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Picard groups and strongly graded coalgebras

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Abstract

In this paper we study strongly graded coalgebras and its relation to the Picard group. A classification theorem for this kind of coalgebras is given via the second Doi's cohomology group. The strong Picard group of a coalgebra is introduced in order to characterize those graded coalgebras with strongly graded dual ring. Finally, for a Hopf algebra H we also characterize the H^*-Galois coextensions with dual H-Galois extension solving the question proposed in Dăscălescu et al., J. Algebra 178 (1995) 400–413. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the theory of coalgebras and Hopf algebras, the notion of graded coalgebra appears in a natural way and many important examples exist (symmetric algebras, path coalgebras, dual of group algebras, etc.). These coalgebras are normally graded by $\mathbb{Z}$ and the components of negative degree are zero. A study of coalgebras graded by an arbitrary group has been carried out in [4–6,12,14]. The notion of strongly graded coalgebra plays a special role in this theory since they present several interesting differences with the dual concept of strongly graded ring. It emphasizes that a strongly graded coalgebra is necessarily graded by a finite group, and that the graded dual ring is not, in general, strongly graded. Another reason to go deeply into the study of this kind of coalgebras is the connection to Hopf–Galois coextensions and the Brauer group of a coalgebra, see [7,20], respectively.

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Let G be a finite group with identity element e and $C=\bigoplus_{\sigma\in G} C_\sigma$ a strongly graded coalgebra. As a consequence of Dade's theorem for coalgebras (see [12, Theorem 5.4]) all C_σ are invertible (C_e, C_e)-bicomodules. Hence the theory of the Picard group for coalgebras, developed in [19], applies to the study of strongly graded coalgebras. In this paper we investigate strongly graded coalgebras using this link and Picard group's techniques.

After reviewing Morita–Takeuchi theory and the Picard group of a coalgebra, in Section 2 we classify the isomorphism types of strongly graded coalgebras with isomorphic degree e components (Theorem 3.6). These are represented by the second Doi's cohomology group $H^2(Z(C_e),(kG)^*)$ where $Z(C_e)$ denotes the cocenter of the degree e component. The class group and the generalized Rees coalgebra associated to a coflat monomorphism is also introduced in this section. In Section 3 we define a new subgroup of the Picard group, the strong Picard group, which is related to the strong equivalences studied by I-Peng Lin [10]. This subgroup allows us to solve the problem of characterizing the graded coalgebras with strongly graded dual ring (Theorem 4.6). A graded coalgebra $C=\bigoplus_{\sigma\in G} C_\sigma$ has strongly graded dual ring $R=\bigoplus_{\sigma\in G} R_\sigma$ if and only if it is strongly graded and all C_σ belong to $Pic^s(C_e)$, the strong Picard group of C_e. When $[C_\sigma]\in Out(C_e)\subseteq Pic^s(C_e)$, the group of outer automorphisms, then the strongly graded coalgebra is indeed a graded crossed coproduct (Theorem 4.8). As a consequence of these results, we give an example of coalgebra where the strong Picard group is a proper subgroup of the Picard group.

Noticing that strongly graded coalgebras (resp. rings) are Galois coextensions (resp. extensions) by the group giving the grading, the answer of the above problem leads us to a more general problem which was proposed in [5]. Let H be a finite dimensional Hopf algebra and C/D an H^*-Galois coextension. When is the dual extension an H-Galois extension? In Theorem 4.11 we solve this problem. A coextension of coalgebras C/D is H^*-Galois if and only if the dual extension C^*/D^* is H-Galois and C is finitely cogenerated as D-comodule.

2. Preliminaries

Throughout k is a fixed ground field. All coalgebras, algebras, vector spaces, and unadorned $\otimes$ are over k. For general facts on coalgebras and comodules we refer to [16]. C always denotes a coalgebra with comultiplication Δ_C and counit ε_C. $\mathcal{M}^C$ is the category of right C-comodules, and for $M\in\mathcal{M}^C$, $\rho_M: M\to M\otimes C$ is the C-comodule structure map. Given $M,N\in\mathcal{M}^C$, $Com_{-C}(M,N)$ is the space of all C-colinear maps from M to N. For a (C,D)-bicomodule M, $\rho^+: M\to M\otimes D$ denotes the right D-comodule structure map and $\rho^-: M\to C\otimes M$ denotes the left C-comodule structure map.

Morita theory for coalgebras (see [17]): Let M be a right C-comodule and N a left C-comodule with structure maps ρ_M and ρ_N, respectively. The *cotensor product* $M\,\square_C\,N$ is the kernel of the map

$$\rho_M\otimes 1-1\otimes\rho_N: M\otimes N\to M\otimes C\otimes N.$$

The functors $M\,\square_C -$ and $-\,\square_C N$ are left exact and preserve direct sums. If M is a (C,D)-bicomodule and N a (D,E)-bicomodule, then $M\,\square_D N$ is a (C,E)-bicomodule with comodule structures induced by those of M and N.

