Nonstationary Two-Stage Multisplitting Methods for Symmetric Positive Definite Matrices

ZHONG-YUN LIU
Department of Mathematics, Shanghai University
Shanghai, 200436, P.R. China
kunyus@yahoo.com  zyliu640263.net

LU LIN
Department of Mathematics, Xiamen University
Xiamen 361005, P.R. China

CHUN-CHAO SHI
Department of Mathematics, Fudan University
Shanghai, 200433, P.R. China

(Received August 1999; accepted September 1999)

Communicated by D. J. Rose

Abstract—Nonstationary synchronous two-stage multisplitting methods for the solution of the symmetric positive definite linear system of equations are considered. The convergence properties of these methods are studied. Relaxed variants are also discussed. The main tool for the construction of the two-stage multisplitting and related theoretical investigation is the diagonally compensated reduction (cf. [1]). © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Nonstationary two-stage multisplitting, Diagonal compensation reduction, Block diagonal conformable.

1. INTRODUCTION

Consider the solution of a large linear system of equations

\[ Ax = b \] (1)

on parallel computers, where \( A \) is symmetric positive definite (SPD). The multisplitting method was introduced by O'Leary and White [2] and further studied by many authors, see e.g., [2–6]. Furthermore, the relationship between the two-stage method (cf. [7]) and the multisplitting

Supported by the State Major Key Project for Basic Researches and Doctoral Point Foundation of China and NSF of Hu-Nan Province of China.

The authors would like to acknowledge the referees for reading carefully the manuscript and making some useful suggestions, which led to many improvements in the original manuscript.
method was considered by Szyld and Jones in [8]. As a result, the two-stage multisplitting method was naturally introduced. Recently, parallel, synchronous, and asynchronous two-stage multisplitting methods have been presented and widely studied, see e.g., [9-11]. As a special case of two-stage and multisplitting methods, nonstationary multisplitting method is also investigated, see e.g., [3,12,13]. But the attention was mainly concentrated on monotone matrix and H-matrix. Only a little attention was focused on an SPD matrix, see e.g., [2,4,7].

In this paper, we will investigate the convergence of (relaxed) nonstationary two-stage multisplitting methods (cf. [10]) for symmetric positive definite matrices. In particular, the construction of the multisplitting and related theoretical investigation, which are based on the diagonal compensation reduction of nonnegative off-diagonal entries of \( A \)—a technique originally developed and analyzed by Axelsson and Kolotilina in [1], are different from the traditional iterative methods based on P-regular splitting and P-regular splitting theorem (cf. [13,14]).

2. PRELIMINARIES

We begin with some basic notation (cf. [13-15]).

A matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) is called a Z-matrix if \( a_{ij} \leq 0 \) for \( i \neq j \). If \( A \) is a nonsingular Z-matrix and \( A^{-1} \geq 0 \), then \( A \) is called an M-matrix.

A splitting \( A = M - N \) of \( A \) is called regular if \( M^{-1} \geq 0 \) and \( N \geq 0 \), weak regular if \( M^{-1} \leq 0 \) and \( M^{-1}N \geq 0 \), convergent if \( \rho(M^{-1}N) < 1 \). Here \( \rho(C) \) denotes the spectral radius of the matrix \( C \).

For any matrix \( A \in \mathbb{R}^{n \times n} \), the matrix \( A^T \) denotes the transpose. Similarly, the vector \( x^T \) denotes the transpose of a vector \( x \in \mathbb{R}^n \). Let \( A \in \mathbb{R}^{n \times n} \) be an SPD matrix, then \( (x, y) = x^T A y \) defines an inner product on \( \mathbb{R}^n \). Therefore, \( \|x\|_A = (x^T A x)^{1/2} \) is a vector norm on \( \mathbb{R}^n \). The matrix norm induced by that vector norm is also denoted by \( \| \cdot \|_A \).

For a two-stage multisplitting of \( A \), the description can be found in [8,11].

DEFINITION 1. A collection of pentads of matrices \((M_k, F_k, G_k, N_k, E_k)\), \( k = 1, \ldots, K \), satisfying

(i) \( A = M_k - N_k \), \( k = 1, \ldots, K \), are \( K \) splittings of \( A \),
(ii) \( \sum_{k=1}^K E_k = I \),
(iii) \( M_k = F_k - G_k \) is a splitting of \( M_k \),

is called a two-stage multisplitting of \( A \).

