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Exact Solution of Variable Coefficient Mixed Hyperbolic Partial Differential Problems

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Abstract—This paper is concerned with the construction of exact series solution of mixed variable coefficient hyperbolic problems.

Keywords—Hyperbolic mixed problem, Exact solution.

1. INTRODUCTION

In recent papers [1,2], exact series solutions of certain hyperbolic mixed problems have been given using separation of variables technique. Here we consider a variable coefficient mixed hyperbolic system of the form

$$U_{xx}(x,t) - q(x)U(x,t) = \frac{1}{r(t)} U_{tt}(x,t), \qquad 0 < x < p, \quad t > 0, \tag{1.1}$$

$$a_1 U(0,t) + b_1 U_x(0,t) = 0, \qquad t > 0, \tag{1.2}$$

$$a_2 U(p,t) + b_2 U_x(p,t) = 0, \qquad t > 0, \tag{1.3}$$

$$U(x,0) = f(x), \quad U_t(x,0) = g(x), \qquad 0 \le x \le p, \tag{1.4}$$

where $q(x)$ is real, $r(t)$ >0 and

$$|a_1| + |b_1| > 0, \qquad |a_2| + |b_2| > 0. \tag{1.5}$$

Conditions on the coefficients and data functions are given in Section 2 in order to guarantee an exact series solution of problem (1.1)–(1.5).

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2. EXACT SERIES SOLUTION

Following the ideas developed in [2], we propose a candidate series solution of problem (1.1)–(1.4) of the form

$$U(x,t) = \sum_{n\geq 1} \{a_n z_n(t) + b_n \omega_n(t)\} \varphi_n(x), \tag{2.1}$$

where $\{\varphi_n(x)\}$ is the eigenfunction system of the Sturm-Liouville problem

$$\begin{aligned} \varphi''(x) + (\lambda - q(x))\varphi(x) = 0, \qquad 0 < x < p, \\ a_1\varphi(0) + b_1\varphi'(0) = 0, \\ a_2\varphi(p) + b_2\varphi'(p) = 0, \end{aligned} \tag{2.2}$$

and if λ_n is the n^{th} eigenvalue of (2.2), then the pair $\{z_n, \omega_n\}$ are the solutions of

$$Y_n''(t) + \lambda_n r(t) Y_n(t) = 0, \qquad t > 0, \tag{2.3}$$

satisfying

$$z_n(0) = 1, \quad z_n'(0) = 1; \qquad \omega_n(0) = 0, \quad \omega_n'(0) = 1; \tag{2.4}$$

and

$$a_n = \frac{\int_0^p f(x)\varphi_n(x)\,dx}{\int_0^p (\varphi_n(x))^2\,dx}, \quad b_n = \frac{\int_0^p g(x)\varphi_n(x)\,dx}{\int_0^p (\varphi_n(x))^2\,dx}, \qquad n \geq 1. \tag{2.5}$$

Let us denote

$$q_1 = \min\{q(x);\ 0 \leq x \leq p\}, \qquad q_2 = \max\{q(x);\ 0 \leq x \leq p\}. \tag{2.6}$$

By [1; 3, p. 264], the n^{th} eigenvalue of problem (2.2) satisfies

$$\frac{n^2\pi^2}{p^2} + q_1 \leq \lambda_n \leq \frac{(n+1)^2\pi^2}{p^2} + q_2, \qquad n \geq 1. \tag{2.7}$$

If $r(t)$ is continuously differentiable in $0 \leq t \leq T$ and

$$\begin{aligned} M(T) &= \max\{r(t);\ 0 \leq t \leq T\}, \\ M'(T) &= \max\{r'(t);\ 0 \leq t \leq T\}, \\ m(T) &= \min\{r(t);\ 0 \leq t \leq T\}, \end{aligned} \tag{2.8}$$

taking n_0 large enough so that $\lambda_n > 0$ for $n \geq n_0$, then by Theorem 4 of [4] it follows that

$$\begin{aligned} |z_n(t)| &\leq L, & |z_n'(t)| &\leq L\sqrt{\lambda_n M(T)}, & 0 \leq t \leq T, \\ |\omega_n(t)| &\leq \frac{L}{\sqrt{\lambda_n r(0)}}, & |\omega_n'(t)| &\leq L\sqrt{\frac{M(T)}{r(0)}}, & n \geq n_0, \end{aligned} \tag{2.9}$$

where

$$L = \sqrt{\frac{r(0)}{m(T)}} \exp\left(\frac{TM'(T)}{2m(T)}\right). \tag{2.10}$$

Note that by (2.3), (2.7), and (2.9) for $n \geq n_0$ one gets

$$\begin{aligned} |z_n''(t)| &\leq LM(t)\left[\left(\frac{(n+1)\pi}{p}\right)^2 + q_2\right], & 0 \leq t \leq T, \\ |\omega_n''(t)| &\leq \frac{LM(T)}{\sqrt{r(0)}}\left[\left(\frac{(n+1)\pi}{p}\right)^2 + q_2\right]^{1/2}, & 0 \leq t \leq T. \end{aligned} \tag{2.11}$$

By (2.2) and (2.7) we have

$$|\varphi_n''(x)| \le \left[\frac{(n+1)\pi}{p}\right]^2 |\varphi_n(x)|, \qquad 0 \le x \le p. \tag{2.12}$$

