n$ matrix ring $M_n(R)$ over R is a right CS-ring.*

Theorem 4.2. *Let R be a right n-hereditary ring such that $Q^r_{\max}(R)$ is directly finite and let $M_n(R)$ $(n > 1)$ be right weakly selfinjective. Then R is right weakly selfinjective.*

Proof. Because R is n-hereditary, $M_n(R)$ is right PP [6, Exercise 12, p. 23]. Thus by Corollary 3.5, $M_n(R)$ is a right CS-ring. Therefore, by Lemma 4.1, R^n is CS as a right R-module. Consequently R_R is right CS. By Corollary 3.5, R is right weakly selfinjective. □

In particular, for a von Neumann regular ring, we can show that $M_n(R)$, $n > 1$, is right weakly selfinjective if and only if R is right selfinjective. First, we prove the following lemma.

Lemma 4.3. *Let R be a von Neumann regular ring. Then the following are equivalent*:

(1) *R is right weakly selfinjective.*
(2) *R is right R-tight.*
(3) *R is right CS.*
(4) *R is right continuous.*

Proof. (1) ⇒ (2) is obvious. (2) ⇒ (3) follows by Theorem 3.2. (3) ⇔ (4) follow by von Neumann regularity of R. We will prove (4) ⇒ (1). Assume R is right continuous. Then $R = R_1 \times R_2$ where R_1 is right selfinjective and R_2 is an abelian regular continuous ring [9, Theorem 13.17]. So, without any loss of generality, assume R is abelian regular continuous. Then $Q^r_{\max}(R) = Q^l_{\max}(R)$ is also abelian regular. Thus $Q^r_{\max}(R)$ is unit regular and hence by Corollary 3.5, R is right weakly selfinjective. □

The theorem that follows generalizes the results of Al-Huzali, Jain, and López–Permouth [1, Theorem 2.11] and Tannan [13, Theorem 3.6].

Theorem 4.4. *Let R be a von Neumann regular ring. Then the following are equivalent for $n > 1$:*

(1) *$M_n(R)$ is right weakly selfinjective.*
(2) *$M_n(R)$ is right $M_n(R)$-tight*
(3) *$M_n(R)$ is a right CS-ring.*
(4) *R is right selfinjective.*

Proof. (1) ⇔ (2) ⇔ (3) follow by Lemma 4.3. Since a regular ring is right CS if and only if it is right continuous, (3) ⇒ (4) follows by the fact that $M_n(R)$ is right continuous if and only if R is right selfinjective [9, Corollary 13.19]. (4) ⇒ (1) is trivial. □

We now proceed to obtain necessary and sufficient conditions for R^n_R to be a CS-module when R does not possess an infinite set of nonzero orthogonal idempotents. For the convenience of the reader, we state below some results that will be used latter.

Theorem 4.5 [5, 12.2, p. 105]. *A ring R is a right nonsingular right CS-ring if and only if R is a Baer ring such that every nonessential right ideal has nonzero right annihilator.*

Theorem 4.6 [5, Corollary 12.7]. *A ring R is a right and left nonsingular right and left CS-ring if and only if R is a Baer ring for which right and left maximal quotient rings coincide. In other words, the class of rings which are both Baer and Utumi is precisely the class of nonsingular CS-rings.*

Theorem 4.7 [3, Lemma 8.4]. *Let R be a right PP ring which does not possess any infinite set of orthogonal idempotents. Then R is a left PP ring, each right or left annihilator in R is generated by an idempotent, and acc and dcc hold for right annihilators.*

Lemma 4.8. *Let R be a right nonsingular right CS-ring and let S be a ring such that $R_R \subset_e S_R$. Then S is right CS.*

Proof. Let K be a closed right ideal of S. Then $K \cap R$ is a closed right ideal of R. Since R is right CS, $K \cap R = eR$ for some idempotent e in R. We claim that $(1-e)K = 0$. Let $a \in K$. Since $R_R \subset_e S_R$, there exists an essential right ideal E of R such that $0 \neq aE \subset R$. Thus $aE \subset K \cap R = eR$. But then $(1-e)aE = 0$. Since R_R and hence S_R is nonsingular, $(1-e)a = 0$. Hence $(1-e)K = 0$. Consequently $K \subset eS$. As $eR \subset_e eS$ and $eR \subset K$, $K = eS$, because K is closed. □

We now prove our next main result.

Theorem 4.9. *Let R be a ring with no infinite set of nonzero orthogonal idempotents* (*in particular, if* $\text{u.dim}(R_R) < \infty$) *and let $n > 1$ be any positive integer. Then the following are equivalent*:

(1) *R_R^n is a nonsingular CS right R-module.*
(2) *$M_n(R)$ is right weakly selfinjective and right PP.*
(3) *$M_n(R)$ is Utumi and Baer.*
(4) *R is Utumi and right n-hereditary.*
(5) *Left side versions of* (1)–(4).