The two-stage multisplitting algorithm is as follows.

ALGORITHM 1. STATIONARY TWO-STAGE MULTISPLITTING. Given the initial vector \( x^0 \) and a fixed positive integer \( p \),

for \( j = 0, 1, \ldots, \), until convergence,

for \( k = 1 \) to \( K \),

\[ y_k^0 = x^j, \]

for \( i = 0 \) to \( p - 1 \),

\[ F_k y_k^{i+1} = G_k y_k^i + N_k x^j + b, \]  

\[ x^{j+1} = \sum_{k=1}^K E_k y_k^i, \]

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).

In Algorithm 1, a relaxation parameter \( \omega > 0 \) can be introduced by replacing the computation of \( y_k^{i+1} \) in (2) with the equation (cf. [9])

\[ F_k y_k^{i+1} = \omega (G_k y_k^i + N_k x^j + b) + (1 - \omega) F_k y_k^i \]  

When the number of the inner iterations \( p \) varies for \( j, k \), the outer iteration and the processor indices, i.e., when \( p = p(j, k) \) in Algorithm 1, we have a nonstationary two-stage multisplitting algorithm (Algorithm 2).
or $x^{j+1}$ in (3) with the equation (cf. [5])

$$x^{j+1} = \omega \sum_{k=0}^{K} E_k y_k^{p_j} + (1 - \omega) x^j. \tag{5}$$

Then we obtain a inner or outer relaxed nonstationary two-stage multisplitting algorithm, denoted by Algorithm 3 or Algorithm 4, respectively.

We rewrite Algorithm 2 in the following form:

$$x^{j+1} = \sum_{k=1}^{K} E_k \left[ (F_{k}^{-1} G_{k})^{p(j,k)} x^{j} + \sum_{i=0}^{p(j,k)-1} (F_{k}^{-1} G_{k})^{i} F_{k}^{-1} (N_{k} x^{j} + b) \right]. \tag{6}$$

The iteration matrix corresponding to (6) is

$$T(j) = \sum_{k=1}^{K} E_k \left[ (F_{k}^{-1} G_{k})^{p(j,k)} + \sum_{i=0}^{p(j,k)-1} (F_{k}^{-1} G_{k})^{i} F_{k}^{-1} N_{k} \right]. \tag{7}$$

It is well known that for the stationary case, i.e., $T(j) = T$, the method (6) converges for any initial vector $x^0$ if and only if $\rho(T) < 1$; and for the nonstationary case, using the error analysis, the method (6) is convergent for any initial vector $x^0$ if and only if $\lim_{j \to \infty} T(j) T(j-1) \cdots T(0) = 0$ (cf. [9,10,12,16]).

**Definition 2.** (See [17].) A two-stage multisplitting $(M_k, F_k, G_k, E_k, N_k, \Theta_k, \gamma_k)$ of $A$, where $M_k, F_k, E_k$ are square block diagonal matrices, denoted by $M_k = \text{diag}(M_{k_{11}}^{(1)})$, $F_k = \text{diag}(F_{k_{11}}^{(1)})$, $E_k = \text{diag}(E_{k_{11}}^{(1)})$, respectively, and satisfying

(i) the size of $M_{k_{ij}}^{(1)}$ coincides with that of $F_{k_{ij}}^{(1)}$, $i = 1, \ldots, K$,

(ii) $M_{k_{ij}}^{(1)} = F_{k_{ij}}^{(1)} - G_{k_{ij}}^{(1)}$, $i, k = 1, \ldots, K$,

is called a block diagonal conformable two-stage multisplitting.

**Definition 3.** (See [1].) Let $A$ be an SPD matrix, then the matrix $\hat{A} = B + D$ is called a diagonally compensated reduced matrix of $A$, where $B = A - R$ with $R$ symmetric and nonnegative (called the reduced entry matrix) and $D$ is a diagonal matrix (called the diagonal compensation matrix) satisfying $D v = R v$ for a positive vector $v$ selected arbitrarily.

