Endpoint Strichartz estimates for the Klein–Gordon equation in two space dimensions and some applications

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Abstract

We prove the endpoint Strichartz estimates for the Klein–Gordon equation in mixed norms on the polar coordinates in two space dimensions. As an application, similar endpoint estimates for the Schrödinger equation in two space dimensions are shown by using the non-relativistic limit. The existence of global solutions for the cubic nonlinear Klein–Gordon equation in two space dimensions for small data is also shown.

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1. Introduction

The Strichartz estimate for the Klein–Gordon equation,

\[ \partial_t^2 u - \Delta u + u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \]  

is the estimate of the following form:
\[ \|U(t)f\|_{L^p_t L^q_x} \lesssim \|f\|_{H^s}, \quad (1.2) \]

where \( U(t) = (\nabla)^{-1} \sin(t(\nabla)) \) with \( \langle \nabla \rangle = \sqrt{1 - \Delta} \) is the fundamental solution of (1.1), \( p \in [2, \infty] \), \( q \in [2, 2n/(n-2)] \) (\( q \in [2, \infty] \) if \( n = 1, 2 \)), and \( s \) satisfy:

\[ \frac{1}{p} = \frac{n-1+\theta}{2} \left( \frac{1}{2} - \frac{1}{q} \right), \quad s = \frac{n+1+\theta}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - 1, \quad (1.3) \]

with \((p, q) \neq (2, \infty)\). In particular the estimate,

\[ \|U(t)f\|_{L^2_t L^{2/(n-2)}_x} \lesssim \|f\|_{H^{-(n-2)/2}}, \quad (1.4) \]

seems to be false. For the related result, see Corollary 3.2 below.

**Remark 1.1.** More generally, the estimate (1.2) holds for the exponents satisfying the conditions,

\[ \frac{1}{p} = \frac{n-1+\theta}{2} \left( \frac{1}{2} - \frac{1}{q} \right), \quad s = \frac{n+1+\theta}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - 1, \]

where \( \theta \in [0, 1] \). The parameter \( \theta \) comes from the decay estimate:

\[ \|U(t)\Delta_j f\|_{L^q_x} \lesssim |t|^{(n-1+\theta)(1/2-1/q)}2^{(n+1+\theta)(1/2-1/q)-1} \|\Delta_j f\|_{L^q_y}, \]

where \( \{\Delta_j\}_{j=0}^\infty \) denotes the Littlewood–Paley decomposition (see e.g. [13]). In this paper we are concerned with the case \( \theta = 1 \), which means that the corresponding Strichartz estimate is based on the best decay estimate. For the case \( \theta = 0 \), see [11].

The Strichartz estimate is also known for the wave equation and the Schrödinger equation. As for the wave equation, the estimate corresponding to (1.4) is, in three space dimensions,

\[ \|\nabla^{-1} \sin(t(\nabla)) f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2_x}, \quad (1.5) \]

which is also known to be false [8]. However, Klainerman and Machedon [8, Proposition 4] showed that if \( f \) is radial, then the estimate (1.5) holds. They observed that under the assumption of radial symmetry the estimate (1.5) is just the Hardy–Littlewood maximal function estimate.

As for the Schrödinger equation, the estimate corresponding to (1.4) is, in two space dimensions,

\[ \|e^{it\Delta} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2_x}, \quad (1.6) \]

which is known to be false as we mentioned above [14]. (Montgomery and Smith [14] also showed that the estimates (1.5), (1.6) are false even if we replace \( L^\infty \) by BMO.) In this case, Tao [21] showed that by introducing the mixed norms on the polar coordinates \( x = r\omega \) with \( r > 0, \omega \in S^1 \),

\[ \|e^{it\Delta} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{L^2} \quad (1.7) \]

holds without assuming radial symmetry on \( f \), where

\[ \|g\|_{L^\infty_r L^q_\omega} = \sup_{r > 0} \left( \int_{S^1} |g(r\omega)|^q \, d\sigma(\omega) \right)^{1/q}. \]

His method is based on the expansion by spherical harmonics.

Recently, Machihara, Nakamura, Nakanishi, and Ozawa [11] refined the estimate in the three-dimensional wave case. Namely, they showed that
|∇|^{-1} \sin (t|∇|) f \|_{L^2_t L^q_x} \lesssim \sqrt{q} \| f \|_{L^2_x} \quad (1.8)

holds for $1 \leq q < \infty$. They observed that the estimate (1.8) can be reduced to the $T^*T$-version of the Hardy–Littlewood maximal function estimate. (See Lemma 2.5 below.) They also showed that a similar estimate for the Klein–Gordon equation in three space dimensions holds, which is easier since $t^{-3/2}$ decay of the fundamental solution can be used.

The purpose of this paper is to show the endpoint estimate of the form (1.8) for the Klein–Gordon equation in two space dimensions. From the $T^*T$ argument, the estimate is equivalent to,

$$
\left\| \int_0^t \langle \nabla \rangle^{-2} \cos ((t-s)\langle \nabla \rangle) F(s) \, ds \right\|_{L^2_t L^\infty_x L^q_y} \lesssim q \| F \|_{L^2_t L^1_x L^q_y}.
$$

To show the estimate, we divide the integral kernel $\langle \nabla \rangle^{-2} \cos(t \langle \nabla \rangle)$ into two parts, namely, the kernel having the support in $\{(t, x) \mid |x| > t/2\}$, and the kernel having the support in $\{(t, x) \mid |x| \leq t/2\}$. Then, by using explicit representations of the kernels, we observe that the former one is treated similarly to the three-dimensional wave case. Meanwhile, the operator having the latter kernel, which is not appeared in the three-dimensional wave case due to the strong Huygens principle, is estimated in $L^2_t L^\infty_x$ without averaging over the sphere. To derive such estimate we employ the method of the stationary phase and the asymptotic expansion of the oscillatory integrals associated with the latter kernel.

This paper is organized as follows. We describe our main result and the proof in Section 2. In Section 3, we show that a similar estimate for the Schrödinger equation holds as an application of our main result. In Section 4, we briefly describe applications to the existence of global solutions to the cubic nonlinear Klein–Gordon equations for small data.

**Notation.** For $A, B \geq 0$, we denote $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively. For a function $m$, we define the operator $m(\nabla)$ as $m(\nabla) = \mathcal{F}^{-1}m(\xi)\mathcal{F}$. Finally, we define $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, and $\langle \xi \rangle_m = (m^2 + |\xi|^2)^{1/2}$ for $m > 0$. Note that $\langle \xi \rangle_1 = \langle \xi \rangle$.

### 2. Main result

In this section, we state our main result with the proof. We denote,

$$
\dot{U}(t) = \cos[t \langle \nabla \rangle], \quad U(t) = \langle \nabla \rangle^{-1} \sin[t \langle \nabla \rangle],
$$

the fundamental solution of the Klein–Gordon equation (1.1) and its time derivative. Our main result is the following:

**Theorem 2.1.** Let $n = 2$ and let $1 \leq q < \infty$. Then, the following estimates hold:

$$
\left\| \dot{U}(t) f \right\|_{L^2_t L^\infty_x L^q_y} \lesssim \sqrt{q} \| f \|_{H^1_x}, \quad (2.1)
$$

$$
\left\| U(t) g \right\|_{L^2_t L^\infty_x L^q_y} \lesssim \sqrt{q} \| g \|_{L^2_x}. \quad (2.2)
$$

Moreover, we have:

$$
\left\| \int_0^t \dot{U}(t-s)F(s) \, ds \right\|_{L^2_t L^\infty_x L^q_y} \lesssim \sqrt{q} \| F \|_{L^1_t H^1_x}, \quad (2.3)
$$

$$
\left\| \int_0^t U(t-s)G(s) \, ds \right\|_{L^2_t L^\infty_x L^q_y} \lesssim \sqrt{q} \| G \|_{L^1_t L^2_x}. \quad (2.4)
$$
**Remark 2.2.**

(1) By the Sobolev embedding on $S^1$, the estimates in Theorem 2.1 imply that the $L^2_t L^\infty_x$ estimate holds if we assume a slight angular regularity on the data. For example, from the estimate (2.2) we have:

$$
\left\| U(t)g \right\|_{L^2_t L^\infty_x} \lesssim \left\| \langle \nabla S^1 \rangle^\delta U(t)g \right\|_{L^2_t L^\infty_x} \lesssim \sqrt{q} \left\| \langle \nabla S^1 \rangle^\delta g \right\|_{L^2},
$$

where $\delta > 1/q$, $\langle \nabla S^1 \rangle^\delta = (I - \Delta_S^{1/2})^{\delta/2}$, $\Delta_S$ is the Laplace–Beltrami operator on $S^1$.

(2) For the Klein–Gordon equation,

$$
\partial^2_t u - \Delta u + m^2 u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^2,
$$

where $m > 0$, the fundamental solution becomes:

$$
U_m(t) = \langle \nabla \rangle^{-1} \sin[t \langle \nabla \rangle].
$$

Since $U_m(t)g(x) = m^{-1}(U(mt)g(\cdot/m))(mx)$, applying (2.2) we obtain:

$$
\left\| U_m(t)g \right\|_{L^2_t L^\infty_x} \lesssim \left\| \left( U(mt)g(\cdot/m) \right)(m\omega) \right\|_{L^2_t L^\infty_x} \lesssim m^{-3/2} \left\| \langle \nabla \rangle f(\cdot/m) \right\|_{L^2} \lesssim \sqrt{q/m} \left\| g(\cdot/m) \right\|_{L^2}.
$$

Thus, if we use the conserved energy defined by:

$$
E(u) = \int_{\mathbb{R}^2} \left| \partial_t^2 u \right| + |\nabla u|^2 + m^2 |u|^2 \, dx
$$

as in [11], the estimates (2.1) and (2.2) are written as the following estimate for the solution $u$ of (2.5),

$$
\| u \|_{L^2_t L^\infty_x L^q_\omega} \lesssim \sqrt{q/m} E(u)^{1/2}.
$$

(3) By the Christ–Kiselev lemma ([1], [21, Lemma 3.1]), the estimates (2.3), (2.4) also hold if we replace $\| F \|_{L^p_t L^q_x}$ on the right-hand side by $\| F \|_{L^p_t B^{q'}_2}$ for exponents $\tilde{p}, \tilde{q}, s$ satisfying the conditions (1.3), where $B^{q'}_2$ denotes the Besov space. However, we cannot replace it by $\| F \|_{L^p_t L^q_x}$, which is excluded from the range of the application of the Christ–Kiselev lemma.