Proof. (1) ⇒ (2). By Lemma 4.1, $M_n(R)$ is a right CS-ring. Since R contains no infinite set of orthogonal idempotents and is right CS, it is folklore that $\text{u.dim}(R_R) < \infty$ and thus the right maximal quotient ring $Q^r_{\max}(R)$ of R (and hence $Q^r_{\max}(M_n(R))$) is semisimple artinian. For the sake of completeness, we may sketch the proof of the fact that $\text{u.dim}(R_R) < \infty$. Assume $\text{u.dim}(R_R)$ is infinite and let K be a closed right ideal of infinite uniform dimension. Because K is closed, $K = eR$ for some idempotent $e \in R$. Write $K = K_1 \oplus L_1$ where $\text{u.dim}(K_1)$ is infinite and $L_1 \neq 0$. Let $e = k_1 + l_1$. Then $k_1^2 = k_1$, $l_1^2 = l_1$, $k_1 l_1 = 0$, and $l_1 k_1 = 0$. Repeating this process with K_1 and so on, we produce an infinite set of orthogonal idempotents, a contradiction. Thus by Corollary 3.5, $M_n(R)$ is right weakly selfinjective and right PP.

(2) ⇒ (1) follows by Corollary 3.5 and Lemma 4.1. Thus (1) ⇔ (2).

(2) ⇒ (3) Let $S = M_n(R)$, $Q = Q^r_{\max}(R)$. Also, as proved in the proof of (1) ⇒ (2), Q is semisimple artinian. Let $0 \neq q \in Q$. Consider the element

$$x = \begin{bmatrix} q & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in Q^r_{\max}(S) = M_n(Q).$$

Because S is right weakly selfinjective, there exists $y = (y_{ij}) \in Q^r_{\max}(S)$ such that $\mathrm{r.ann}_S(y) = 0$ and $x \in yS$. Since $Q^r_{\max}(S)$ is von Neumann regular, y is left invertible. As observed above, $Q^r_{\max}(S)$ is directly finite and hence y is invertible. Thus there exists $p = (p_{ij}) \in Q^r_{\max}(S)$ such that $py = 1 = yp$. Now $x \in yS$ implies $px \in S$. Thus

$$(p_{ij})\begin{bmatrix} q & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in S,$$

and so p_{i1}, $p_{i1}q \in R$ for all i with $1 \leqslant i \leqslant n$. If each $p_{i1}q = 0$, then using $yp = 1$, we obtain

$$\begin{bmatrix} q & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} = yp\begin{bmatrix} q & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$= (y_{ij})\begin{bmatrix} 0 & * & * & \dots & * \\ 0 & * & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots \\ 0 & * & * & \dots & * \end{bmatrix}.$$

This implies that $q = 0$, a contradiction. Hence $p_{i1}q \neq 0$ for some i, and this implies that ${}_RR \subset_e {}_RQ$. We now claim that R is left nonsingular. So, let $a \in R$ with $\mathrm{l.ann}_R(R) \subset_e {}_RR$. Since $\mathrm{l.ann}_R(a) \subset \mathrm{l.ann}_Q(a)$, it follows that $\mathrm{l.ann}_Q(a)$ is essential in ${}_RQ$ and hence in ${}_QQ$. But Q is von Neumann regular. Therefore $a = 0$. Thus R is left nonsingular. Since ${}_RR \subset_e {}_RQ$, we get $Q \subset Q^l_{\max}(R)$. Now ${}_RR$ is essential in $Q^l_{\max}(R)$ and $R \subset Q$. Therefore ${}_RQ \subset_e {}_RQ^l_{\max}(R)$ and so ${}_QQ \subset {}_QQ^l_{\max}(R)$. Since Q is left selfinjective, $Q = Q^\ell_{\max}(R)$. Thus R is Utumi and hence S is Utumi. Since S is right nonsingular and right CS, it is Baer by Theorem 4.5.

(3) $\Rightarrow$ (1) follows by Theorem 4.6.

(3) $\Rightarrow$ (4) Since $M_n(R)$ is Utumi, so is R. As $M_n(R)$ is right PP, R is right n-hereditary.

(4) $\Rightarrow$ (3). Since R is right n-hereditary and hence right PP and R has no infinite set of nonzero orthogonal idempotents, R is Baer by Theorem 4.7. Thus R is left and right CS by Theorem 4.6. So as explained in the proof of (1) $\Rightarrow$ (2), $\mathrm{u.dim}(R_R) < \infty$, and $\mathrm{u.dim}({}_RR) < \infty$. Therefore, the same holds for $M_n(R)$. In particular, $M_n(R)$ does not possess any infinite set of nonzero orthogonal idempotents. Since R is right n-hereditary, $M_n(R)$ is right PP [6, Exercise 12, p. 23] and so by Theorem 4.7, $M_n(R)$ is Baer. Since R is Utumi, $M_n(R)$ is also Utumi.