**Remark 1.** (See [4].) We note that if the reduced matrix $B$ is a $Z$-matrix (e.g., all positive offdiagonal entries of $A$ are reduced by the reduced entry matrix $R$), then $\hat{A} = D + B$ is a $Z$ matrix, too. Since $\hat{A}$ is positive semidefinite, $\hat{A}$ is a Stieltjes matrix (i.e., $\hat{A}$ is an SPD Z-matrix). Hence, $\hat{A}$ is an SPD $M$-matrix (cf. [15].) Thus, the problem of constructing a convergent splitting for an SPD matrix, owing to Lemma 2, can always be reduced to that for a Stieltjes matrix which is an SPD $M$-matrix. Henceforth, when we construct a diagonally compensated reduced matrix $\hat{A}$ from an SPD matrix $A$, we will always make $\hat{A}$ be a Stieltjes matrix.

### 3. MAIN RESULT

**Lemma 1.** Let $A$ be SPD, let $\hat{A}$ be a diagonally compensated reduced Stieltjes matrix of $A$. Let the splitting $\hat{A} = M - \hat{N}$ be regular and $M = F - G$ be weak regular, where $M$ and $F$ are SPD matrices. Then the matrix $M_T = M(I - (F^{-1} G)^p)^{-1}$ is SPD. Furthermore, the resulting two-stage method corresponding to the diagonally compensated reduced two-stage splitting $A = M - N$, $M = F - G$ is convergent for any initial vector $x^0$ and for any number $p \geq 1$ of inner iterations.

**Proof.** Note that the iteration matrix corresponding to the two-stage splitting $\hat{A} = M - \hat{N}$, $M = F - G$ is $T = T_\nu = (F^{-1} G)^p + \sum_{i=0}^{p-1} (F^{-1} G)^i F^{-1} \hat{N}$. By Theorem 4.2 in [11], we yield $\rho(T) < 1$
and it induces a unique weak regular splitting $\hat{A} = M_T - N_T$, where $M_T = M(I - (F^{-1}G)^p)^{-1}$. The iteration matrix corresponding to the two-stage splitting $\hat{A} = M - N$, $M = F - G$ is

$I = T_p = (F^{-1}G)^p + \sum_{i=0}^{p-1} (F^{-1}G)^i F^{-1}$. It induces the splitting $A = M_T - N_T$, where $M_T = M(I - (F^{-1}G)^p)^{-1}$. Obviously, $M_T = M_T$ and $M_T^{-1} = \sum_{i=0}^{p-1} (F^{-1}G)^i F^{-1}$ is symmetric.

Now we show that $M_T$ is positive definite. For $p = 1$, $M_T^{-1} = F$ is positive definite. For $p = 2$,

$$M_{T_2} = M \left(I - (F^{-1}G)^2\right)^{-1}$$

$$= M + M(F^{-1}G)^2 + \cdots + M(F^{-1}G)^{2k} + \cdots$$

$$= M + GF^{-1}MF^{-1}G + \cdots + (GF^{-1})^k M(F^{-1}G)^k + \cdots$$

Each term after the first term is positive semidefinite. Thus, $M_{T_2}$ are both positive definite, where $M_{T_2} = F^{-1} + F^{-1}GF^{-1}$. Note that for $p \geq 1$, $M_T^{-1}$ can be represented by the sum of such terms which are $(F^{-1}G)^i M_{T_2}^i (GF^{-1})$ or $(F^{-1}G)^i F^{-1}(GF^{-1})^i$, for $i = 0, 1, \ldots, p - 1$, they are all symmetric positive semidefinite. Hence, $M_{T_2}$ and $M_T$ are both positive definite. By the Theorem 2.1 in [1], we obtain $\rho(T_p) < 1$.

**Theorem 2.** Let $A$ be SPD, let $\hat{A}$ be a diagonally compensated reduced Stieltjes matrix of $A$. Let the two-stage multisplitting $(M_k, F_k, G_k, N_k, E_k)$ with the splitting $\hat{A} = M_k - N_k$ regular and the splitting $M_k = F_k - G_k$ weak regular, for $k = 1, \ldots, K$, be block diagonal conformable, where $M_k$ and $F_k, k = 1, \ldots, K$, are SPD matrices. If either $N$ or $G$ is positive definite, then the resulting nonstationary two-stage multisplitting method (Algorithm 2) corresponding to the diagonally compensated reduced two-stage multisplitting $(M_k, F_k, G_k, N_k, E_k)$ is convergent for any initial vector $x^0$ and any sequence of numbers $p(j, k) \geq 1$.

