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Automorphisms of Hilbert space effect algebras ☆

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Abstract

We consider bijections of the Hilbert space effect algebra that preserve the algebraic structures in one direction and have some other properties. It is shown that if $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ is conditionally multiplicative and conditionally additive, then ϕ is implemented by a unitary or antiunitary operator on $\mathscr{H}$. We also show that 2-local ortho-order automorphisms on $\mathscr{E}(\mathscr{H})$ are of the same form if $\dim \mathscr{H} \geqslant 3$.

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Keywords: Hilbert space effects; 2-local ortho-order automorphism

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. Denote by $\mathscr{B}(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$. The operator interval $\mathscr{E}(\mathscr{H})$ of all positive operators in $\mathscr{B}(\mathscr{H})$ which are bounded by the identity I is called the Hilbert space effect algebra. Effect algebras play an important role in the mathematical foundations of quantum mechanics.

The effect algebra $\mathscr{E}(\mathscr{H})$ can be equipped with several algebraic operations. For example, one can define a partial addition on it. Namely, if $A, B \in \mathscr{E}(\mathscr{H})$ and $A +$

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$B \in \mathscr{E}(\mathscr{H})$, then one can set $A \oplus B = A + B$. Moreover one can define natural partial ordering $\leqslant$ which comes from the usual ordering on the set of self-adjoint operators on $\mathscr{H}$ and one can also define the operation of so-called orthocomplementation by $\perp$: $A \mapsto I - A$. Finally, as for the multiplicative operation on $\mathscr{E}(\mathscr{H})$, note that in general $A, B \in \mathscr{E}(\mathscr{H})$ does not imply that $AB \in \mathscr{E}(\mathscr{H})$. However we have $ABA \in \mathscr{E}(\mathscr{H})$. This multiplication is sometimes called Jordan triple product. Also, if A and B commute, then $AB \in \mathscr{E}(\mathscr{H})$.

Because of the importance of effect algebra, it is a natural problem to study the isomorphisms of the mentioned structures. In a series of papers [5–7,9,10], the authors studied these isomorphisms. In fact, they studied bijections $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ that preserve the algebraic structures in both directions and obtained many significant and interesting results.

In this note we consider such bijections that preserve the algebraic structures in one direction and have some other properties. Especially we consider conditions for the bijections to be implemented by unitary or antiunitary operators.

Let us fix some notations and terminologies. Let $\mathscr{P}(\mathscr{H})$ denote the set of all projections on $\mathscr{H}$. A map ϕ is said to be orthoadditive on $\mathscr{P}(\mathscr{H})$ if for every pair of orthogonal projections P, Q we have $\phi(P + Q) = \phi(P) + \phi(Q)$. By an antiunitary operator we mean a norm preserving conjugate-linear bijection of the underlying Hilbert space $\mathscr{H}$. For $x, y \in \mathscr{H}$ we denote by $x \otimes y$ the operator defined by $(x \otimes y)(z) = \langle z, y\rangle x (z \in \mathscr{H})$.

2. Results

Our first theorem is proved by borrowing the idea of [10, Theorem 2].

Theorem 1. *Suppose that $\mathscr{H}$ is a separable Hilbert space with* $\dim \mathscr{H} \geqslant 3$. *Let $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ be a bijective map satisfying*

$$[AB = BA] \Rightarrow [\phi(AB) = \phi(A)\phi(B) = \phi(B)\phi(A)],$$

and

$$[AB = BA \text{ and } A + B \in \mathscr{E}(\mathscr{H})] \Rightarrow [\phi(A + B) = \phi(A) + \phi(B)],$$

for $A, B \in \mathscr{E}(\mathscr{H})$. Then there exists an either unitary or antiunitary operator U on $\mathscr{H}$ such that $\phi(A) = UAU^*(A \in \mathscr{E}(\mathscr{H}))$.