(5) $\Leftrightarrow$ (1) follows by the symmetry of conditions in (3). □

Remark 4.1. We may remark that the statements (1)–(3) in Theorem 4.9 are equivalent if we replace the hypothesis that R has no infinite set of orthogonal idempotents by a weaker hypothesis that $Q^r_{\max}(R)$ is a left selfinjective ring.

The corollary that follows answers an open question on finding necessary and sufficient conditions for $(R \times R)_R$ to be CS where R is any domain.

Corollary 4.10. *The following are equivalent for a domain* R:

(i) R^2_R *is CS.*
(ii) R *is right* 2*-hereditary two-sided Ore domain.*
(iii) *Left side versions of* (i) *and* (ii).

Before we give the next application, we prove another key lemma on reduced rings (rings with no nonzero nilpotent elements) that is also of independent interest.

Lemma 4.11. *Let R be a reduced ring such that R is right* 2*-hereditary and every nonessential principal right ideal has a nonzero right* (= *left*) *annihilator. Then $Q^r_{\mathrm{cl}}(R)$ exists and is von Neumann regular.*

Proof. As R is reduced, $xy = 0$ if and only if $yx = 0$. Let $a \in R$. Since aR is projective, $\mathrm{r.ann}_R(a) = eR$, $e = e^2$ is central because R is reduced. We claim that (1) $a + e$ is a regular element. For, if $d(a+e) = 0$ then $d(a+e)e = 0$. Thus $de = 0$ and hence $da = 0$. This gives $d = de = 0$, proving $a + e$ is regular.

Next, we claim that (2) if for some $a, b \in R$, $aR + bR \subset_e R$, and $\mathrm{r.ann}_R(a) = eR$ then $a + eb$ is a regular element. We note that $a \in (1 - e)R$ and $aR + bR \subset (1 - e)R + ebR \subset_e R$. If $(a + eb)c = 0$ then by multiplying with e, $ebc = 0$ and so $ac = 0$. Then $c \in \mathrm{r.ann}_R(a) = eR$, and thus $c(1 - e)R = 0$. Now $ebc = 0$ implies $cebR = 0$. Thus c annihilates the essential right ideal $(1 - e)R + ebR$, proving $c = 0$ because R is right (as well as left) nonsingular.

We now prove that the intersection $aR \cap bR$ of any two principal essential right ideals aR and bR contains a regular element. Since $aR + bR$ is projective, the exact sequence

$$0 \longrightarrow aR \cap bR \xrightarrow{f} aR \times bR \longrightarrow aR + bR \longrightarrow 0,$$

where $f(x) = (x, -x)$, splits and so $aR \cap bR$ is 2-generated right ideal, say $cR + dR$, and is essential. Thus by claim (2) above $c + de$ is a regular element where $\mathrm{r.ann}_R(c) = eR$, proving our claim.

Finally, we prove that $Q^r_{\mathrm{cl}}(R)$ exists and is von Neumann regular. To show the existence, we proceed to prove the right Ore condition. Let $p, q \in R$ where p is regular. Let $\mathrm{r.ann}_R(q) = (1 - u)R$, $u = u^2$. Then $q = qu \in uR$ and q is regular in the ring uR. Also pu is regular in the ring uR. Since each nonessential right ideal in R has a nonzero right annihilator, the same holds in the ring direct summand uR. Thus $qR = quR$ and puR are essential right ideals in uR and hence by the result proved in the previous paragraph $qR \cap puR$ contains a regular element, say x. Then $x = qd = puy$ for some $y, d \in uR$. Clearly, d is regular in uR and so $\mathrm{r.ann}_R(d) = (1 - u)R$. By claim (1), $d + (1 - u)$ is regular in R. Therefore, $p(uy) = qd = q(d + 1 - u)$, proving right Ore condition.

To prove $Q = Q^r_{\mathrm{cl}}(R)$ is von Neumann regular, let $a \in R$ and $\mathrm{r.ann}_R(a) = eR$, $e = e^2$. Recall $a + e$ is regular and so $(a + e)^{-1} \in Q$, and $a(a + e)^{-1} = 1 - e$. This gives

$a(a+e)^{-1}a = (1-e)a = a$. Thus for any $ab^{-1} \in Q$, we have $ab^{-1}[b(a+e)^{-1}]ab^{-1} = ab^{-1}$. This completes the proof. □

As a consequence of the above lemma, we have the following interesting corollary.