For the proof of Theorem 2.1 the following explicit representations of the fundamental solution play an important role.

**Lemma 2.3.** Let $n = 2$. We have $U(t) f = E(t) * f$, with

$$
E(t, x) = \frac{1}{2\pi} \left( t^2 - r^2 \right)^{1/2} \cos \sqrt{t^2 - r^2},
$$

where $r = |x|$.

**Proof.** Let $v(t) = U(t) f$ be a solution to

$$
\Box v + v = 0, \quad v(0) = 0, \quad \partial_t v(0) = f,
$$

where $\Box = \partial^2_t - \Delta$.
where $\Box_n$ denotes the d’Alembertian in $n$ space dimensions. Then $w(t, \tilde{x}) = e^{ix_3} v(t, x)$ solves,

$$\Box_3 w = 0, \quad w(0) = 0, \quad \partial_t w(0) = e^{ix_3} f,$$

where $\tilde{x} = (x, x_3) \in \mathbb{R}^3$. By using the Kirchhoff formula for the wave equation in three space dimensions, we have:

$$e^{ix_3} v(t, x) = \frac{t}{4\pi} \int_{S^2} e^{i(x_3 - t\omega)} f(x - t\omega) d\sigma(\omega),$$

where $\tilde{\omega} = (\omega, \omega_3) \in S^2$. Thus, we obtain:

$$v(t, x) = \frac{t}{4\pi} \int_{S^2} e^{-i\omega_3} f(x - t\omega) d\sigma(\omega)$$

which implies (2.6). $\Box$

For the proof of Theorem 2.1 we consider the operators:

$$T_{\pm} F = \int_{0}^{\infty} (\nabla)^{-1} (e^{is\nabla} \pm e^{-is\nabla}) F(s) \, ds.$$

Then, $U(t)$ and $\dot{U}(t)$ are represented by using the dual operators of $T_{\pm}$. Indeed,

$$T_{\pm}^* f(t) = \pm (\nabla)^{-1} (e^{it\nabla} \pm e^{-it\nabla}) f,$$

and thus we have:

$$U(t) g = -\frac{1}{2t} T_{-}^* g, \quad \dot{U}(t) f = \frac{1}{2} T_{+}^* (\nabla) f.$$

So, the estimates,

$$\|T_{\pm}^* f\|_{L^2_{t} L^\infty_{\nabla} L^q_{\omega}} \lesssim q \|f\|_{L^2}, \quad (2.7)$$

imply the estimates (2.1), (2.2). From the $T^* T$ argument, the estimates (2.7) are equivalent to the estimates

$$\|T_{\pm}^* T_{\pm} F\|_{L^2_{t} L^\infty_{\nabla} L^q_{\omega}} \lesssim q \|F\|_{L^2_{t} L^1_{\nabla} L^{q'}_{\omega}}. \quad (2.8)$$
Here, we have:

\[
T^* \pm T \pm F(t) = \int_0^\infty \langle \nabla \rangle^{-2} \left( \cos((t-s) \langle \nabla \rangle) \pm \cos((t+s) \langle \nabla \rangle) \right) F(s) \, ds
\]

\[
= \int_\mathbb{R} \langle \nabla \rangle^{-2} \cos((t-s) \langle \nabla \rangle) \tilde{F}(s) \, ds
\]

\[
= \int_\mathbb{R} \mathcal{K}(t-s) \tilde{F}(s) \, ds,
\]

where \( \tilde{F} \) denotes the even or odd extension of \( F \) to \( \mathbb{R} \times \mathbb{R}^2 \), and

\[
\mathcal{K}(t) = \langle \nabla \rangle^{-2} \cos\{t \langle \nabla \rangle\}.
\]

In the following, we devote to show the estimate (2.8). We first consider the explicit representation of \( \mathcal{K}(t) \).

**Lemma 2.4.** We have \( \mathcal{K}(t) f = \mathcal{K}(t) \ast f \), with

\[
K(t, x) = \frac{1}{2\pi} \int_{|r| \vee r} \frac{1}{s^2 - r^2} \sin \sqrt{s^2 - r^2} \, ds - \chi_{\{r < |t|\}} \frac{1}{|t|} \sin \sqrt{t^2 - r^2}
\]

\[
= \frac{1}{2\pi} \int_{r \vee r} \frac{1}{r} \int_0^{1 \vee (r/|t|)} \sin \left( \frac{r}{\rho} \sqrt{1 - \rho^2} \right) \, d\rho - \chi_{\{r < |t|\}} \frac{1}{|t|} \sin \sqrt{t^2 - r^2},
\]

where \( r = |x|, a \lor b = \max\{a, b\}, a \land b = \min\{a, b\} \), and \( \chi_A \) denotes the characteristic function of the set \( A \).

**Proof.** It suffices to consider the case where \( t > 0 \). We first observe that \( \mathcal{K}(t) \) is written as

\[
\mathcal{K}(t) = \langle \nabla \rangle^{-2} \cos\{t \langle \nabla \rangle\} = \int_t^\infty \langle \nabla \rangle^{-1} \sin\{s \langle \nabla \rangle\} \, ds.
\]

Then, by using Lemma 2.3 we obtain:

\[
K(t, x) = \frac{1}{2\pi} \int_t^\infty (s^2 - r^2)^{-1/2} \cos \sqrt{s^2 - r^2} \, ds
\]

\[
= \frac{1}{2\pi} \int_{r \vee r} (s^2 - r^2)^{-1/2} \cos \sqrt{s^2 - r^2} \, ds.
\]

Now we apply integration by parts to obtain:

\[
K(t, x) = -\frac{1}{2\pi} \int_{r \vee r} \frac{1}{s} \partial_s \left( \sin \sqrt{s^2 - r^2} \right) \, ds
\]

\[
= \frac{1}{2\pi} \int_{r \vee r} \frac{1}{s} \sin \sqrt{s^2 - r^2} \, ds - \chi_{\{r < |t|\}} \frac{1}{|t|} \sin \sqrt{t^2 - r^2}.
\]

The second representation in the lemma is derived by a change of variable \( s = r/\rho \).

To show the estimate (2.8) we divide \( K(t, x) \) into two parts, supported inside and outside of the cone \( \{|x| = |t|/2\} \), and then we apply the different representation in Lemma 2.4 as follows:
\[ K(t, x) = \chi_{|r|>\frac{|t|}{2}} K(t, x) + \chi_{|r|<\frac{|t|}{2}} K(t, x) \]

\[ = \frac{1}{2\pi} \chi_{|r|>\frac{|t|}{2}} \frac{1}{r} \int_0^\infty \sin \left( \frac{r}{\rho} \sqrt{1-\frac{r^2}{1} \frac{r^2}{1}} \right) d\rho \]

\[ + \frac{1}{2\pi} \chi_{|r|<\frac{|t|}{2}} \int_{|t|}^\infty \frac{1}{\rho^2} \sin \sqrt{s^2 - r^2} ds \]

\[- \frac{1}{2\pi} \chi_{|r|<\frac{|t|}{2}} \frac{1}{|t|} \sin \sqrt{t^2 - r^2} \]

\[ \equiv K_1(t, x) + K_2(t, x) - K_3(t, x). \]

We first consider the estimate of the operator having the kernel \( K_1 \). Since

\[ \left| K_1(t, x) \right| \lesssim \frac{1}{r} \chi_{|r|>\frac{|t|}{2}}, \]

we are able to treat this kernel similarly to the 3D wave case as in [11]. In fact, denoting \( x = r\omega \) with \( r > 0, \omega \in S^1 \) we have:

\[ \left| (K_1(t) * f)(r\omega) \right| \lesssim \int_{\mathbb{R}^2} \frac{1}{|r\omega - y|} f(y) dy \]

\[ = \int_0^\infty \int_{S^1} \frac{1}{|r\omega - l\theta|} f(l\theta) l d\sigma(\theta) dl \]

\[ \equiv \int_0^\infty (\Omega(t, r, l) f(l\cdot))(\omega) l dl, \]

where

\[ (\Omega(t, r, l) g) (\omega) = \int_{S^1} \frac{1}{|r\omega - l\theta|} g(\theta) d\sigma(\theta). \]

Then, we have the following estimate,

\[ \| \Omega(t, r, l) g \|_{L^q_{\omega}} \lesssim \frac{1}{r} h \left( \frac{|r|}{r \vee l} \right) \| g \|_{L^{q'}_{\omega}}, \quad (2.9) \]

for \( 2 \leq q \leq \infty \), where

\[ h(\lambda) = \lambda^{-1+2(1-\epsilon)/q} \chi_{[\lambda<4]} \]

for small \( \epsilon > 0 \). Here, we notice that \( \| h \|_{L^1} = C q \). The estimate (2.9) is obtained by interpolating the following two estimates,

\[ \| \Omega(t, r, l) g \|_{L^\infty_{\omega}} \lesssim \frac{1}{|r|} \chi_{|r|_{\frac{1}{r \vee 4}} < 4} \| g \|_{L^1_{\omega}}, \quad (2.10) \]

\[ \| \Omega(t, r, l) g \|_{L_2^{\infty}_{\omega}} \lesssim \frac{1}{|r|^{\epsilon}} \chi_{|r|_{\frac{1}{r \vee 4}} < 4} \| g \|_{L^2_{\omega}}, \quad (11.1) \]

To derive (2.10) we estimate,

\[ \left| (\Omega(t, r, l) g)(\omega) \right| \leq \frac{2}{|r|} \int_{S^1} \chi_{|2(r\vee l)| > |l/2|} |g(\theta)| d\sigma(\theta) = \frac{2}{|r|} \chi_{|l|_{\frac{1}{r \vee 4}} < 4} \| g \|_{L^1_{\omega}}. \]
To derive (2.11) we rewrite the integral on \( S^1 \) in the following way. We denote \( e = (1, 0) \) and \( \omega = Ae \) with \( A \in SO(2) \). Then,

\[
(\Omega(t, r, l)g)(Ae) = \int_{SO(2)} \frac{\chi_{|re-\omega|>|r/2|}}{|re-\omega|} g(ABe) \, d\mu(B),
\]

where \( \mu \) is the standard Haar measure on \( SO(2) \). Thus,

\[
\| \Omega_1(t, r, l)g \|_{L^2_\omega} \leq \frac{1}{|t|^\varepsilon} \| g \|_{L^2_\omega} \int_{SO(2)} \frac{\chi_{|2(r\vee l)>|t/2|}}{|re-\omega|^{1-\varepsilon}} \, d\mu(B)
\]

\[
\lesssim \frac{1}{|t|^\varepsilon} \frac{1}{(r \vee l)^{1-\varepsilon}} X_{\frac{|\omega|}{r \vee l} < 4} \| g \|_{L^2_\omega},
\]

because

\[
\int_{S^1} \frac{1}{|re-\omega|^{1-\varepsilon}} \, d\sigma(\theta) \lesssim \frac{1}{(r \vee l)^{1-\varepsilon}}.
\]