A comodule $M \in \mathscr{M}^C$ is called *quasi-finite* if $Com_{-C}(N,M)$ is finite dimensional for any finite dimensional comodule $N \in \mathscr{M}^C$. Let M be a (C,D)-bicomodule. Recall from [17, Proposition 1.10] that M_D is quasi-finite if and only if the cotensor functor $-\,\square_C M : \mathscr{M}^C \to \mathscr{M}^D$ has a left adjoint functor, denoted by $h_{-D}(M,-)$, and called the *co-hom functor*. That is, for comodules $N \in \mathscr{M}^D$ and $P \in \mathscr{M}^C$,

$$Com_{-C}(h_{-D}(M,N),P) \cong Com_{-D}(N, P\,\square_C M), \tag{1}$$

where

$$h_{-D}(M,N) = \varinjlim_{\lambda} Com_{-D}(N_\lambda, M)^*$$

and $\{N_\lambda\}$ is a directed system of finite dimensional subcomodules of N such that $N = \varinjlim_{\lambda} N_\lambda$.

When $M = N$ then $h_{-D}(M,M)$ is denoted by $e_{-D}(M)$ and it becomes a coalgebra, called the *co-endomorphism* coalgebra. Let θ denote the canonical D-colinear map $N \to h_{-D}(M,N) \otimes N$ which corresponds to the identity map $h_{-D}(M,N) \to h_{-D}(M,N)$ in (1) when $C = k$. The comultiplication of $e_{-D}(M)$ corresponds to $(1 \otimes \theta)\theta : M \to e_{-D}(M) \otimes e_{-D}(M) \otimes M$ in (1), and the counit corresponds to the identity map 1_M. Let M be a (C,D)-bicomodule such that M is quasi-finite as D-comodule. Then there is a unique coalgebra map $\lambda : e_{-D}(M) \to C$ such that the $\rho^- = (\lambda \otimes 1)\theta$.

A *Morita–Takeuchi context* consists of coalgebras C, D, bicomodules ${}_C M_D$, ${}_D N_C$ and bicolinear maps $f : C \to M\,\square_D N$ and $g : D \to N\,\square_C M$ such that the following diagrams are commutative:

$$\begin{array}{ccc} M & \xrightarrow{\rho_M^+} & M\square_D D \\ \big\downarrow{\scriptstyle \rho_M^-} & & \big\downarrow{\scriptstyle 1\square g} \\ C\square_C M & \xrightarrow{f\square 1} & M\square_D N\square_C M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\rho_N^+} & N\square_C C \\ \big\downarrow{\scriptstyle \rho_N^-} & & \big\downarrow{\scriptstyle 1\square f} \\ D\square_D N & \xrightarrow{g\square 1} & N\square_C M\square_D N \end{array}$$

The context is said to be *strict* if both f and g are injective (equivalently, isomorphisms). The following result, due to Takeuchi, characterizes the equivalences between two categories of comodules [17, Proposition 2.5, Theorem 3.5]:

Theorem 2.1. *Let C,D be coalgebras.*

(a) *If $F : \mathscr{M}^C \to \mathscr{M}^D$ is a left exact linear functor that preserves direct sums, then there exists a (C,D)-bicomodule M such that $F(-) \cong -\,\square_C M$.*

(b) *Let M be a (C,D)-bicomodule. The following assertions are equivalent*:

   (i) *The functor $-\,\square_C M : \mathscr{M}^C \to \mathscr{M}^D$ is an equivalence.*

(ii) *M is a quasi-finite injective cogenerator as a right D-comodule and* $e_{-D}(M) \cong C$ *as coalgebras.*

(iii) *There is a strict Morita–Takeuchi context* (C, D, M, N, f, g).

When the conditions hold, the inverse equivalence is given by $-\square_D N : \mathscr{M}^D \to \mathscr{M}^C$, *where N denotes the* (D, C)*-bicomodule* $h_{-D}(M, D)$. *In this case C and D are called Morita–Takeuchi equivalent coalgebras.*

The Picard group (see [19]): A (C, C)-bicomodule M is called *invertible* if there is a (C, C)-bicomodule N and two bicomodule isomorphisms $f : C \cong M \square_C N$ and $g : C \cong N \square_C M$ such that (C, C, M, N, f, g) is a Morita–Takeuchi context. Then M is a quasi-finite injective cogenerator and $N \cong h_{-C}(M, C)$ as (C, C)-bicomodules. Equivalently, M is invertible if and only if the functor $-\square_C M : \mathscr{M}^C \to \mathscr{M}^C$ defines a Morita–Takeuchi equivalence.