Proof. Note that ϕ maps projections to projections, preserves the order on the set of projections and it is orthoadditive on $\mathscr{P}(\mathscr{H})$. By [2], the restriction $\phi|_{\mathscr{P}(\mathscr{H})}$ of ϕ to $\mathscr{P}(\mathscr{H})$ can be extended to a bounded linear operator $\tilde{\phi} : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$. By a standard argument, it follows that $\tilde{\phi}$ is a Jordan $*$-homomorphism. First assume that $\mathscr{H}$ is separable and infinite dimensional. Let $x \in \mathscr{H}$ be with $\|x\| = 1$. As ϕ is surjective, there is an element $A \in \mathscr{E}(\mathscr{H})$ such that $\phi(A) = x \otimes x$. By the conditional multiplicativity, we have $\phi(A^2) = \phi(A)^2 = x \otimes x = \phi(A)$. As ϕ is injective,

A is a projection. Hence the image of $\tilde{\phi}$ contains a rank one projection. Similarly, the identity I is contained in the range of $\tilde{\phi}$. Since the range of $\tilde{\phi}$ contains a rank-one operator and an operator with dense range, it follows that $\tilde{\phi}$ is bijective by [7, Theorem 1]. Secondly assume that $\dim \mathscr{H} = n$ is finite. Since M_n is a simple ring, it is simple as a Jordan ring. So, $\ker \tilde{\phi} = \{0\}$. Hence $\tilde{\phi}$ is surjective. Since any Jordan $*$-automorphism on $\mathscr{B}(\mathscr{H})$ is an either $*$-automorphism or $*$-antiautomorphism, $\tilde{\phi}$ is an either $*$-automorphism or $*$-antiautomorphism. The structures of these maps are well-known. In fact, they are of the forms

$$A \mapsto UAU^*, \quad A \mapsto VA^*V^*,$$

where U is a unitary and V is an antiunitary operator on $\mathscr{H}$. So, ϕ is of the form

$$\phi(P) = UPU^* \quad (P \in \mathscr{P}(\mathscr{H})),$$

where U is an either unitary or antiunitary operator on $\mathscr{H}$. We may assume without loss of generality that $\phi(P) = P$. For $0 \leqslant \lambda \leqslant 1$, we have $\phi(\lambda A) = \phi(\lambda I)\phi(A) = \phi(A)\phi(\lambda I)$. So, $\phi(\lambda I)$ commutes with every element of $\mathscr{E}(\mathscr{H})$ and hence there exists a function $f : [0, 1] \to [0, 1]$ such that $\phi(\lambda I) = f(\lambda)I$. Note that f is multiplicative, injective and strictly increasing. Also, if $\lambda, \mu \in [0, 1]$ and $0 \leqslant \lambda + \mu \leqslant 1$, then $f(\lambda + \mu) = f(\lambda) + f(\mu)$. Since every strictly increasing function on $\mathbb{R}$ has at most countable discontinuities, it follows that f has at least one point of continuity in $[0, 1]$. From $f(\lambda + \mu) = f(\lambda) + f(\mu)$, it is easy to show that f is continuous. Hence $f([0, 1])$ is an interval. As $f(0) = 0$ and $f(1) = 1$, we have that f is surjective. It is well-known that any continuous bijection $g : [0, 1] \to [0, 1]$ which is multiplicative is of the form $g(\lambda) = \lambda^\rho$ for some $\rho > 0$. As f is additive, we have $\rho = 1$, so $f(\lambda) = \lambda$. Hence it follows that $\phi(\lambda P) = \phi(\lambda I)\phi(P) = f(\lambda)\phi(P) = \lambda P$.

Let $A \in \mathscr{E}(\mathscr{H})$ and denote its spectral measure on the Borel sets of $[0, 1]$ by E_A. Let $0 < \lambda_1 < \lambda_2 \cdots < \lambda_n < 1$ be arbitrary. Denote

$$P_1 = E_A([0, \lambda_1]),\ P_2 = E_A((\lambda_1, \lambda_2]), \ldots,$$

$$P_n = E_A((\lambda_{n-1}, \lambda_n]),\ P_{n+1} = E_A((\lambda_n, 1]).$$

Set

$$A_\lambda = 0P_1 + \lambda_1 P_2 + \cdots + \lambda_n P_{n+1},$$

and

$$A^\lambda = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n + 1P_{n+1}.$$