Therefore, we derive the estimate (2.9), and then we obtain:

\[
\left\| \int_{\mathbb{R}} K_1(t-s) \ast \tilde{F}(s) \, ds \right\|_{L^2_t L^q_{\omega}} \lesssim \int_{\mathbb{R}} \int_{S^1} \| \Omega(t-s, r, l) \tilde{F}(s, l) \|_{L^q_{\omega}} \, ds \, dl
\]

\[
\lesssim \left\| \int_{\mathbb{R}} \int_{S^1} \frac{1}{r \vee l} h\left( \frac{|t-s|}{r \vee l} \right) \| \tilde{F}(s, l) \|_{L^q_{\omega}} \, ds \, dl \right\|_{L^1_t L^\infty}.
\]

Now we apply the following lemma, the \( T^*T \) version of the Hardy–Littlewood maximal function estimate, which is due to Machihara, Nakamura, Nakanishi, and Ozawa [11]. In the lemma below, we forget about the polar coordinates and \( L^p_r \) just denotes \( L^p(0, \infty; dr) \).

**Lemma 2.5.** (See [11, Lemma 3.1].) Let \( h(\lambda) \) be a nonnegative nonincreasing integrable function on \( (0, \infty) \). Then the following estimate holds:

\[
\left\| \int_{\mathbb{R}} h\left( \frac{|t-s|}{r \vee l} \right) G(s, l) \, ds \right\|_{L^1_t L^\infty} \lesssim \| h \|_{L^1} \| G \|_{L^2_t L^1_l}.
\]

Applying this lemma we finally obtain:

\[
\left\| \int_{\mathbb{R}} K_1(t-s) \ast \tilde{F}(s) \, ds \right\|_{L^2_t L^q_{\omega}} \lesssim \| h \|_{L^1} \| F \|_{L^2_t L^q_{\omega}} \lesssim q \| F \|_{L^2_t L^q_{\omega}}.
\]

Here, we notice that the case \( 1 \leq q < 2 \) is easily reduced to the case \( q = 2 \) by using the Hölder inequality.

We next consider the estimate of the operator having the kernel \( K_2 \). To derive the estimate, we apply the integration by parts again. For \( r < |t|/2 \),

\[
K_2(t, x) = -\frac{1}{2\pi} \int_{|t|}^\infty \frac{\sqrt{s^2 - r^2}}{s^3} \partial_s (\cos \sqrt{s^2 - r^2}) \, ds
\]

\[
= \frac{1}{2\pi} \left\{ \frac{\sqrt{t^2 - r^2}}{|t|^3} \cos \sqrt{t^2 - r^2} + \int_{|t|}^\infty \frac{-2s^2 + 3r^2}{s^4\sqrt{s^2 - r^2}} \cos \sqrt{s^2 - r^2} \, ds \right\}.
\]
Since $\sqrt{s^2 - r^2} \gtrsim |t|$ for $s > |t|$, $r < |t|/2$, we have:

$$|K_2(t, x)| \lesssim \frac{1}{|t|^2} + \frac{1}{|t|} \int \frac{ds}{s^2} \lesssim \frac{1}{|t|^2}.$$

Meanwhile, for $|t| < 1$ we estimate $K_2$ directly to obtain:

$$|K_2(t, x)| \lesssim \left( \int \frac{1}{s^2} \sqrt{s^2 - r^2} ds + \int \frac{ds}{s^2} \right) \lesssim \log \frac{1}{|t|} + 1.$$

Thus we obtain:

$$|K_2(t, x)| \lesssim \chi_{|t|<1} \left( \log \frac{1}{|t|} + 1 \right) + \chi_{|t|\geq 1} \frac{1}{|t|^2} \equiv k(t).$$

Note that $k \in L^1(\mathbb{R})$. Since

$$\left| (K_2(t - s) \ast \tilde{F}(s))(x) \right| \leq \int_{\mathbb{R}^2} \left| K_2(t - s, x - y) \tilde{F}(s, y) \right| dy \lesssim \int_{\mathbb{R}^2} k(t - s) \left| \tilde{F}(s, y) \right| dy = k(t - s) \| \tilde{F}(s, \cdot) \|_{L^1},$$

applying the Young inequality we obtain:

$$\left\| \int_{\mathbb{R}} K_2(t - s) \ast \tilde{F}(s) ds \right\|_{L^2_t L^\infty_x} \lesssim \left\| \int_{\mathbb{R}} k(t - s) \left\| \tilde{F}(s) \right\|_{L^1} ds \right\|_{L^2_t} \lesssim \| k \|_{L^1_t} \| \tilde{F} \|_{L^2_t L^1_x} \lesssim \| F \|_{L^2_t L^1_x}.$$

We finally consider the estimate of the operator with the kernel $K_3$. We first divide $K_3$ into two parts,

$$K_3(t, x) = (1 - \rho(x/t))K_3(t, x) + \rho(x/t)K_3(t, x), \quad (2.12)$$

where $\rho \in C_0^\infty(\mathbb{R}^n)$ is a radial function which satisfies $\rho(x) = 1$ for $|x| \leq 1/16$, $\rho(x) = 0$ for $|x| > 1/8$, $0 \leq \rho \leq 1$, and $\rho(x) = \rho_0(|x|)$. Then, it is easy to see that the first term on the right-hand side of (2.12) is bounded by a constant multiple of,

$$\frac{1}{r} \chi_{|r|>|t|/16}.$$

Thus, similarly as $K_1$, we are able to estimate the operator having this kernel. So, we set:

$$\tilde{K}_3(t, x) = \rho(x/t)K_3(t, x),$$

and devote to the estimate of the operator having the kernel $\tilde{K}_3$. For that operator a better estimate holds.

**Lemma 2.6.** We have,

$$\left\| \tilde{K}_3 *_{t,x} \tilde{F} \right\|_{L^2_t L^\infty_x} \lesssim \| F \|_{L^2_t L^1_x}, \quad (2.13)$$

where we denote by $*_{t,x}$ the convolution with respect to $t$ and $x$. 
Once Lemma 2.6 were shown, the proof of the estimate (2.8) would be completed. So, in most of the rest of this section we devote ourselves to the proof of Lemma 2.6.

**Proof of Lemma 2.6.** For the proof of Lemma 2.6 we divide $\tilde{K}_3$ into three parts, which corresponds to the decomposition of the frequency space in space and time:

$$\begin{align*}
\{ |\xi| > 1/4 \}, & \quad \{ |\xi| \leq 1/2, |\tau| - 1 \geq 1/8 \}, & \quad \{ |\xi| \leq 1/2, |\tau| - 1 \leq 1/4 \}.
\end{align*}$$

Precisely, we decompose $\tilde{K}_3$ as follows:

$$\tilde{K}_3(t, x) = \mathcal{F}^{-1}(1 - \rho(\xi/4))\mathcal{F}\tilde{K}_3 + \mathcal{F}^{-1}(1 - \rho_0((|\tau| - 1)/2))\mathcal{F}\varphi \ast \tilde{K}_3$$

$$+ \mathcal{F}^{-1}\rho((|\tau| - 1)/2)\rho(\xi/4)\mathcal{F}_{t,x} \tilde{K}_3$$

$$\equiv \tilde{K}_{3,1}(t, x) + \tilde{K}_{3,2}(t, x) + \tilde{K}_{3,3}(t, x),$$

where $\rho$ is the same one appeared in (2.12), and $\varphi$ is defined by $\hat{\varphi}(\xi) = \rho(\xi/4)$. Then, it suffices to show that the estimate (2.13) holds for each $\tilde{K}_{3,j}$ instead of $\tilde{K}_3$.

We first consider the estimate on $\tilde{K}_{3,1}$.

**Claim 1.** We have:

$$\| \tilde{K}_{3,1}(t, x) \|_{L^\infty_t L^\infty_x} \lesssim (t)^{-2}, \quad t \in \mathbb{R}. \quad (2.14)$$

If Claim 1 were proved, then applying Claim 1 and the Young inequality we would immediately obtain:

$$\left\| \int_{\mathbb{R}} \tilde{K}_{3,1}(t-s) * \tilde{F}(s) \, ds \right\|_{L^2_t L^\infty_x} \lesssim \left\| \int_{\mathbb{R}} \tilde{K}_{3,1}(t-s) \| \tilde{F}(s) \|_{L^1_x} \, ds \right\|_{L^2_t}$$

$$\lesssim \left\| \int_{\mathbb{R}} \frac{1}{(t-s)^2} \| F(s) \|_{L^1_x} \, ds \right\|_{L^2_t}$$

$$\lesssim \| F \|_{L^2_t L^1_x}.$$

**Proof of Claim 1.** To show Claim 1 we prove the decay estimate of $\mathcal{F}[\tilde{K}_{3,1}(t)]$ by using the stationary phase method:

$$\mathcal{F}[\tilde{K}_{3,1}(t)](\xi) = (1 - \rho(\xi/4)) \frac{1}{2\pi |t|} \int e^{ix \cdot \xi} \rho\left(\frac{x}{t}\right) \sin \sqrt{t^2 - r^2} \, dx$$

$$= (1 - \rho(\xi/4)) \frac{|t|}{4\pi i} \int e^{i|t|(y \cdot \xi + \sqrt{1 - |y|^2})} - e^{i|t|(y \cdot \xi - \sqrt{1 - |y|^2})} \rho(y) \, dy$$

$$\equiv (1 - \rho(\xi/4)) \frac{|t|}{4\pi i} (I^+(t, \xi) - I^-(t, \xi)).$$

Then, the phase functions of the oscillatory integral $I^\pm(t, \xi)$ are,

$$\phi_\pm(y, \xi) = y \cdot \xi \pm \sqrt{1 - |y|^2},$$

respectively, and they satisfy:

$$|\nabla_y \phi_\pm(y, \xi)| \geq |\xi| - \frac{|y|}{\sqrt{1 - |y|^2}} \geq \left(1 - \frac{4}{\sqrt{63}}\right)|\xi|, \quad (2.15)$$

for $|\xi| \geq 1/4$ and $|y| \leq 1/8$, which implies $|y| \leq |\xi|/2$. Here, these conditions come from the support property of $\rho$. Moreover, we have:

$$|\nabla_y \phi_\pm(y, \xi)| \lesssim |\xi|, \quad |\partial_y^\alpha \phi_\pm(y, \xi)| \lesssim 1, \quad |\alpha| \geq 2 \quad (2.16)$$

for $|\xi| \geq 1/4$ and $|y| \leq 1/8$. Since

$$e^{i|r|\phi_{\pm}(y, \xi)} = \frac{1}{i|r|} \frac{\nabla_y \phi_{\pm}(y, \xi)}{|\nabla_y \phi_{\pm}(y, \xi)|^2} \cdot \nabla_y e^{i|r|\phi_{\pm}(y, \xi)},$$

we observe that

$$I^{\pm}(t, \xi) = \frac{1}{i|r|} \int_{\mathbb{R}^2} \nabla_y e^{i|r|\phi_{\pm}(y, \xi)} \cdot \nabla_y \phi_{\pm}(y, \xi) \rho(y) dy$$

$$= -\frac{1}{i|r|^3} \sum_{j=1}^2 \int_{\mathbb{R}^2} e^{i|r|\phi_{\pm}(y, \xi)} \partial_j \left( \frac{\partial_j \phi_{\pm}(y, \xi)}{|\nabla_y \phi_{\pm}(y, \xi)|^2} \rho(y) \right) dy$$

$$= \frac{(-1)^3}{i|r|^3} \sum_{j,k,l} \int_{\mathbb{R}^2} e^{i|r|\phi_{\pm}(y, \xi)} \partial_j \frac{\partial_k \phi_{\pm}}{|\nabla_y \phi_{\pm}|^2} \partial_l \frac{\partial_j \phi_{\pm}}{|\nabla_y \phi_{\pm}|^2} \partial_j \partial_k \partial_l \rho(y) dy.$$

According to (2.15), (2.16), the worst term with respect to the decay on $|\xi|$ is, for example,

$$\left| \frac{(-1)^3}{i|r|^3} \int_{\mathbb{R}^2} e^{i|r|\phi_{\pm}(y, \xi)} \partial_j \frac{\partial_k \phi_{\pm}}{|\nabla_y \phi_{\pm}|^2} \partial_l \frac{\partial_j \phi_{\pm}}{|\nabla_y \phi_{\pm}|^2} \partial_j \partial_k \partial_l \rho(y) dy \right| \lesssim \frac{1}{|r|^3 |\xi|^3}.$$

Therefore, we have:

$$|\mathcal{F}[\tilde{K}_{3,1}(t)](\xi)| = C|r| \left| (1 - \rho(\xi/4)) (I^+(t, \xi) - I^-(t, \xi)) \right| \lesssim \frac{1}{|r|^2 |\xi|^3}.$$

Thus, we obtain:

$$\|\tilde{K}_{3,1}(t)\|_{L^\infty_x} \lesssim \|\mathcal{F}[\tilde{K}_{3,1}(t)]\|_{L^1_t} \lesssim 1/|r|^2.$$

Meanwhile, it is easy to see that

$$\|\tilde{K}_{3,1}(t)\|_{L^\infty_x} \leq \|\tilde{K}_3(t)\|_{L^\infty_x} + \|\varphi \ast \tilde{K}_3(t)\|_{L^\infty_x} \lesssim 1.$$

Combining the above estimates we obtain (2.14). \hfill \Box

We next consider the estimate on the operator having the kernel $\tilde{K}_{3,2}$. We begin with stating the following claim:

**Claim 2.** We have:

$$\|\mathcal{F}_t[\tilde{K}_{3,2}](\tau, x)\|_{L^\infty_t \mathbb{R}^2} \lesssim |x|^{-1} (|x|)^{-2}, \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{2.17}$$

Once Claim 2 were proved, then by applying the Sobolev embedding we would obtain:

$$\|\tilde{K}_{3,2} \ast_{t,x} F\|_{L^2_t L^\infty_x} = \|\tilde{K}_{3,2} \ast_{t,x} \tilde{\varphi} \ast \tilde{F}\|_{L^2_t L^\infty_x}$$

$$\leq \sum_{|\alpha| \leq 2} \|\mathcal{F}_t[\tilde{K}_{3,2} \ast_{t,x} (\partial^\alpha_x \tilde{\varphi} \ast \tilde{F})]\|_{L^2_t L^2_x}$$

$$= \sum_{|\alpha| \leq 2} \|\mathcal{F}_t[\tilde{K}_{3,2}](\tau) \ast \mathcal{F}_t[\partial^\alpha_x \tilde{\varphi} \ast \tilde{F}](\tau)\|_{L^2_t L^2_x},$$

where $\tilde{\varphi}$ is defined by $\mathcal{F}\tilde{\varphi}(\xi) = \rho(\xi/8)$ so that $\tilde{\varphi} \equiv \varphi$, and we have applied Plancherel’s theorem with respect to the time variable. Now we apply Claim 2 to obtain:
Proof of Claim 2. We derive the decay estimate on $\mathcal{F}_t[\tilde{K}_{3, 2}]$ by using the stationary phase method again:

$$
\mathcal{F}_t[\tilde{K}_{3, 2}](\tau, x) = \frac{1}{2\pi} \left(1 - \rho_0 \left(\frac{|\tau| - \frac{1}{2}}{2}\right)\right) \int_{|r| \geq 8r} e^{-ir\tau} \frac{1}{|r|} \rho \left(\frac{x}{r}\right) \sin \sqrt{r^2 - r^2} \, dt
$$

$$
= \frac{1}{2\pi} \left(1 - \rho_0 \left(\frac{|\tau| - \frac{1}{2}}{2}\right)\right) \left\{ \int_{8}^{\infty} e^{-ir\tau} s^{-1} \rho_0 (1/s) \sin (r \sqrt{s^2 - 1}) \, ds + \int_{8}^{\infty} e^{-ir\tau} s^{-1} \rho_0 (1/s) \sin (r \sqrt{s^2 - 1}) \, ds \right\}.
$$

So, the problem is reduced to the estimate of the oscillatory integral of the form,

$$
I(r, \tau) = \int_{8}^{\infty} e^{\pm ir\phi_{\pm}(s, \tau)} s^{-1} \rho_0 (1/s) \, ds,
$$

for $|\tau| - 1 \geq 1/8$, where $\phi_{\pm}(s, \tau) = s \tau \pm \sqrt{s^2 - 1}$. The phase functions $\phi_{\pm}$ satisfy,

$$
|\partial_s \phi_{\pm}(s, \tau)| \geq |\tau| - \frac{s}{\sqrt{s^2 - 1}} = |\tau| - 1 - \left(\frac{s}{\sqrt{s^2 - 1}} - 1\right) \geq \frac{9}{8} - \frac{8}{\sqrt{63}} > 0,
$$

$$
|\partial_s^l \phi_{\pm}(s, \tau)| \leq C_l s^{-l + 1}, \quad l \geq 2.
$$

for $|\tau| - 1 \geq 1/8, s \geq 8$. Thus, integrating by parts we obtain:

$$
I(r, \tau) = \frac{-1}{\pm ir} \int_{8}^{\infty} e^{\pm ir\phi_{\pm}(s, \tau)} \partial_s \left(\frac{1}{\partial_s \phi_{\pm}(s, \tau)} s^{-1} \rho_0 (1/s)\right) \, ds
$$

$$
= \frac{(-1)^3}{(\pm ir)^3} \int_{8}^{\infty} e^{\pm ir\phi_{\pm}(s, \tau)} \partial_s \left\{ \frac{1}{\partial_s \phi_{\pm}} \partial_s \left(\frac{1}{\partial_s \phi_{\pm}} s^{-1} \rho_0 (1/s)\right) \right\} \, ds. \quad (2.19)
$$

Combining with the estimate $|I(r, \tau)| \lesssim 1/r$ which resulted from (2.19), we obtain (2.17). \(\square\)

We finally consider the estimate on the operator having the kernel $\tilde{K}_{3, 3}$. We begin with stating the following claim.

Claim 3. We have:

$$
\|\mathcal{F}_t [\tilde{K}_{3, 3}]\|_{L^\infty} \lesssim 1. \quad (2.20)
$$
Once Claim 3 were proved, then by the Sobolev embedding we would obtain:
\[ \| \tilde{K}_{3,3} \ast_{t,x} \tilde{F} \|_{L^2_t L^\infty_x} = \| \tilde{K}_{3,2} \ast_{t,x} \tilde{\varphi} \ast \tilde{F} \|_{L^2_t L^\infty_x} \]
\[ \lesssim \sum_{|\alpha| \leq 2} \| \tilde{K}_{3,3} \ast_{t,x} (\partial_x^\alpha \tilde{\varphi} \ast \tilde{F}) \|_{L^2_t L^2_x} \]
\[ = \sum_{|\alpha| \leq 2} \| \mathcal{F}_{t,x}[\tilde{K}_{3,3}] \mathcal{F}_{t,x}[\partial_x^\alpha \tilde{\varphi} \ast \tilde{F}] \|_{L^2_t L^2_x} \]
\[ \lesssim \sum_{|\alpha| \leq 2} \| \mathcal{F}_{t,x}[\partial_x^\alpha \tilde{\varphi} \ast \tilde{F}] \|_{L^2_t L^2_x} \]
\[ = \sum_{|\alpha| \leq 2} \| \partial_x^\alpha \tilde{\varphi} \ast \tilde{F} \|_{L^2_t L^2_x} \lesssim F \|_{L^2_t L^1_x}, \]
where \( \tilde{\varphi} \) is the same one appeared in (2.18).