Now we are concerned in bounding the eigenfunctions $\varphi_n(x)$ as well as the Sturm-Liouville coefficients a_n, b_n, defined by (2.5). In order to prove that (2.1) gives a rigorous solution of problem (1.1)–(1.4), it is sufficient to show that series (2.1) as well as

$$\sum_{n\ge1} \{a_n z_n(t) + b_n \omega_n(t)\} \varphi_n''(x), \tag{2.13}$$

$$\sum_{n\ge1} \{a_n z_n''(t) + b_n \omega_n(t)\} \varphi_n(x), \tag{2.14}$$

are locally uniformly convergent in a rectangle $[0,p] \times [t_0-\delta, t_0+\delta] = R(t_0,\delta)$ for $0 < \delta < t_0 < t_0 + \delta \le T$. All these series will be uniformly convergent in $R(t_0,\delta)$ if the series $\sum_{n\ge1}\{n^2|a_n| + n|b_n|\}n^2|\varphi_n(x)|$ is uniformly convergent for $0 \le x \le p$. Thus, we are interested in conditions which guarantee the uniform convergence of series

$$\sum_{n\ge1} n^4 |a_n||\varphi_n(x)|, \quad \sum_{n\ge1} n^3|b_n||\varphi_n(x)|, \qquad 0 \le x \le p. \tag{2.15}$$

Let us suppose that

$$q(x) \text{ is a function of bounded variation in } 0 \le x \le p, \tag{2.16}$$

let us denote by $V_q(0,p)$ the total variation of $q(x)$ in $[0,p]$ and take n_1 large enough so that

$$\lambda_n > q_2, \qquad \text{for all } n \ge n_1. \tag{2.17}$$

By applying Prufer transformation [3,5] to problem (2.2) and using Theorem 4.1 of [4], it follows that

$$\begin{aligned} &|\varphi_n(x)| \le C_q M_q, && |\varphi_n'(x)| \le C_q M_q \sqrt{\lambda_n - q(p)}, && 0 \le x \le p, \\ &C_q = \sqrt{1 + \frac{1}{\lambda_n - q(0)}}, && M_q = \exp\left(\frac{V_q(0,p)}{2\left[(n\pi/p)^2 - V_q(0,p)\right]}\right), && n \ge n_1. \end{aligned} \tag{2.18}$$

Note that by (2.18) the series appearing in (2.15) is uniformly convergent in $0 \le x \le p$ if coefficients a_n, b_n satisfy

$$|a_n| = O\left(n^{-6}\right), \qquad |b_n| = O\left(n^{-5}\right)|, \qquad n \to \infty. \tag{2.19}$$

By Theorem 5 of [5, p. 273], and the proof of Theorem 4 of [2], condition (2.19) holds true if function $f(x)$ and $g(x)$ satisfy the following conditions:

$$\begin{aligned} a_1 f(0) + b_1 f'(0) = 0, \qquad a_2 f(p) + b_2 f'(p) = 0, \\ a_1 g(0) + b_1 g'(0) = 0, \qquad a_2 g(p) + b_2 g'(p) = 0, \end{aligned} \tag{2.20}$$

$f(x)$ is six times differentiable, $f^{(6)}(x)$ is piecewise continuous and has a bounded variation in $[0,p]$ with $f^{(2i)}(0) = f^{(2i)}(p) = 0$, for $i = 1,2,3$. (2.21)

$g(x)$ is five times differentiable, $g^{(5)}(x)$ is piecewise continuous and has a bounded variation in $[0,p]$ with $g^{(2i)}(0) = g^{(2i)}(p) = 0$, for $i = 1,2$. (2.22)

Summarizing, the following result has been established.

THEOREM 1. *Consider real valued functions $f(x)$, $g(x)$, $q(x)$, and $r(t)$ where $r(t)$ is continuous and positive, $q(x)$ satisfies (2.16) and $f(x)$, $g(x)$ satisfy conditions (2.20)–(2.22). If $\{z_n, \omega_n\}$ are solutions of (2.3) satisfying (2.4) and a_n, b_n are defined by (2.5), then $U(x,t)$ given by (2.1) is a solution of problem (1.1)–(1.5).*

REFERENCES

1. L. Jódar and M.D. Roselló, Mixed problems for separate variable coefficient wave equation. The non-Dirichlet case. Continuous numerical solutions with *a priori* error bounds, *Mathl. Comput. Modelling* **30** (9/10), 1–22, (1999).
2. P. Almenar and L. Jódar, Mixed problems for separated variable coefficient wave equations. Analytic-numerical solutions with *a priori* error bounds, *J. Comput. Applied Math.* **134**, 301–323, (2001).
3. H. Sagan, *Boundary and Eigenvalues Problems in Mathematical Physics*, Dover, New York, (1989).
4. L. Jódar and A.E. Posso, Analytic numerical approximation with *a priori* error bound for the wave equation with time dependent coefficient, *Mathl. Comput. Modelling* **29** (6), 1–14, (1999).
5. G. Birkhoff and G.C. Rota, *Ordinary Differential Equations*, Third Edition, John-Wiley, New York, (1978).