By the Borel function calculus, it is easy to see that there are operators $B, C \in \mathscr{E}(\mathscr{H})$ such that $AB = BA$, $A^\lambda C = CA^\lambda$ and $A_\lambda = AB$, $A = A^\lambda C$. By the conditional additivity, we have $A_\lambda = \phi(A_\lambda)$. Hence it follows that

$$A_\lambda = \phi(A_\lambda) = \phi(AB) = \phi(A)\phi(B) \leqslant \phi(A).$$

Also,

$$\phi(A) = \phi(A^\lambda)\phi(C) \leqslant \phi(A^\lambda) = A^\lambda.$$

As these inequalities hold for every division $\lambda_1, \ldots, \lambda_n$ of [0, 1] appearing as above and as A_λ, A^λ approximate A in the operator norm, we deduce that

$$A \leqslant \phi(A) \quad \text{and} \quad \phi(A) \leqslant A.$$

Consequently, we have $\phi(A) = A (A \in \mathscr{E}(\mathscr{H}))$ and this completes the proof. □

In [8], the author studied bijective maps $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ which preserve the order and zero product in both directions, i.e., which satisfy

$$A \leqslant B \Leftrightarrow \phi(A) \leqslant \phi(B) \quad (A, B \in \mathscr{E}(\mathscr{H})),$$

and

$$AB = 0 \Leftrightarrow \phi(A)\phi(B) = 0 \quad (A, B \in \mathscr{E}(\mathscr{H})).$$

Theorem 2. *Suppose that $\mathscr{H}$ is a separable Hilbert space with* $\dim \mathscr{H} \geqslant 3$. *Let $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ be a bijective map which preserves the order in both directions. If*

$$[AB = 0] \Rightarrow [\phi(A)\phi(B) = 0] \quad (A, B \in \mathscr{E}(\mathscr{H}))$$

then there exists an either unitary or antiunitary operator U on $\mathscr{H}$ and a real number $\rho < 1$ such that with the function $f_\rho(x) = \frac{x}{x\rho + (1-\rho)} (x \in [0, 1])$ we have

$$\phi(A) = U f_\rho(A) U^* \quad (A \in \mathscr{E}(\mathscr{H})).$$

Here, $f_\rho(A)$ denotes the image of the function f_ρ under the continuous function calculus belonging to the operator A.

Proof. Recall that every bijection of the effect algebra of a Hilbert space which preserves the order in both directions preserves the projections as well as their ranks in both directions (see [4, Theorem 5.8]). Let $P, Q \in \mathscr{P}(\mathscr{H})$ be orthogonal projections. As $\phi(P)\phi(Q) = 0$, $\phi(P) + \phi(Q) = \phi(R)$ for some projection R. Then $\phi(P) \leqslant \phi(R)$, $\phi(Q) \leqslant \phi(R)$ and we have $P + Q \leqslant R$. Since ϕ is monotone it follows that $\phi(P + Q) \leqslant \phi(R)$. It is clear that $\phi(P) + \phi(Q) \leqslant \phi(P + Q)$. Hence $\phi(P) + \phi(Q) = \phi(P + Q)$. Since ϕ is orthoadditive on $\mathscr{P}(\mathscr{H})$, there exists an either unitary or antiunitary operator V on $\mathscr{H}$ such that $\phi(P) = VPV^* (P \in \mathscr{P}(\mathscr{H}))$, as in the first part of the proof of Theorem 1. In particular, two projections are orthogonal if and only if their images are orthogonal. We assert that for $A, B \in \mathscr{E}(\mathscr{H})$, $\phi(A)\phi(B) = 0$ implies $AB = 0$. Let R be the range projection of A (i.e., the projection onto the closure of the range of A). Then R is the infimum of the set of all projections which are greater than or equal to A. Since ϕ preserves the order in both directions, it follows that $\phi(R)$ is the range projection of $\phi(A)$. Note that for any $A, B \in \mathscr{E}(\mathscr{H})$, we have $AB = 0$ if and only if the range projections are orthogonal. From these facts, we have that $\phi(A)\phi(B) = 0$ implies $AB = 0$. Hence ϕ is a bijection preserving order and zero product in both directions. The conclusion follows from [8, Theorem 1]. □