**Proof of Claim 3.** Since
\[ \mathcal{F}_{t,x}[\tilde{K}_{3,3}](\tau, \xi) = \rho_0\left( \left| \tau \right| - 1 \right) / 2 \rho(\xi/4) \mathcal{F}_{t,x}[\tilde{K}_{3,3}](\tau, \xi), \]
it suffices to show that \( \mathcal{F}_{t,x}[\tilde{K}_{3,3}](\tau, \xi) \) is bounded on
\[ D = \{ (\tau, \xi) \mid \left| \tau \right| - 1 \leq 1/4, |\xi| \leq 1/2 \}. \] (2.21)

Introducing the convergence factor,
\[ \mathcal{F}_{t,x}[\tilde{K}_{3,3}] = \lim_{\varepsilon \to 0} \mathcal{F}_{t,x}[\varepsilon^{-\varepsilon} \sqrt{t^2 - |\tau|^2} \tilde{K}_{3,3}] \text{ in } S'(\mathbb{R} \times \mathbb{R}^2), \]
we calculate \( \mathcal{F}_{t,x}[\varepsilon^{-\varepsilon} \sqrt{t^2 - |\tau|^2} \tilde{K}_{3,3}] \) for \( \varepsilon > 0 \) as follows:
\[ \mathcal{F}_{t,x}[\varepsilon^{-\varepsilon} \sqrt{t^2 - |\tau|^2} \tilde{K}_{3,3}](\tau, \xi) \]
\[ = \frac{1}{4\pi} \int_{\mathbb{R} \times \mathbb{R}^2} \varepsilon^{-i\tau \cdot x - \varepsilon \sqrt{t^2 - |\tau|^2} \cdot t} \sin \sqrt{t^2 - |x|^2} \rho(x/t) \, dt \, dx \]
\[ = \frac{1}{4\pi} \int_0^\infty \int_{|x| \leq 1/8} \varepsilon^{-i\tau \cdot x - \varepsilon \sqrt{t^2 - |\tau|^2} \cdot t} \sin \sqrt{t^2 - |x|^2} \rho(x/t) \, dt \, dx \]
\[ + \frac{1}{4\pi} \int_0^\infty \int_0^\infty \varepsilon^{i\tau \cdot x - \varepsilon \sqrt{t^2 - |\tau|^2} \cdot t} \sin \sqrt{t^2 - |x|^2} \rho(x/t) \, dt \, dx \]
\[ = \frac{1}{4\pi} \int_0^\infty \int_{\mathbb{R}^2} \varepsilon^{-i(\lambda - i\lambda(\tau, \xi)) \cdot \Omega}(\lambda, \Omega_0)^{-\varepsilon} \sin \lambda \rho(\Omega'/\Omega_0) \lambda^2 \, d\lambda \, d\Omega \]
\[ + \frac{1}{4\pi} \int_0^\infty \int_{\mathbb{R}^2} \varepsilon^{i(\lambda + i\lambda(\tau, \xi)) \cdot \Omega}(\lambda, \Omega_0)^{-\varepsilon} \sin \lambda \rho(\Omega'/\Omega_0) \lambda^2 \, d\lambda \, d\Omega \]
\[ = \frac{1}{4\pi} \left( \int_0^\infty \int_0^\infty \varepsilon^{-i\lambda - \varepsilon \lambda \cdot \Omega}(\lambda, \Omega_0)^{-\varepsilon} \sin \lambda \rho(\lambda'H_{-}(\lambda, \tau, \xi)) \, d\lambda \right) \]
\[ + \int_0^\infty \varepsilon^{i\lambda + \varepsilon \lambda \cdot \Omega}(\lambda, \Omega_0)^{-\varepsilon} \sin \lambda \rho(\lambda'H_{+}(\lambda, \tau, \xi)) \, d\lambda. \]
where \( \text{sgn}(\tau) \) denotes the sign of \( \tau \in \mathbb{R} \), and we have introduced the hyperbolic polar coordinates,

\[
(t, x) = \lambda \Omega, \quad \lambda = \sqrt{t^2 - |x|^2}, \quad \Omega = (\Omega_0, \Omega') = \left( \frac{t}{\lambda}, \frac{x}{\lambda} \right) \in \mathbb{H}^2,
\]

and \( dt \, dx = \lambda^2 \, d\lambda \, d\Omega \). See e.g. [2]. Here, \( \mathbb{H}^2 = \{(t, x) \in (0, \infty) \times \mathbb{R}^2 | t^2 - |x|^2 = 1\} \) and we set:

\[
H_\pm(\lambda, \tau, \xi) = \int_{\mathbb{H}^2} e^{\pm i \lambda \cdot (\Omega - \text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2})} \frac{1}{\Omega_0} \rho(\Omega'/\Omega_0) \, d\Omega
\]

\[
= \int_{\mathbb{R}^2} e^{\pm i \lambda \cdot (y, \xi - \text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2})} \frac{1}{(y/\langle y \rangle)^2} \rho(y/\langle y \rangle) \, dy,
\]

(2.22)

for \( \lambda > 0, (\tau, \xi) \in D \). Therefore, we obtain:

\[
F_{t,x} \left[ e^{-\varepsilon \sqrt{t^2 - |x|^2}} \tilde{K}_3 \right](\tau, \xi) = \frac{1}{8\pi i} \left( \int_0^\infty e^{-\varepsilon \lambda - i \lambda [\text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2} - 1]} \lambda H_-(\lambda, \tau, \xi) \, d\lambda
\]

\[
- \int_0^\infty e^{-\varepsilon \lambda - i \lambda [\text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2} + 1]} \lambda H_+(\lambda, \tau, \xi) \, d\lambda
\]

\[
+ \int_0^\infty e^{-\varepsilon \lambda + i \lambda [\text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2} - 1]} \lambda H_+(\lambda, \tau, \xi) \, d\lambda
\]

\[
- \int_0^\infty e^{-\varepsilon \lambda + i \lambda [\text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2} + 1]} \lambda H_-(\lambda, \tau, \xi) \, d\lambda
\]

\[
\equiv \frac{1}{8\pi i} \left( I_{\varepsilon 1}^+ (\tau, \xi) - I_{\varepsilon 2}^+ (\tau, \xi) + I_{\varepsilon 3}^+ (\tau, \xi) - I_{\varepsilon 4}^+ (\tau, \xi) \right).
\]

(2.23)

For the estimate of the right-hand side of (2.23) the estimates of the oscillatory integrals \( H_\pm(\lambda, \tau, \xi) \) defined in (2.22) play an important role. The asymptotic behaviors of \( H_\pm(\lambda, \tau, \xi) \) as \( \lambda \to \infty \) are characterized by the behavior of the phase function,

\[
\phi_\pm(y, \tau, \xi) = \pm \left\{ (\tau, \xi) \cdot (y, \xi) - \text{sgn}(\tau) \sqrt{\tau^2 - |\xi|^2} \right\}.
\]

(2.24)

Since

\[
\nabla_y \phi_\pm(y, \tau, \xi) = \pm \left\{ \frac{\tau y}{|y|} + \xi \right\},
\]

and we are concerned with the region (2.21), we observe that \( \phi_\pm \) may have a stationary point \( y^0(\tau, \xi) \) on the support of the amplitude function \( (y) \cdot (y/\langle y \rangle)^{-2} \rho(y/\langle y \rangle) \). In that case \( \phi_\pm \) satisfy \( (\nabla_y \phi_\pm)(y^0, \tau, \xi) = 0 \), i.e. \( y^0 \) satisfies:

\[
\frac{\tau y^0}{|y^0|} + \xi = 0.
\]

(2.25)

We notice that \( \phi_\pm \) also satisfy:

\[
\phi_\pm(y^0, \tau, \xi) = 0.
\]

In fact, from (2.25) we derive two equations,

\[
\tau \frac{|y^0|^2}{|y^0|} + \xi \cdot y^0 = 0, \quad \tau \frac{\xi \cdot y^0}{|y^0|^2} + |\xi|^2 = 0.
\]
From these we obtain:
\[
\langle y_0 \rangle = \frac{|\tau|}{\sqrt{\tau^2 - |\xi|^2}}, \quad \text{and} \quad y^0 = -\frac{\text{sgn}(\tau)\xi}{\sqrt{\tau^2 - |\xi|^2}}.
\] (2.26)

Thus, we have:
\[
\phi_\pm(y^0, \tau, \xi) = \pm\{\tau \langle y_0 \rangle + \xi \cdot y^0 - \text{sgn}(\tau)\sqrt{\tau^2 - |\xi|^2}\} = 0.
\]

From these observations the asymptotic behaviors of \(H_\pm(\lambda, \tau, \xi)\) are summarized in the next lemma.

**Lemma 2.7.** Let \(H_\pm(\lambda, \tau, \xi)\) be defined in (2.22) and let \((\tau, \xi)\) satisfy (2.21). Then, for \(N \in \mathbb{N} \cup \{0\}\) we have:
\[
H_\pm(\lambda, \tau, \xi) = \sum_{j=0}^{N} a^j_\pm(\tau, \xi)\lambda^{-1-j} + R^N_\pm(\lambda, \tau, \xi), \quad \lambda > 0,
\] (2.27)

where the remainder terms \(R^N_\pm\) satisfy,
\[
\sup_{(\tau, \xi) \in D} \left| \partial^k \lambda R^N_\pm(\lambda, \tau, \xi) \right| \leq C\lambda^{-2-N-k}
\] (2.28)

for any \(\lambda > 0\). In particular, we have,
\[
a^0_\pm(\tau, \xi) = \pm\frac{2\pi i}{|\tau|} \left( \frac{\xi}{|\tau|} \right),
\] (2.29)
\[
a^1_\pm(\tau, \xi) = a^1(\tau, \xi),
\] (2.30)

for some \(a^1\) which is bounded on \(D\).