The *Picard group of C*, denoted by *Pic*(C), was introduced in [19] and it is defined as the set of all bicomodule isomorphism classes $[M]$ of invertible (C, C)-bicomodules. *Pic*(C) becomes a group with the multiplication $[M][N] = [M \square_C N]$, identity element $[C]$, and the inverse of $[M]$ is $[M]^{-1} = [h_{-C}(M, C)]$. In view of Theorem 2.1(a), the Picard group represents the set of self-equivalences of $\mathscr{M}^C$.

The cocenter (see [18]): Let C be a coalgebra. If we view C as a right C^e-comodule ($C^e = C^{op} \otimes C$), then C is quasi-finite. The co-endomorphism coalgebra $Z(C) = e_{-C^e}(C)$, called *the cocenter of C*, verifies the following properties:

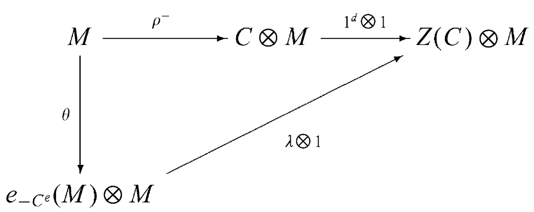
(i) $Z(C)$ is a cocommutative coalgebra with a surjective coalgebra map $1^d : C \to Z(C)$ which is cocentral, i.e.,

$$\sum_{(c)} 1^d(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} 1^d(c_{(2)}) \otimes c_{(1)} \quad \forall c \in C.$$

(ii) For any cocentral coalgebra map $f : C \to D$, there exists a unique coalgebra map $g : Z(C) \to D$ such that $f = g1^d$.

Proposition 2.2. *Let M be an invertible* (C, C)*-bicomodule. Then* $e_{-C^e}(M) \cong Z(C)$.

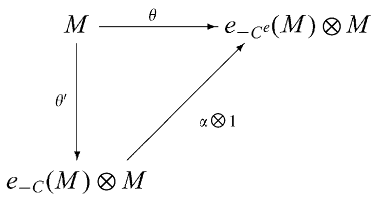
Proof. M is a $(Z(C), C^e)$-bicomodule with the structure maps $\omega^+ = (\tau \otimes 1)(\rho^- \otimes 1)\rho^+$ and $\omega^- = (1^d \otimes 1)\rho^-$, where τ is the twist map. By the universal property of the co-endomorphism coalgebra, there is a unique coalgebra map $\lambda : e_{-C^e}(M) \to Z(C)$ such that the following diagram commutes:



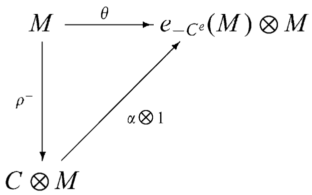
We are going to prove that λ is an isomorphism. M is a $(e_{-C^e}(M), C)$-bicomodule via θ and ρ^+, and a $(e_{-C^e}(M), C^{op})$-bicomodule via θ and $\tau\rho^-$. Then, $(\theta\otimes 1)\rho^+=(1\otimes\rho^+)\theta$, and $(\theta\otimes 1)\tau\rho^- = (1\otimes\tau)(1\otimes\rho^-)\theta$. Applying $(1\otimes\tau)$ to the last equality, we obtain

$$(1\otimes\rho^-)\theta = (1\otimes\tau)(\theta\otimes 1)\tau\rho^- = (\tau\otimes 1)(1\otimes\theta)\rho^-. \qquad (*)$$

By the universal property of the co-endomorphism coalgebra $e_{-C}(M)$, there is a unique coalgebra map $\alpha : e_{-C}(M) \to e_{-C^e}(M)$ such that the following diagram is commutative (θ' is the canonical map associated to $e_{-C}(M)$):



Since M is invertible, we may identify $e_{-C}(M)$ with C and θ' with ρ^-. Then



We check that α is cocentral.

$$\begin{aligned}
[(\alpha\otimes 1)\tau\Delta\otimes 1]\rho^- &= (\alpha\otimes 1\otimes 1)(\tau\otimes 1)(\Delta\otimes 1)\rho^- \\
&= (\alpha\otimes 1\otimes 1)(\tau\otimes 1)(1\otimes\rho^-)\rho^- \\
&= (\tau\otimes 1)(1\otimes\alpha\otimes 1)(1\otimes\rho^-)\rho^- \\
&= (\tau\otimes 1)(1\otimes\theta)\rho^- \\
&= (1\otimes\rho^-)\theta \quad by\ (*) \\
&= (1\otimes\rho^-)(\alpha\otimes 1)\rho^- \\
&= (\alpha\otimes 1\otimes 1)(1\otimes\rho^-)\rho^- \\
&= (\alpha\otimes 1\otimes 1)(\Delta\otimes 1)\rho^- \\
&= [(\alpha\otimes 1)\Delta\otimes 1]\rho^-.
\end{aligned}$$

It follows that $(\alpha\otimes 1)\tau\Delta=(\alpha\otimes 1)\Delta$, i.e., α is cocentral. Now, by the universal property of the cocenter, there is a unique coalgebra map $\beta : Z(C) \to e_{-C^e}(M)$ such that $\alpha=1^d\beta$.