**Proof.** For each outer iteration $j$ and for a fixed $k$, we note that the iteration matrix corresponding to the two-stage splitting $\hat{A} = M_k - N_k$, $M_k = F_k - G_k$ is $T_k(j) = (F_k^{-1}G_k)^{p(j,k)} + \sum_{i=0}^{p(j,k)} (F_k^{-1}G_k)^i F_k^{-1}N_k$. From the hypothesis and by Lemma 2.3, Theorem 4.2 in [18], we get $\rho(T_k(j)) < 1$ and the induced splitting $\hat{A} = M_{T_k(j)} - N_{T_k(j)}$ of $\hat{A}$ by $T_k(j)$ is weak regular, where $M_{T_k(j)} = M_k(I - (F_k^{-1}G_k)^{p(j,k)})^{-1}$ is symmetric positive definite. Therefore, by Theorem 1 in [2], the two-stage multisplitting method reduces to a convergent multisplitting method, and such a multisplitting of $\hat{A}$ can be viewed as a single splitting $\hat{A} = G^{-1}(j) - G^{-1}(j)I(j)$, where $G(j) = \sum_{k=1}^{K} E_k M_{T_k(j)}^{-1}$.

Similarly, the iteration matrix of the two-stage splitting $A = M_k - N_k$, $M_k = F_k - G_k$ is $T_k(j) = (F_k^{-1}G_k)^{p(j,k)} + \sum_{i=0}^{p(j,k)-1} (F_k^{-1}G_k)^i F_k^{-1}N_k$. By Lemma 1, $\rho(T_k(j)) < 1$. Thus, by Lemma 2.3 in [18], $T_k(j)$ induces a unique single splitting $A = M_{T_k(j)} - N_{T_k(j)}$, where $M_{T_k(j)} = M_k(I - (F_k^{-1}G_k)^{p(j,k)})^{-1} = M_{T_k(j)}$. Thus, the two-stage multisplitting method (6) reduces to a two-stage multisplitting method such a multisplitting of $A$ can be regarded as a single splitting $A = G^{-1}(j) - G^{-1}(j)T(j)$, where $G(j) = \sum_{k=1}^{K} E_k M_{T_k(j)}^{-1} = \hat{G}(j)$ is nonsingular and $T(j) = \sum_{k=1}^{K} E_k T_k(j)$.

Note that for each outer iteration $j$, the iteration matrix $T(j) = I - \sum_{k=1}^{K} E_k \sum_{i=0}^{p(j,k)-1} (F_k^{-1}G_k)^i F_k^{-1}A$ is similar to $\tilde{T}(j) = I - A^{1/2} \sum_{k=1}^{K} E_k \sum_{i=0}^{p(j,k)-1} (F_k^{-1}G_k)^i F_k^{-1}A^{1/2}$. Obviously, $\tilde{T}(j)$ is a symmetric matrix. Hence, we have $\lambda(T(j)) = \lambda(\tilde{T}(j))$ and $||T(j)||_A = ||\tilde{T}(j)||_2$. Note that $M_{T_k(j)}^{-1}$, for $p(j, k) \geq 1$, can be represented by the sum of such terms which are $(F_k^{-1}G_k)^i M_{T_k(j)}^{-1} (G_kF_k^{-1})^i$ or $(F_k^{-1}G_k)^i F_k^{-1}(G_kF_k^{-1})^i$, for $i = 0, 1, \ldots, p(j, k) - 1$, they are all symmetric positive semidefinite. Hence,

$$\frac{x^T \tilde{T}(j) x}{x^T x} \leq \begin{cases} \frac{x^T A^{1/2} \sum_{k=1}^{K} E_k F_k^{-1} A^{1/2} x}{x^T x}, & p(j, k) = 1, \\ \frac{x^T A^{1/2} \sum_{k=1}^{K} E_k M_{T_k(j)}^{-1} A^{1/2} x}{x^T x}, & p(k, k) \geq 2, \end{cases}$$
where \( M_{kT_2}^{-1} = F_k^{-1} + F_k^{-1}G_kF_k^{-1} \). With the same argument, we yield easily

\[
\frac{x^T \bar{T}(j)x}{x^T x} \geq 1 - \frac{x^T A^{1/2} \sum_{k=1}^{K} E_k M_k^{-1} A^{1/2} x}{x^T x}.
\]