**Proof.** We first summarize the properties of the phase function \(\phi_\pm\) defined in (2.24). As we state before, \(\phi_\pm\) satisfies:
\[
\nabla \phi_\pm(y, \tau, \xi) = 0 \iff y = y^0(\tau, \xi),
\]
and \(\phi(y^0, \tau, \xi) = 0\), where \(y^0(\tau, \xi)\) is the one in (2.26). Then, we have:
\[
\text{Hess} \phi_\pm(y, \tau, \xi) = \tau \begin{pmatrix}
\frac{1}{\langle y \rangle} - \frac{\xi_1^2}{\langle y \rangle^3} & -\frac{\xi_1 \xi_2}{\langle y \rangle^3} \\
-\frac{\xi_1 \xi_2}{\langle y \rangle^3} & \frac{1}{\langle y \rangle} - \frac{\xi_2^2}{\langle y \rangle^3}
\end{pmatrix},
\] (2.31)
\[
H_\pm := \text{Hess} \phi_\pm(y^0(\tau, \xi)) = \pm \text{sgn}(\tau) \sqrt{\frac{\tau^2 - |\xi|^2}{\tau^2}} \begin{pmatrix}
\tau^2 - \xi_1^2 & -\xi_1 \xi_2 \\
-\xi_1 \xi_2 & \tau^2 - \xi_2^2
\end{pmatrix},
\]
and eigenvalues of \(H_\pm\) are,
\[
\pm \text{sgn}(\tau) \sqrt{\frac{\tau^2 - |\xi|^2}{\tau^2}}, \quad \pm \text{sgn}(\tau) \frac{(\tau^2 - |\xi|^2)^{3/2}}{\tau^2},
\]
and thus
\[
\inf_{(\tau, \xi) \in D} |\det H_\pm| = \inf_{(\tau, \xi) \in D} \frac{(\tau^2 - |\xi|^2)^2}{\tau^2} > 0.
\] (2.32)

Moreover, we observe that
\[
\inf_{(\tau, \xi) \in D} |\nabla \phi_\pm(y, \tau, \xi)| \geq C|y - y^0|, \quad \sup_{(\tau, \xi) \in D} |\partial^\alpha \phi_\pm(y, \tau, \xi)| \leq C_\alpha, \quad |\alpha| \geq 1,
\] (2.33)

hold for \(y \in \text{supp} \rho(\cdot/|\cdot|) \subset \{|y| \leq 1/4\}\). These estimates are essential to derive the uniform bound on the remainder term (2.28). We will give the proof of the first estimate in (2.33) here, since the second one is easy. Since
\[ \partial_j \phi_\pm(y, \tau, \xi) = \partial_j \phi_\pm(y, \tau, \xi) - \partial_j \phi_\pm(y^0, \tau, \xi) \]
\[ = \sum_{k=1}^{2} \int_0^1 (\partial_j \partial_k \phi_\pm)(y^0 + t(y - y^0)) \, dt (y_k - y^0_k), \]
we have:

\[ \left| \partial_1 \phi_\pm(y, \tau, \xi) \right| \geq \left| \int_0^1 (\partial_1^2 \phi_\pm)(y^0 + t(y - y^0)) \, dt \right| |y_1 - y_1^0| \]
\[ - \left| \int_0^1 (\partial_1 \partial_2 \phi_\pm)(y^0 + t(y - y^0)) \, dt \right| |y_2 - y_2^0| \]
\[ \geq |\tau| \int_0^1 \frac{1 + y_2(t)^2}{(y(t))^3} \, dt |y_1 - y_1^0| - |\tau| \int_0^1 \frac{|y_1(t)y_2(t)|}{(y(t))^3} \, dt |y_2 - y_2^0| \]
\[ \geq |\tau| \int_0^1 \frac{1}{(y(t))^3} \, dt |y_1 - y_1^0| - \frac{|\tau|}{16} \int_0^1 \frac{1}{(y(t))^3} \, dt |y_2 - y_2^0|, \]
where \( y(t) = y^0 + t(y - y^0), |y|, |y^0| \leq 1/4, \) and thus \( |y(t)| \leq 1/4. \) Combining with the similar estimate on \( |\partial_2 \phi_\pm(y, \tau, \xi)|, \) we obtain:

\[ |\nabla \phi_\pm(y, \tau, \xi)| \geq 2^{-1/2} \left( |\partial_1 \phi_\pm(y, \tau, \xi)| + |\partial_2 \phi_\pm(y, \tau, \xi)| \right) \]
\[ \geq 2^{-1/2} |\tau| \left( \int_0^1 \frac{1}{(y(t))^3} \, dt |y - y^0| - \frac{\sqrt{2}}{16} \int_0^1 \frac{1}{(y(t))^3} \, dt |y - y^0| \right) \]
\[ \geq \frac{|\tau|}{2} \int_0^1 \frac{1}{(y(t))^3} \, dt |y - y^0| \]
\[ \geq C |y - y^0| \]

for \((\tau, \xi) \in D, |y|, |y^0| \leq 1/4.\)

Now we divide \( H_\pm(\lambda, \tau, \xi) \) into two parts,

\[ H_\pm(\lambda, \tau, \xi) = \int_{\mathbb{R}^2} e^{i \lambda \phi_\pm(y, \tau, \xi)} y^{2} \rho(y/\langle y \rangle) \psi_\delta(y) \, dy \]
\[ + \int_{\mathbb{R}^2} e^{i \lambda \phi_\pm(y, \tau, \xi)} y^{2} \rho(y/\langle y \rangle) (1 - \psi_\delta(y)) \, dy \]
\[ \equiv H_1^\pm(\lambda, \tau, \xi) + H_2^\pm(\lambda, \tau, \xi), \]
where \( \psi_\delta \) is a smooth cutoff function satisfying

\[ \psi_\delta(y) = \begin{cases} 1, & |y - y^0| \leq \delta/2, \\ 0, & |y - y^0| \geq \delta. \end{cases} \]

for some \( \delta > 0 \) which is determined later. Then, by (2.33) and [20, VIII, Proposition 5] we obtain:

\[ H_2^\pm(\lambda, \tau, \xi) \leq C(\delta) \lambda^{-l}, \quad \lambda > 0, \]

for any \( l \in \mathbb{N}. \) Note that estimates (2.33) enable us to take \( C(\delta) \) independently of \((\tau, \xi) \in D.\)
So, the asymptotic behavior of $H_\pm(\lambda, \tau, \xi)$ is determined by $H_\pm^1(\lambda, \tau, \xi)$. To derive its asymptotic expansion we refer to [20, VIII, Proposition 6]. Here, $\delta > 0$ is determined to guarantee Morse’s lemma, which can be chosen independently of $(\tau, \xi) \in D$ because it is determined depending only on (2.31) and (2.32). Note that the uniform bound of the remainder term (2.28) is assured by (2.33).

Finally, we make some remarks on the coefficients of the expansion. The representation (2.29) is also due to [20, VIII, Proposition 6]. For the representation (2.30) see [20, VIII, 5.1]. Precisely, applying Morse’s lemma we have:

$$H_\pm^1(\lambda, \tau, \xi) = \int e^{i\frac{1}{2}(H_\pm w, w)}(y(w))^{-2} \rho(y(w)/|y(w)|) \psi_\delta(y(w)) \left| \frac{\partial(y)}{\partial(w)} \right| dw$$

$$\equiv \int e^{i\frac{1}{2}(H_\pm w, w)} \tilde{\rho}(w) dw$$

$$\sim \sum_{j=0}^\infty a_\pm^j(\tau, \xi) \lambda^{-1-j},$$

where

$$a_\pm^j(\tau, \xi) = 2\pi |\det H_\pm|^{-1/2} e^{\pm \pi i/2} \frac{i^j}{2j!} (\Delta_{H_\pm} \tilde{\rho})(y^0(\tau, \xi)),$$

$$\Delta_{H_\pm} = \sum_{j,k} h_{\pm}^{j,k} \partial_j \partial_k, \quad \text{with } H_\pm^{-1} = (h_{\pm}^{j,k}).$$

Here, we notice that $a_\pm^j$ are bounded on $D$, because $D \ni (\tau, \xi) \mapsto y^0(\tau, \xi)$ and $\rho, \phi_\delta$ are smooth and bounded. Moreover, since $e^{\pm \pi i/2} = \pm i$ and $H_\pm^{-1} = -H_\pm^{-1}$ we have:

$$a_+^j(\tau, \xi) = a_-^j(\tau, \xi),$$

which is crucial for the argument below. \square

In what follows we only consider the case $\tau > 0$, since the case $\tau < 0$ is similar. When $\tau > 0$, we easily observe that $I_2^\varepsilon(\tau, \xi)$ and $I_3^\varepsilon(\tau, \xi)$ on the right-hand side of (2.23) is bounded independently of $\varepsilon$. For example, applying the integration by parts we have:

$$\lim_{\varepsilon \to 0} I_2^\varepsilon(\tau, \xi) = \int_0^1 e^{-i\lambda(\sqrt{\tau^2 - |\xi|^2} + 1)} \lambda H_-(\lambda, \tau, \xi) d\lambda + \frac{e^{-i(\sqrt{\tau^2 - |\xi|^2} + 1)}}{i(\sqrt{\tau^2 - |\xi|^2} + 1)} H_-(1, \tau, \xi)$$

$$+ \frac{1}{i(\sqrt{\tau^2 - |\xi|^2} + 1)} \int_1^\infty e^{-i\lambda(\sqrt{\tau^2 - |\xi|^2} + 1)} \partial_\lambda (\lambda H_-(\lambda, \tau, \xi)) d\lambda.$$
For the first terms we apply integration by parts and Lemma 2.7 to obtain:

\[ I_{1,1}(\tau, \xi) = \left[ e^{-\varepsilon \lambda - i \alpha \lambda} \frac{e^{-i \alpha \lambda}}{i \alpha} \right]_{-i \alpha}^{0} \int_{-\varepsilon}^{\varepsilon} e^{-\varepsilon \lambda - i \alpha \lambda} H_-(\lambda, \tau, \xi) d\lambda \]

\[ = \frac{2\pi}{i \alpha |\alpha|} e^{-\varepsilon \lambda - i \alpha \lambda} H_-(2\pi/|\alpha|, \tau, \xi) \]

\[ + \frac{1}{i \alpha |\alpha|} \int_{-\varepsilon}^{\varepsilon} e^{-i \alpha \lambda} (R_0^0(\lambda, \tau, \xi) + \lambda \partial_\lambda R_0^0(\lambda, \tau, \xi)) d\lambda \]

\[ \to \frac{2\pi}{i \alpha |\alpha|} H_-(2\pi/|\alpha|, \tau, \xi) \]

\[ + \frac{1}{i \alpha |\alpha|} \int_{-\varepsilon}^{\varepsilon} e^{-i \alpha \lambda} (R_0^0(\lambda, \tau, \xi) + \lambda \partial_\lambda R_0^0(\lambda, \tau, \xi)) d\lambda \]  

(2.34)

as \( \varepsilon \to 0 \). Here, we notice that by Lemma 2.7 the second term in (2.34) is bounded by,

\[ C \int_{-\varepsilon}^{\varepsilon} \lambda^{-2} d\lambda = \frac{C}{2\pi}. \]

Similarly, we have:

\[ \lim_{\varepsilon \to 0} I_{1,1}^e(\tau, \xi) = -\frac{2\pi}{i \alpha |\alpha|} H_+(2\pi/|\alpha|, \tau, \xi) + O(1). \]