We see that β is the inverse of λ. We know that $\theta=(\alpha\otimes 1)\rho^-$ and $(\lambda\otimes 1)\theta=(1^d\otimes 1)\rho^-$. Then

$$(\beta\lambda\otimes 1)\theta=(\beta\otimes 1)(1^d\otimes 1)\rho^-=(\beta 1^d\otimes 1)\rho^-=(\alpha\otimes 1)\rho^-=\theta,$$

$$\begin{aligned}(\lambda\beta 1^d\otimes 1)\rho^- &= (\lambda\otimes 1)(\beta 1^d\otimes 1)\rho^-=(\lambda\otimes 1)(\alpha\otimes 1)\rho^-=(\lambda\otimes 1)\theta\\ &=(1^d\otimes 1)\rho^-.\end{aligned}$$

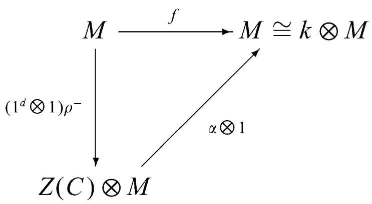
It follows that $\beta\lambda=1_{e_{-C^e}(M)}$ and $\lambda\beta 1^d=1^d$ which implies $\lambda\beta=1_{Z(C)}$. Hence $e_{-C^e}(M)\cong Z(C)$. □

Proposition 2.3. *Let M be an invertible (C,C)-bicomodule. If $f:M\to M$ is a (C,C)-bicomodule map, then there is a unique linear map $\alpha:Z(C)\to k$ such that*

$$f(m)=\sum_{(m)}\langle\alpha,1^d(m_{(-1)})\rangle m_{(0)}\quad\forall m\in M.$$

Moreover, if f is an isomorphism, then α is a unit in $Z(C)^*$.

Proof. By the above lemma, we may identify $e_{-C^e}(M)$ with $Z(C)$ and θ with $(1^d\otimes 1)\rho^-$. M is a right C^e-comodule and f a C^e-comodule map. There is a unique linear map $\alpha:Z(C)\to k$ such that the following diagram is commutative:



This gives the first part. As $f=1_M$ then the map α is the counit $\varepsilon:Z(C)\to k$. Suppose that f is an isomorphism and $\alpha':Z(C)\to k$ is the associated linear map to f^{-1},

$$\begin{aligned}m=f^{-1}f(m)&=\sum_{(m)}\langle\alpha,1^d(m_{(-1)})\rangle f^{-1}(m_{(0)})\\ &=\sum_{(m)}\langle\alpha,1^d(m_{(-2)})\rangle\langle\alpha',1^d(m_{(-1)})\rangle m_{(0)}\\ &=\sum_{(m)}\langle\alpha\cdot\alpha',1^d(m_{(-1)})\rangle m_{(0)},\end{aligned}$$

where $\cdot$ denotes the convolution product in $Z(C)^*$. On the other hand,

$$m=1_M(m)=\sum_{(m)}\langle\varepsilon,1^d(m_{(-1)})\rangle m_{(0)}.$$

From the uniqueness, it follows that $\alpha\cdot\alpha'=\varepsilon$. Similarly, $\alpha'\cdot\alpha=\varepsilon$. □

Corollary 2.4. *Let C be a coalgebra. The dual $Z(C)^*$ is equal to the center of C^*, $Z(C^*)$.*

Proof. Let $c^* \in C^*$ and $d^* \in Z(C)^*$. Then, for any $c \in C$,

$$\langle c^* \cdot d^*, c\rangle = \sum_{(c)} \langle c^*, c_{(1)}\rangle \langle d^*, 1^d(c_{(2)})\rangle$$

$$= \sum_{(c)} \langle c^*, c_{(2)}\rangle \langle d^*, 1^d(c_{(1)})\rangle = \langle d^* \cdot c^*, c\rangle,$$

where we have used that 1^d is cocentral. Hence $Z(C)^* \subseteq Z(C^*)$.