A map $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ is called a 2-local ortho-order automorphism if for every $A, B \in \mathscr{E}(\mathscr{H})$ there is an ortho-order automorphism $\phi_{A,B}$ of $\mathscr{E}(\mathscr{H})$ (that is, an automorphism of $\mathscr{E}(\mathscr{H})$ with respect to the relations $\leqslant$ and $\perp$) for which $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$. The notion of 2-locality was introduced by Šemrl [11] who obtained the first results on 2-local automorphisms and 2-local derivations on $\mathscr{B}(\mathscr{H})$. In [1, Theorem 3], the authors showed that every local automorphism of $\mathscr{E}(\mathscr{H})$ is an automorphism for separable infinite dimensional Hilbert space $\mathscr{H}$. In the following theorem, we show that this holds true for finite dimensional cases, too. In [5, Proposition], the author obtained the general form of 2-local automorphisms of the orthomodular poset $\mathscr{P}(\mathscr{H})$ for separable infinite dimensional $\mathscr{H}$.

Theorem 3. *Suppose that $\mathscr{H}$ is a separable Hilbert space with* $\dim \mathscr{H} \geqslant 3$. *Let $\phi : \mathscr{E}(\mathscr{H}) \to \mathscr{E}(\mathscr{H})$ be a 2-local ortho-order automorphism. Then there exists an either unitary or antiunitary operator V on $\mathscr{H}$ such that $\phi(A) = VAV^*(A \in \mathscr{E}(\mathscr{H}))$.*

Proof. Let $A, B \in \mathscr{E}(\mathscr{H})$. Then by [3], there is an either unitary or antiunitary operator U, depending on A and B, such that $\phi(A) = UAU^*$ and $\phi(B) = UBU^*$. Hence the restriction $\phi|_{\mathscr{P}(\mathscr{H})} : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ of ϕ to the set of projections is a 2-local ortho-order automorphism. First assume that $\dim \mathscr{H} = \infty$. By [5], there exists an either unitary or antiunitary operator V on $\mathscr{H}$ such that $\phi(P) = VPV^*(P \in \mathscr{P}(\mathscr{H}))$. Secondly assume that $\dim \mathscr{H} = n < \infty$. From the 2-local property of ϕ, it follows that $\tau(\phi(A)\phi(B)^*) = \tau(AB^*)$ for every $A, B \in \mathscr{E}(\mathscr{H})$, where τ is the trace functional of M_n. Hence if $A, B, C, A + B \in \mathscr{E}(\mathscr{H})$, then by the linearity of τ we have

$$\tau[(\phi(A + B) - \phi(A) - \phi(B))\phi(C)^*] = 0,$$

and hence

$$\tau[(\phi(A + B) - \phi(A) - \phi(B))(\phi(A + B) - \phi(A) - \phi(B))^*] = 0,$$

from which it follows that ϕ is partially additive. In particular, if P and Q are orthogonal projections, we have $\phi(P + Q) = \phi(P) + \phi(Q)$. So, the restriction $\phi|_{\mathscr{P}(\mathscr{H})} : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ of ϕ to $\mathscr{P}(\mathscr{H})$ extends to a continuous linear map $\tilde{\phi} : M_n \to M_n$ by [2]. As in the proof of Theorem 1, $\tilde{\phi}$ is an either $*$-automorphism or $*$-antiautomorphism. Hence as in the infinite dimensional case, there exists an either unitary or antiunitary operator V on $\mathscr{H}$ such that $\phi(P) = VPV^*(P \in \mathscr{P}(\mathscr{H}))$. Let $x \in \mathscr{H}$ be with $\|x\| = 1$, $A \in \mathscr{E}(\mathscr{H})$ and U be an either unitary or antiunitary operator that corresponds to the projection $x \otimes x$ and to the operator A by the 2-locality of ϕ. We then have

$$\begin{aligned}
\langle Ax, x\rangle Ux \otimes Ux &= U(x \otimes x)U^*UAU^*U(x \otimes x)U^* \\
&= \phi(x \otimes x)\phi(A)\phi(x \otimes x) \\
&= V(x \otimes x)V^*\phi(A)V(x \otimes x)V^* \\
&= \langle \phi(A)Vx, Vx\rangle Vx \otimes Vx \\
&= \langle V^*\phi(A)Vx, x\rangle Vx \otimes Vx,
\end{aligned}$$

from which it follows that $\phi(A) = VAV^*(A \in \mathscr{E}(\mathscr{H}))$, completing the proof. □

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