Thus,

\[ \lim_{\varepsilon \to 0} \left( I_{1,1}^e(\tau, \xi) - I_{4,1}^e(\tau, \xi) \right) \]

\[ = \frac{2\pi}{i \alpha |\alpha|} \left( H_-(2\pi/|\alpha|, \tau, \xi) + H_+(2\pi/|\alpha|, \tau, \xi) \right) + O(1) \]

\[ = \frac{2\pi}{i \alpha |\alpha|} \left( a_0^0(\tau, \xi) |\alpha|/2\pi + R_0^0(2\pi/|\alpha|, \tau, \xi) + a_+^0(\tau, \xi) |\alpha|/2\pi + R_+^0(2\pi/|\alpha|, \tau, \xi) \right) + O(1) \]

\[ = \frac{2\pi}{i \alpha |\alpha|} \left( R_0^0(2\pi/|\alpha|, \tau, \xi) + R_+^0(2\pi/|\alpha|, \tau, \xi) \right) + O(1). \]

In the above calculation first terms of the asymptotic expansions of \( H_\pm(\lambda, \tau, \xi) \) are cancelled out due to (2.29). Therefore, applying (2.28) we obtain:

\[ \left| \lim_{\varepsilon \to 0} \left( I_{1,1}^e(\tau, \xi) - I_{4,1}^e(\tau, \xi) \right) \right| \leq \frac{2\pi}{i \alpha |\alpha|^2} C \left( \frac{2\pi}{|\alpha|} \right)^{-2} + C \leq C. \]

For the estimate of \( I_{1,2}^e(\tau, \xi) \) and \( I_{4,2}^e(\tau, \xi) \) we only consider the case \( 2\pi/|\alpha| > 1 \), since the case \( 2\pi/|\alpha| \leq 1 \) is easy. In fact, for such case it is sufficient to apply the trivial estimates,

\[ |H_\pm(\lambda, \tau, \xi)| \leq C, \quad \lambda \geq 0. \]

In the case \( 2\pi/|\alpha| > 1 \), applying Lemma 2.7 we have:

\[ \lim_{\varepsilon \to 0} I_{1,2}^e(\tau, \xi) = \int_{0}^{2\pi/|\alpha|} e^{-i \alpha \lambda} H_-(\lambda, \tau, \xi) d\lambda. \]
\[
= \int_1^{2\pi} e^{-ia\lambda} (a_0^0(\tau, \xi)\lambda^{-1} + a_0^1(\tau, \xi)\lambda^{-2} + R^1_\lambda(\lambda, \tau, \xi)) d\lambda
\]
\[
+ \int_0^{2\pi} e^{-ia\lambda} (a_0^0(\tau, \xi)\lambda^{-1} + R^0_\lambda(\lambda, \tau, \xi)) d\lambda
\]
\[
= \int_0^{2\pi} e^{-ia\lambda} a_0^0(\tau, \xi) d\lambda + \int_1^{2\pi} e^{-ia\lambda} a_1^0(\tau, \xi)\lambda^{-1} d\lambda
\]
\[
+ \int_0^{2\pi} e^{-ia\lambda} \lambda R^1_\lambda(\lambda, \tau, \xi) d\lambda + \int_0^{2\pi} e^{-ia\lambda} \lambda R^0_\lambda(\lambda, \tau, \xi) d\lambda
\]
\[
\equiv I^1_{1,2}(\tau, \xi) + I^2_{1,2}(\tau, \xi) + I^3_{1,2}(\tau, \xi) + I^4_{1,2}(\tau, \xi).
\]

Then, it is easy to see that
\[
I^1_{1,2}(\tau, \xi) = a_0^0(\tau, \xi) \left[ \frac{1}{-i\alpha} \right]_0^{2\pi} = 0
\]
and
\[
|I^1_{1,2}(\tau, \xi)| \leq \int_1^{\infty} C\lambda^{-2} d\lambda = C.
\]

Moreover, since
\[
|R^0_\lambda(\lambda, \tau, \xi)| = |H^0_\lambda(\lambda, \tau, \xi) - a_0^0(\tau, \xi)\lambda^{-1}| \leq C\lambda^{-1}, \quad 0 < \lambda \leq 1,
\]
we have:
\[
|I^4_{1,2}(\tau, \xi)| \leq \int_0^{1} \lambda |R^0_\lambda(\lambda, \tau, \xi)| d\lambda \leq C.
\]

Thus, we have:
\[
\lim_{\varepsilon \to 0} I^\varepsilon_{1,2}(\tau, \xi) = I^2_{1,2}(\tau, \xi) + O(1).
\]

Similarly, we have:
\[
\lim_{\varepsilon \to 0} I^\varepsilon_{4,2}(\tau, \xi) = \int_1^{2\pi} e^{ia\lambda} a_1^1(\tau, \xi)\lambda^{-1} d\lambda + O(1)
\]
\[
\equiv I^2_{4,2}(\tau, \xi) + O(1).
\]

Finally, from (2.30) we observe that
\[
I^2_{1,2}(\tau, \xi) - I^2_{4,2}(\tau, \xi) = \int_1^{2\pi} e^{-ia\lambda} a_1(\tau, \xi)\lambda^{-1} d\lambda - \int_1^{2\pi} e^{ia\lambda} a_1(\tau, \xi)\lambda^{-1} d\lambda.
\]
Therefore, since we are concerned with the case $|\alpha| < 2\pi$, we conclude that

$$
|I_{1,2}^2(\tau, \xi) - I_{4,2}^2(\tau, \xi)| \leq C \int_0^{2\pi} d\lambda' = 2\pi C.
$$

This completes the proof of Claim 3. □

Accordingly, this completes the proof of Lemma 2.6. □

In the rest of this section, we remark that the retarded estimates (2.3), (2.4) are immediately follow from (2.1), (2.2), respectively. For example, we have:

$$
\left\| \int_0^t U(t-s)F(s) ds \right\|_{L^2_t L^\infty_r L^q_\omega} \leq \left\| \chi_{[s<t]}U(t-s)F(s) \right\|_{L^2_t L^\infty_r L^q_\omega} ds
$$

$$
\leq \int_0^\infty \left\| U(t')F(s) \right\|_{L^2_t(0,\infty;L^\infty_r L^q_\omega)} ds
$$

$$
\leq \int_0^\infty \sqrt{q} \left\| F(s) \right\|_{L^2_x} ds = \sqrt{q} \left\| F \right\|_{L^1_t L^2_x}.
$$

3. Endpoint Strichartz estimates for the 2D Schrödinger equation

In this section, we prove the endpoint Strichartz estimates for the two-dimensional Schrödinger equation as an application of Theorem 2.1. Concerning this problem, Tao [21] showed that the estimate,

$$
\left\| e^{it\Delta} f \right\|_{L^2_t L^\infty_r H^s_\omega} \lesssim \left\| f \right\|_{L^2},
$$

(3.1)

holds for small $s > 0$, where $H^s_\omega$ denotes the Sobolev space on $S^1$. However, Machihara, Nakamura, Nakanishi, and Ozawa [11, Theorem 5.1] pointed out the fact that $s \leq 1/3$ is necessary for the estimate (3.1), which implies we cannot take $q > 6$ in (3.2) below by using the Sobolev embedding and (3.1). The following theorem states that the estimate holds for all $q \geq 1$.

**Theorem 3.1.** Let $n = 2$ and let $1 \leq q < \infty$. Then, the following estimate holds:

$$
\left\| e^{it\Delta/2} f \right\|_{L^2_t L^\infty_r L^q_\omega} \lesssim \sqrt{q} \left\| f \right\|_{L^2}.
$$

(3.2)

The proof of Theorem 3.1 is based on the argument of non-relativistic limit to reduce the problem to the Klein–Gordon case. See e.g. [22]. The Klein–Gordon equation is generally given by:

$$
\frac{1}{c^2} \partial^2_t u - \Delta u + \frac{m^2 c^2}{\hbar} u = 0,
$$

(3.3)

where $c > 0$ is the speed of light, $m > 0$ is the mass, and $\hbar > 0$ is the Planck constant. Considering the modulated function $v = e^{imc^2t/\hbar} u$, we observe that $v$ solves the following modulated equation:
\[
\frac{1}{c^2} \partial_t^2 v - \frac{2im}{\hbar} \partial_t v - \Delta v = 0.
\]

Then, taking the non-relativistic limit \(c \to \infty\), we easily observe that the modulated equation converges to the Schrödinger equation

\[
i \partial_t v + \frac{\hbar}{2m} \Delta v = 0.
\]

By using this relation between the Klein–Gordon equation and the Schrödinger equation, Theorem 3.1 is derived as a simple application of Theorem 2.1.

**Proof of Theorem 3.1.** To begin with, by rescaling we normalize the constants as \(m = \hbar = 1\). For the proof of the estimate (3.2) it suffices to consider \(f \in S(\mathbb{R}^2)\). Now we consider the Cauchy problem of the modulated equation:

\[
\begin{aligned}
\begin{cases}
c^{-2} \partial_t^2 v - 2i \partial_t v - \Delta v = 0, \\
v(0) = f, \quad \partial_t v(0) = g,
\end{cases}
\end{aligned}
\]

for some \(g \in S(\mathbb{R}^2)\). The solution of this problem is given by:

\[
\psi(t) = e^{itc^2} \left\{ \cos(tc(\nabla)_c) f - \frac{ic}{\nabla c} \sin(tc(\nabla)_c) \right\} + e^{itc^2} \frac{1}{c(\nabla)_c} \sin(tc(\nabla)_c) g.
\]

In the following, we show the uniform boundedness of \(\{\psi\} \) by using Theorem 2.1. Since \(\cos(tc(\nabla)_c) f(x) = \{\cos(tc^2(\nabla)) f(\cdot/c)\}(cx)\), we have:

\[
\| \cos(tc(\nabla)_c) f \|_{L^2_t L^\infty_x L^q} \leq \| \{\cos(tc^2(\nabla)) f(\cdot/c)\} \|_{L^2_t L^\infty_x L^q} \leq c^{-1} \sqrt{q} \| f(\cdot/c) \|_{H^1} \leq \sqrt{q} \| f \|_{L^2} + c^{-1} \| f \|_{H^1}.
\]