It is well known that $Com_{-C}(C,C) \cong C^*$ via the map $c^* \mapsto \Theta_{c^*}$ defined by $\Theta_{c^*}(c) = \sum_{(c)} \langle c^*, c_{(1)}\rangle c_{(2)}$. By restricting this map to $Z(C^*)$, there is an isomorphism between $Z(C^*)$ and $Com_{C-C}(C,C)$, the space of (C,C)-bicomodule maps from C into itself. Given $d^* \in Z(C^*)$, consider the (C,C)-bicomodule map Θ_{d^*}. By Lemma 2.3, there is a unique linear map $\beta \in Z(C)^*$ such that $\Theta_{d^*}(c) = \sum_{(c)} \langle \beta, 1^d(c_{(1)})\rangle c_{(2)}$. Then, $\sum_{(c)} \langle d^*, c_{(1)}\rangle c_{(2)} = \sum_{(c)} \langle (1^d)^*(\beta), c_{(1)}\rangle c_{(2)}$. It follows that $d^* = (1^d)^*(\beta)$, then $d^* \in Z(C)^*$. □

In the next section we will use the Picard group in the study of graded coalgebras. For this reason, we include a brief paragraph with several facts on graded coalgebras. For other results on graded coalgebras we refer to the reader to [12], [5], or [4].

Graded coalgebras: In the sequel G will denote a group with identity element e. A coalgebra C is called a G-*graded coalgebra* if C admits a decomposition as a direct sum of spaces $C = \bigoplus_{\sigma \in G} C_\sigma$ such that

(i) $\Delta(C_\sigma) \subseteq \sum_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu$ for any $\sigma \in G$;
(ii) $\varepsilon(C_\sigma) = 0$ for any $\sigma \neq e$.

For any $\sigma \in G$ we write $\pi_\sigma : C \to C_\sigma$ for the canonical projection and $i_\sigma : C_\sigma \to C$ for the inclusion map.

(1) If $\sigma, \tau \in G$ there exists a unique linear map $u_{\sigma,\tau} : C_{\sigma\tau} \to C_\sigma \otimes C_\tau$ such that $u_{\sigma,\tau}\pi_{\sigma\tau} = (\pi_\sigma \otimes \pi_\tau)\Delta$. Indeed, $u_{\sigma,\tau} = (\pi_\sigma \otimes \pi_\tau)\Delta i_{\sigma\tau}$.

(2) For any σ, τ, $\lambda \in G$: $(u_{\sigma,\tau} \otimes 1)u_{\sigma\tau,\lambda} = (1 \otimes u_{\tau,\lambda})u_{\sigma,\tau\lambda}$.

(3) If $\sigma \in G$, $(1 \otimes \varepsilon)u_{\sigma,e} = 1$.

(4) If we write $\Delta_e = u_{e,e} : C_e \to C_e \otimes C_e$, then $(C_e, \Delta_e, \varepsilon)$ is a coalgebra and $\pi_e : C \to C_e$ is a coalgebra map. Moreover, C_σ is a (C_e, C_e)-bicomodule with structure maps $\rho^+ = u_{\sigma,e}$ and $\rho^- = u_{e,\sigma}$.

As $u_{\sigma,\tau} : C_{\sigma\tau} \to C_\sigma \otimes C_\tau$ are injective for all $\sigma, \tau \in G$, $C = \bigoplus_{\sigma \in G} C_\sigma$ is called a *strongly graded coalgebra*. In this case all C_σ are invertible (C_e, C_e)-bicomodules with inverse $C_{\sigma^{-1}}$, [12, Corollary 5.5]. Graded crossed coproducts, studied in [5], are a particular case of strongly graded coalgebra. A graded coalgebra $C = \bigoplus_{\sigma \in G} C_\sigma$ is called a *graded crossed coproduct* if for any $\sigma \in G$, there are linear maps $u_\sigma : C_\sigma \to k, v_\sigma : C_{\sigma^{-1}} \to k$ such that

$$\sum_{(c)} u_\sigma(\pi_\sigma(c_{(1)}))v_\sigma(\pi_{\sigma^{-1}}(c_{(2)})) = \sum_{(c)} v_\sigma(\pi_{\sigma^{-1}}(c_{(1)}))v_\sigma(\pi_\sigma(c_{(2)})) = \varepsilon(c), \quad \forall c \in C_e.$$

3. Cofactor sets and cohomology

Let C be a coalgebra and $Pic(C)$ its Picard group. Recall from [19, Theorem 2.10] that there is a group homomorphism $\Phi : Pic(C) \to Aut(Z(C))$ defined as follows: for $[M] \in Pic(C)$, there exists a unique coalgebra automorphism $\Phi_M : Z(C) \to Z(C)$ such that

$$\sum_{(m)} m_{(0)} \otimes \Phi_M 1^d(m_{(1)}) = \sum_{(m)} m_{(0)} \otimes 1^d(m_{(-1)}) \quad \forall m \in M.$$