Corollary 4.12. *Every reduced right 2-hereditary, right CS-ring has a right classical ring of quotients which is von Neumann regular.*

Proof. Let cR be a nonessential right ideal of R. Because R is right CS, cR is essential in some eR, $e = e^2$, $e \neq 1$. This implies $(1-e)c = 0$ and so $\text{r.ann}_R(c) = \text{r.ann}_R(cR) \neq 0$. By Lemma 4.11, $Q^r_{\text{cl}}(R)$ exists and is von Neumann regular. □

Theorems 4.13 and 4.14 give two facts from Al-Huzali–Jain–Lopez [1, Lemma 2.10 and Theorem 3.3)], that will be needed in the proof of Theorem 4.15.

Theorem 4.13 [1, Lemma 2.10]. *Let R be a right nonsingular ring. Then the following are equivalent*:

(i) *R is a right weakly-injective ring.*
(ii) *For all $q_1, q_2 \in Q = Q^r_{\max}(R)$, there exists $c \in R$ such that $q_1, q_2 \in c^{-1}R$. In particular, Q is left classical quotient ring of R.*

Theorem 4.14 [1, Theorem 3.3]. *Let R be a right nonsingular ring. Then the following are equivalent*:

(i) *R is a right weakly-injective ring.*
(ii) *$S = M_n(R)$ is a right weakly-injective ring.*

Theorem 4.15. *Let R be a reduced ring and n be a positive integer greater than* 1. *Then the following are equivalent*:

(1) *$M_n(R)$ is right CS.*
(2) *R is right n-hereditary and $Q^l_{\text{cl}}(R) = Q^r_{\max}(R)$.*
(3) *R is right n-hereditary and right weakly injective.*
(4) *$M_n(R)$ is right weakly injective and right PP.*
(5) *$M_n(R)$ is right weakly selfinjective and right PP.*
(6) *Left side versions of* (1)–(5).

Under any of the equivalent conditions (1)–(6), *R is also an Utumi ring.*

Proof. (1) ⇒ (5). We show that $Q^r_{\max}(R)$ is unit regular. Let A, B be right ideals in R such that $A \cap B = (0)$. Because R is right CS, $A \subset_e eR$, $B \subset_e fR$ where $e = e^2$, $f = f^2$. Then $eR \cap fR = (0)$. Because e and f are central idempotents, $eRfR = 0$. Thus $AB = 0$. It follows that $Q^r_{\max}(R)$ is strongly regular [12, Proposition 21.3)]. Therefore $Q^r_{\max}(M_n(R))$ is unit regular. Then by Corollary 3.5, we obtain (5).

(5) $\Rightarrow$ (1). This follows by Theorem 3.2.

Under (1) or (5) we make the following observation. We already know that $Q^r_{\max}(R)$ is strongly regular. Since $Q^r_{\max}(R)$ is right selfinjective, it is left selfinjective by [9, Corollary 3.9]. Therefore, $Q^r_{\max}(M_n(R)) = M_n(Q^r_{\max}(R))$ is both left and right selfinjective. In particular, it is directly finite by [9, Theorem 9.29]. Next, since $M_n(R)$ is right weakly selfinjective, the same argument as in the proof of (2) $\Rightarrow$ (3) of Theorem 4.9 shows that ${}_RR \subset_e {}_RQ^r_{\max}(R)$ and $Q^r_{\max}(R) = Q^l_{\max}(R)$. Thus R and $M_n(R)$ are Utumi. But $M_n(R)$ is also Baer by (1) and Theorem 4.5. So $M_n(R)$ is left CS by Theorem 4.6. This proves that (1)$\Leftrightarrow$(5)$\Leftrightarrow$ left side versions of (1) and (5).

Now we prove (1) $\Rightarrow$ (2). Since (1) $\Leftrightarrow$ (5), R is right–left n-hereditary, and right–left CS. By Corollary 4.12, both $Q^r_{\text{cl}}(R)$ and $Q^l_{\text{cl}}(R)$ exist and so $Q^r_{\text{cl}}(R) = Q^l_{\text{cl}}(R) = Q_{\text{cl}}(R)$. is von Neumann regular.

Now, $M_n(R) \subset_e M_n(Q_{\text{cl}}(R))$, and so by Lemma 4.8, the ring $M_n(Q_{\text{cl}}(R))$ is a right–left CS-ring. Because $Q_{\text{cl}}(R)$ is von Neumann regular (Corollary 4.12), $Q_{\text{cl}}(R)$ is right–left selfinjective (Theorem 4.4). Therefore, $Q_{\text{cl}}(R) = Q^r_{\max}(R) = Q^l_{\max}(R)$.

(2) $\Rightarrow$ (3) follows by Theorem 4.13 and [6, Exercise 12, p. 23]. (3) $\Rightarrow$ (4) follows by Theorem 4.14. (4) $\Rightarrow$ (5) is obvious. This completes the proof. □

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Further reading

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