Similarly, we have:

\[
\| \frac{i c}{\nabla c} \sin(tc(\nabla)_c) f \|_{L^2_t L^\infty_x L^q} \leq \sqrt{q} \| f \|_{L^2}, \quad \| \frac{1}{c(\nabla)_c} \sin(tc(\nabla)_c) g \|_{L^2_t L^\infty_x L^q} \leq c^{-2} \sqrt{q} \| g \|_{L^2}.
\]

Therefore, we obtain:

\[
\| \psi \|_{L^2_t L^\infty_x L^q} \leq \sqrt{q} \left( \| f \|_{L^2} + c^{-1} \| f \|_{H^1} + c^{-2} \| g \|_{L^2} \right) \leq \sqrt{q} \left( \| f \|_{L^2} + \| f \|_{H^1} + \| g \|_{L^2} \right),
\]

for \(c > 1\). Thus, there exists a subsequence \(\{\psi^{(j)}\}_{j=1}^\infty \subset \{\psi\}\) and \(v \in L^2_t L^\infty_x L^q_{\omega}\) such that

\[
\psi^{(j)} \to v \quad \text{\(\ast\)-weakly in } L^2_t L^\infty_x L^q_{\omega}, \text{ as } j \to \infty.
\]

Moreover, from (3.4),

\[
\| v \|_{L^2_t L^\infty_x L^q} \leq \liminf_{j \to \infty} \| v^{(j)} \|_{L^2_t L^\infty_x L^q} \leq \sqrt{q} \| f \|_{L^2},
\]

holds. Finally, we notice that \(v\) satisfies the Schrödinger equation in the distribution sense. In fact, \(v^{(j)}\) satisfies,

\[
\int_0^\infty \left\{ c_j^{-2} (v^{(j)}, \partial_t^2 \varphi) + 2i (v^{(j)}, \partial_\varphi) - (v, \Delta \varphi) \right\} dt
\]

\[
= c_j^{-2} (g, \varphi(0)) - c_j^{-2} (f, \varphi(0)) + 2i (f, \varphi(0)),
\]
for $\varphi \in \mathcal{D}((0, \infty) \times \mathbb{R}^2)$. Since $\mathcal{D}((0, \infty) \times \mathbb{R}^2) \subset L^2_t L^1_x L^q_\omega$, letting $j \to \infty$ we easily observe that $v$ satisfies the Schrödinger equation with data $f \in \mathcal{S}(\mathbb{R}^2)$. Therefore, we have $v(t) = e^{it\Delta/2} f$. This completes the proof of Theorem 3.1. \hfill $\Box$

From the above argument, we immediately observe that the following result holds. For the notation see Remark 2.2(2).

**Corollary 3.2.** Let $u$ be a solution to the Klein–Gordon equation (2.5). Then, the estimate,
\[
\|u\|_{L^2_t L^\infty_x} \lesssim E(u)^{1/2},
\]
(3.5)
do not hold in general.

In fact, (3.5) implies (2.1) and (2.2). Thus, if (3.5) were true, then the above argument indicates that the estimate,
\[
\|e^{it\Delta/2} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2},
\]
must hold, since the scaling properties of spaces $L^\infty_t L^q_\omega$ and $L^\infty_x$ are the same. However, this contradicts the result of Montgomery-Smith [14].

### 4. Global solutions to the cubic NLKG for small data

In this section, we are concerned with the global solvability of the Cauchy problem on the nonlinear Klein–Gordon equation for small initial data. There are many papers concerning such problem (see, e.g., [6,7,9,18,19]). In particular, we focus on the cubic nonlinearity in two space dimensions here, which can be treated as an application of Theorem 2.1. Although existence of the global solution to such case has been already known even for the quadratic nonlinearity [3,10,16,17], those results require much regularity and decay on the initial data. In the following results, the condition on the initial data is relaxed, especially no decay assumption is required. (For the application to the special quadratic nonlinearity, see [4].)

We consider the following Cauchy problem:
\[
\begin{aligned}
\partial_t^2 u - \Delta u + u &= F(u, \partial u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
u(0, x) &= \epsilon u_0(x), \quad \partial_t u(0, x) = \epsilon u_1(x), \quad x \in \mathbb{R}^2,
\end{aligned}
\]
where $\partial = (\partial_t, \nabla) = (\partial_t, \partial_1, \partial_2)$,
\[
F(u, \partial u) = u^k (\partial_t u)^l (\nabla u)^\alpha, \quad k + l + |\alpha| = 3.
\]

The above Cauchy problem is rewritten as the following integral equation:
\[
u(t) = \epsilon \dot{U}(t) u_0 + \epsilon U(t) u_1 + \int_0^t U(t-t') F(u(t'), \partial u(t')) \, dt'.
\]

Before stating our results, we summarize notation and facts. We define the norm,
\[
\|f\|_{H^m(H^s_\omega)} = \|\langle \nabla S^1 \rangle^s f\|_{H^m},
\]
where $\langle \nabla S^1 \rangle^s = (I - \Delta S^1)^{s/2}$, $\Delta S^1$ is the Laplace–Beltrami operator on $S^1$. We notice that we have the equivalence,
\[
\|f\|_{H^m(H^s_\omega)} \sim \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^2_x H^s_\omega}.
\]

Using equivalent norms defined through local coordinates, the following estimate holds
\[
\|fg\|_{H^{r,p}_\omega} \lesssim \|f\|_{H^{r_1}_{\omega,1}} \|g\|_{L^2_{s_1}} + \|f\|_{L^2_{s_2}} \|g\|_{H^{r_2}_{\omega}},
\]
where $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$ with $q_1, r_2 \neq \infty$. The proofs can be found in [11, §2].
For the nonlinear problem we have the following results. The difference between the following two results comes from the fact that for the estimate of $\|F(u, \partial u)\|_{L^1_tL^2_x}$, $\|\partial u\|_{L^1_tL^\infty_x}$ is required or not, which is determined by the number of $k$ in (4.1).

**Theorem 4.1.** Let $u_0 \in H^1(H^0_\omega)$, $u_1 \in L^2(H^0_\omega)$ for small $\delta > 0$. If $\epsilon > 0$ is sufficiently small, then there exists a unique global solution $u \in C^1_H(H^0_\omega) \cap C^1_t L^2(H^0_\omega)$ to (4.2) with $k = 2, 3$ satisfying:

$$\|\partial u\|_{L^1_tL^2(H^\delta)} + \|u\|_{L^1_tL^2(H^\delta) \cap L^1_tL^\infty_x} < \infty.$$  

**Theorem 4.2.** Let $u_0 \in H^2(H^0_\omega)$, $u_1 \in H^1(H^0_\omega)$ for small $\delta > 0$. If $\epsilon > 0$ is sufficiently small, then there exists a unique global solution $u \in C^1_H(H^0_\omega) \cap C^1_t H^1(H^0_\omega)$ to (4.2) with $k = 0, 1$ satisfying:

$$\|\partial u\|_{L^1_tH^1(H^\delta) \cap L^1_tL^\infty_x} + \|u\|_{L^1_tL^2(H^\delta) \cap L^1_tL^\infty_x} < \infty.$$  

Since the proofs are standard and similar, we briefly sketch only the proof of Theorem 4.1 in the case where $k = 2$.

**Proof of Theorem 4.1.** We define the norm,

$$\|u\|_X = \|\partial u\|_{L^\infty_tL^2(H^\delta_0)} + \|u\|_{L^\infty_tL^2(H^\delta_0)} + \|\langle \nabla \rangle^3 u\|_{L^2_tL^\infty_x},$$

where $\delta > 1/q$, and the space,

$$X = \{ u \in C([0, \infty); H^1(H^0_\omega)) \cap C^1([0, \infty); L^2(H^0_\omega)) \mid \|u\|_X \leq R \},$$

then Theorem 4.1 is proved by using the contraction mapping principle in $X$. Indeed, denoting by $\Phi u$ the right-hand side of (4.2),

$$\|\nabla \Phi u(t)\|_{L^2(H^\delta_0)} + \|\Phi u(t)\|_{L^2(H^\delta_0)} \leq \|\langle \nabla \rangle^3 \Phi u(t)\|_{L^2_x}$$

$$\leq \|\langle \nabla \rangle^3 (\nabla) \Phi u(t)\|_{L^2_x}$$

$$\leq \epsilon \|\langle \nabla \rangle^3 (\nabla) u_0\|_{L^2_x} + \epsilon \|\langle \nabla \rangle^3 u_1\|_{L^2_x} + \int_0^T \|\langle \nabla \rangle^3 F(u(t'), \partial u(t'))\|_{L^2_x} dt'$$

$$\leq \epsilon \|\nabla u_0\|_{L^2(H^\delta_0)} + \epsilon \|u_1\|_{L^2(H^\delta_0)}$$

$$+ \|u_1^2\|_{L^\infty_tL^2_x} \|\langle \nabla \rangle\partial u\|_{L^\infty_tL^2_x} + \|u\|_{L^2_tL^\infty_x} \|\langle \nabla \rangle^3 u\|_{L^2_tL^\infty_x} \|\partial u\|_{L^\infty_tL^2_x}$$

$$\leq \epsilon \|\nabla u_0\|_{H^1(H^\delta_0)} + \epsilon \|u_1\|_{L^2(H^\delta_0)} + \|u\|_X^3,$$

where $1/r = 1/2 - 1/q$. In the last inequality, we used the Sobolev embedding $H^{1/q}(S^1) \hookrightarrow L^r(S^1)$. Similarly, we obtain the same bound for $\|\partial \Phi u\|_{L^\infty_tL^2(H^\delta_0)}$. As an application of Theorem 2.1 we also obtain the same bound for $\|\langle \nabla \rangle^3 \Phi u\|_{L^2_tL^\infty_x}$. Thus, we obtain,

$$\|\Phi u\|_X \leq \epsilon \|u_0\|_{H^1(H^\delta_0)} + \epsilon \|u_1\|_{L^2(H^\delta_0)} + \|u\|_X^3.$$  

Analogously, we obtain,

$$\|\Phi u - \Phi v\|_X \leq \epsilon \|u_0\|_{H^1(H^\delta_0)} + \epsilon \|u_1\|_{L^2(H^\delta_0)} + \|u\|_X^3.$$  

Therefore, choosing $\epsilon$ and $R$ small, we are able to prove that $\Phi$ is a contraction map in $X$.  

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We would like to thank Kenji Nakanishi for pointing out the fact that the Strichartz type estimate for the Klein–Gordon equation implies the corresponding estimate for the Schrödinger equation as a result of the non-relativistic limit [15].
References