Suppose now that G is a finite group and $\Pi : G \to Pic(C)$ is a group homomorphism. For any $\sigma \in G$, we will write $[M_\sigma]$ instead $\Pi(\sigma)$. $Z(C)$ has structure of left $(kG)^*$-comodule coalgebra with the coaction

$$\rho : Z(C) \to (kG)^* \otimes Z(C),\ c \mapsto \sum_{\sigma \in G} p_\sigma \otimes \Phi_{M_{\sigma^{-1}}}(c),$$

where $\{p_\sigma\colon \sigma \in G\}$ is the dual basis of $\{\sigma\colon \sigma \in G\}$. We consider the second Doi's cohomology group $H^2(Z(C),(kG)^*)$ (see [8] or [11]) defined as follows: any linear map $\alpha : Z(C) \to (kG)^* \otimes (kG)^*$ can be expressed as $\alpha(c) = \sum_{x,y \in G} \alpha_{x,y}(c) p_x \otimes p_y$ where $\alpha_{x,y} \in Z(C)^*$ for all $x, y \in G$. α is convolution invertible if and only if $\alpha_{x,y} \in U(Z(C)^*)$ for all $x, y \in G$. Let $Z^2(Z(C),(kG)^*)$ be the set of convolution invertible linear maps $\alpha : Z(C) \to (kG)^* \otimes (kG)^*$ which verifies the *cocycle condition*:

$$\text{(C)} \quad \sum_{(c)} \alpha_{x,yz}(c_{(2)})\alpha_{y,z}(\Phi_{M_{x^{-1}}}(c_{(1)})) = \sum_{(c)} \alpha_{x,y}(c_{(1)})\alpha_{xy,z}(c_{(2)}).$$

A cocycle $\beta \in Z^2(Z(C),(kG)^*)$ is said to be a *coboundary* if there exists a convolution invertible linear map $\alpha : Z(C) \to (kG)^*$, $(\alpha(c) = \sum_{g \in G} \alpha_g(c) p_g\ \forall c \in Z(C))$ such that

$$\beta_{x,y}(c) = \sum_{(c)} \langle \alpha_{xy}, c_{(1)} \rangle \langle \alpha_x^{-1}, c_{(2)} \rangle \langle \alpha_y^{-1}, \Phi_{M_{x^{-1}}}(c_{(3)}) \rangle$$

for all $x, y \in G$, $c \in Z(C)$. The set of coboundaries, denoted by $B^2(Z(C),(kG)^*)$ is a subgroup of $Z^2(Z(C),(kG)^*)$ and the second Doi's cohomology group is defined as

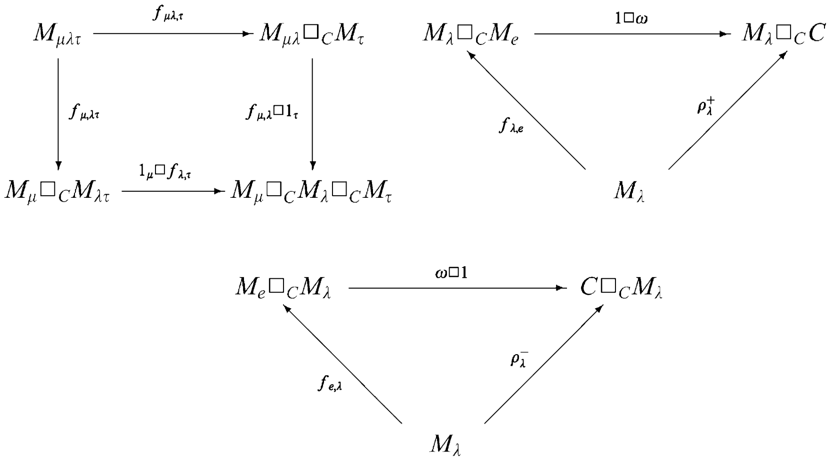
$$H^2(Z(C),(kG)^*) = Z^2(Z(C),(kG)^*)/B^2(Z(C),(kG)^*).$$

For any class $[\alpha] \in H^2(Z(C),(kG)^*)$ we can choose a representative element which verifies the normalized cocycle condition.

$$\text{(CU)} \quad \alpha_{x,e}(c) = \varepsilon(c) \quad \alpha_{e,x}(c) = \varepsilon(c) \quad \forall x \in G,\ c \in Z(C).$$

Definition 3.1. Let $\Pi : G \to Pic(C)$ be a group homomorphism. We set $\Pi(\sigma) = [M_\sigma]$ for any $g \in G$. A cofactor set $\mathscr{F}$ associated to Π is a family $\mathscr{F} = \{f_{\lambda,\mu}\colon \lambda, \mu \in G\}$

consisting of C-bicomodule isomorphisms $f_{\lambda,\mu}: M_{\lambda\mu} \to M_\lambda \,\Box_C\, M_\mu$, $\omega: M_e \to C$ such that the following diagrams commute for all $\lambda, \mu, \tau \in G$.