Ordering the eigenvalues of \( \bar{T}(j) \) in a decreasing order, then, by Theorem 1.3.3 in [14], we have

\[
\lambda_1 = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T \bar{T}(j)x}{x^T x} \leq \left\{ \begin{array}{ll}
1 - \lambda_n \left( \sum_{k=1}^{K} E_k F_k^{-1} A \right), & p(j, k) = 1, \\
1 - \lambda_n \left( \sum_{k=1}^{K} E_k M_k^{-1} A \right), & p(j, k) \geq 2,
\end{array} \right.
\]

and

\[
\lambda_n = \min_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T \bar{T}(j)x}{x^T x} \geq 1 - \lambda_1 \left( \sum_{k=1}^{K} E_k M_k^{-1} A \right) > -1.
\]

Therefore, we get

\[
\|T(j)\|_A = \left\|\bar{T}(j)\right\|_2 = \rho(\bar{T}(j)) = \max(\lambda_1, |\lambda_n|) \leq \theta < 1,
\]

where \( \theta = \max(|1 - \lambda_1(\sum_{k=1}^{K} E_k M_k^{-1} A)|, |1 - \lambda_n(\sum_{k=1}^{K} E_k F_k^{-1} A)|, |1 - \lambda_n(\sum_{k=1}^{K} E_k M_k^{-1} A)|) \).

Thus, by Lemma 2 in [12], we have completed the proof.

**Corollary 3.** Under the assumption of Theorem 2, the resulting inner relaxed nonstationary two-stage multisplitting method (Algorithm 3) corresponding to the diagonally compensated reduced two-stage multisplitting method is convergent for any initial vector \( x^0 \) and any sequence of numbers \( p(j, k) \geq 1, j = 0, 1, \ldots, k = 1, \ldots, K \), provided \( \omega \in (0, 1] \).

**Proof.** Since the equality (4) is equivalent to replacing the splitting \( M_k = F_k - G_k \) by \( M_k = \bar{F}_k - \bar{G}_k \), where \( \bar{F}_k = (1/\omega)F_k \) and \( \bar{G}_k = ((1 - \omega)/\omega)F_k + G_k \). From the hypothesis, \( \bar{F}_k^{-1} \geq 0, \bar{G}_k^{-1} \geq 0 \), and the two-stage multisplitting \( (M_k, \bar{F}_k, \bar{G}_k, N_k, E_k) \) of \( \bar{T} \) is block diagonal conformable. Therefore, by Theorem 2, the proof is completed.

**Corollary 4.** Under the assumption of Theorem 2, then the resulting outer relaxed nonstationary two-stage multisplitting method (Algorithm 4) corresponding to the diagonally compensated reduced two-stage multisplitting is convergent for any initial vector \( x^0 \) and any sequence of numbers \( p(j, k) \geq 1, j = 0, 1, \ldots, k = 1, \ldots, K \). Provided \( \omega \in (0, \omega_0) \) with \( \omega_0 = 2/(1 + \max_j \|T(j)\|_A) \).

**Proof.** The iteration matrix at the \( i \)th outer iteration of Algorithm 4 is \( \bar{T}(j) = \omega T(j) + (1 - \omega)I \). From the hypothesis and by Theorem 2, we have \( \|T(j)\|_A \leq \theta < 1 \). Thus, \( \|\bar{T}(j)\|_A \leq \omega \|T(j)\|_A + (1 - \omega) \leq \omega \theta + (1 - \omega) = \tilde{\theta} < 1 \). Therefore, the proof is completed.

4. **CONCLUSION REMARK**

Different from the multisplitting method and stationary two-stage multisplitting method for the parallel solution of a large linear system of equations, the nonstationary two-stage method may reduce significantly not only computation works and storages, but also communication times and synchronism time. In other words, the presented method may avoid loss of time and efficiency in processor utilization. This is our motivation to develop such a method for symmetric positive definite matrices.
REFERENCES