The set of cofactor sets associated to Π will be denoted by $\mathbf{F}_s(\Pi)$. For $\mathscr{F} \in \mathbf{F}_s(\Pi)$, a strongly graded coalgebra $C\langle \mathscr{F}, \Pi, G\rangle$ may be defined as follows (see [9, p. 39]): as a vector space $C\langle \mathscr{F}, \Pi, G\rangle = \bigoplus_{\sigma\in G} M_\sigma$; for $m \in M_\sigma$, the comultiplication $\Delta(m) = \sum_{\lambda\mu=\sigma} f_{\lambda,\mu}(m)$; the counit $\varepsilon(M_\sigma) = \{0\}$ if $\sigma \neq e$, and $\varepsilon(m) = \varepsilon_C\omega(m)$ if $m \in M_e$.

Following [9], $C\langle \mathscr{F}, \Pi, G\rangle$ is called a *divisorially graded coalgebra* but the definition there is much more general. Divisorially graded coalgebras are defined respect to a torsion theory $\mathscr{T}$ in $\mathscr{M}^C$. Our definition coincides with that when the torsion theory is the trivial one $\mathscr{T} = \{0\}$. In this case the class of divisorially graded coalgebras is just the class of strongly graded coalgebras with component of degree e isomorphic to C.

Example 3.2. (1) (*Garcia Rozas and Torrecillas* [9, Remark, p. 39]). Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a strongly graded coalgebra. The map $\Pi: G \to Pic(C_e)$, $\sigma \mapsto [C_\sigma]$ is a group homomorphism and the family $\mathscr{U} = \{u_{\sigma,\tau}\}_{\sigma,\tau\in G}$ of canonical C_e-bicomodule maps $u_{\sigma\tau}: C_{\sigma\tau} \to C_\sigma \Box_{C_e} C_\tau$ is a cofactor set associated to Π. Moreover, $C = C_e\langle \mathscr{U}, \Pi, G\rangle$ as graded coalgebras.

(2) Let $p: C \to D$ be a surjective coalgebra map. A two-sided D-subbicomodule of C is a subspace X of C such that X is a (D,D)-bicomodule. Let $\mathscr{B}(C,D)$ denote the set of all two-sided D-subbicomodules of C. Any subcoalgebra of C belongs to $\mathscr{B}(C,D)$ by considering it as a (D,D)-bicomodule via p. $X \in \mathscr{B}(C,D)$ is said to be *invertible* if there is $Y \in \mathscr{B}(C,D)$ such that $X \wedge Y = Y \wedge X = Ker(p)$ ($\wedge$ denotes the wedge). We write $Y = X^{-1}$. The set of all invertible $X \in \mathscr{B}(C,D)$, denoted by $\mathscr{I}(C,D)$, is a group under the multiplication $X * Y = X \wedge Y$ with identity element $Ker(p)$. This group is called the *class group* of C respect to D.

There is a canonical map from $\mathscr{I}(C,D)$ to $Pic(D)$ mapping $X \in \mathscr{I}(C,D)$ to $[C/X]$. We check that C/X is an invertible (D,D)-bicomodule. It is a (D,D)-bicomodule with structure maps

$$\rho^+(c+X) = \sum_{(c)} (c_{(1)}+X) \otimes p(c_{(2)}),$$
$$\rho^-(c+X) = \sum_{(c)} p(c_{(1)}) \otimes (c_{(2)}+X) \quad \forall c \in C.$$

We define a map $\eta : C \to C/X \,\square_D\, C/X^{-1}$, $c \mapsto \sum_{(c)} (c_{(1)}+X) \otimes (c_{(2)}+X^{-1})$ which is a (D,D)-bicomodule map with kernel $X \wedge X^{-1} = Ker(p)$. The canonical coalgebra map $\bar{p} : C/Ker(p) \cong D$ is a D-bicomodule map. In this way, we have two D-bicomodule maps

$$f = \eta\bar{p}^{-1} : D \to C/X \,\square_D\, C/X^{-1}, \quad g = \eta'\bar{p}^{-1} : D \to C/X^{-1} \,\square_D\, C/X$$

with $\eta' : C \to C/X^{-1} \,\square_D\, C/X$ similarly defined as η. It is easy to check that $(D, D, C/X, C/X^{-1}, f, g)$ is a strict Morita–Takeuchi context. By Takeuchi [17, Theorem 2.5], f and g are isomorphisms, and so $[C/X] \in Pic(D)$.

We cannot claim, in general, that the map $can : \mathscr{I}(C,D) \to Pic(D)$, $X \mapsto [C/X]$ is a group homomorphism. However, it is true when C is injective as a right D-comodule via p, and the map $\bar{\Delta} : C \to C \,\square_D\, C$, $c \mapsto \sum_{(c)} c_{(1)} \otimes c_{(2)}$ is an isomorphism (this happens if p is a coflat monomorphism, see [13]). Let $X, Y \in \mathscr{I}(C,D)$ and define $\eta : C \to C/X \,\square_D\, C/Y$ as before. The kernel of η is $X \wedge Y$ and denoting by $q_X : C \to C/X$, $q_Y : C \to C/Y$ the canonical projections, the following square is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\eta} & C/X \,\square_D\, C/Y \\ \downarrow{\scriptstyle \bar{\Delta}} & & \downarrow{\scriptstyle 1\square q_Y} \\ C \,\square_D\, C & \xrightarrow{q_X \square 1} & C/X \,\square_D\, C \end{array}$$

Since C and C/X are injective as a left and right D-comodules, respectively, the functors $-\square_D\, C$, $C/X \,\square_D -$ are exact. Thus $q_X \,\square\, 1$ and $1 \,\square\, q_Y$ are surjective. It follows that η is surjective. Hence, $[C/X \wedge Y] = [C/X \,\square_D\, C/Y]$.

Let G be a finite group and $\pi : G \to \mathscr{I}(C,D)$ be a group homomorphism. We set $\pi(\sigma) = X_\sigma$ for all $\sigma \in G$. Let $\Pi = can\pi : G \to Pic(D)$. The family $\mathscr{F} = \{f_{\sigma,\tau}\}_{\sigma,\tau \in G}$ where

$$f_{\sigma,\tau} : C/X_{\sigma\tau} \to C/X_\sigma \,\square_D\, C/X_\tau, \quad (c + X_{\sigma\tau}) \mapsto \sum_{(c)} (c_{(1)} + X_\sigma) \otimes (c_{(2)} + X_\tau)$$

is a cofactor set associated to Π. The strongly graded coalgebra $\mathscr{R}(\Pi) = D\langle \mathscr{F}, \Pi, G\rangle$ is called *the generalized Rees coalgebra associated to* Π.

(3) Let $Aut(C)$ be the group of automorphisms of the coalgebra C. An automorphism $f \in Aut(C)$ is said to be *inner* if there is a unit $u \in C^*$ such that $f(c) = \sum_{(c)} u(c_{(1)})c_{(2)}u^{-1}(c_{(3)})$ for all $c \in C$. The group of inner automorphisms of C, $Inn(C)$, is a normal subgroup of $Aut(C)$ and the factor group $Out(C) = Aut(C)/Inn(C)$ is called

the group of *outer automorphisms of* C. If $f \in Aut(C)$, ${}_fC_1$ is the (C,C)-bicomodule defined as follows: ${}_fC_1 = C$ as right C-comodule and the left comodule structure map is $\rho^- = (f \otimes 1)\Delta$. From [19, Theorem 2.7] we recall that there is an exact sequence:

$$1 \to Inn(C) \to Aut(C) \xrightarrow{\omega} Pic(C), \tag{2}$$

where $\omega(f) = [{}_fC_1]$ for all $f \in Aut(C)$. Hence, ω induces a group monomorphism from $Out(C)$ to $Pic(C)$. We say that C has the *Aut-Pic property* if ω is surjective. In this case $Pic(C) \cong Out(C)$. A study of coalgebras having Aut-Pic property was carried out in [3]. It was proved that basic coalgebras have the Aut-Pic property, see [3, Theorem 2.6]. We remember from [1] that a coalgebra is said to be *basic* if the dual of any simple subcoalgebra is a division algebra. In particular, pointed or cocommutative coalgebras are basic coalgebras, and then these have the Aut-Pic property.

Let G be a finite group, and suppose that there is a group homomorphism $\pi : G \to Aut(C)$. For $\sigma \in G$, we write π_σ instead $\pi(\sigma)$. Let $\Pi : G \to Pic(C)$ be the composite of π with $\omega : Aut(C) \to Pic(C)$. For $\sigma, \tau \in G$, consider the C-bicomodule isomorphism

$$f_{\sigma,\tau} : {}_1C_{\pi_{\sigma\tau}} \to {}_1C_{\pi_\sigma} \square_C {}_1C_{\pi_\tau},\ c \mapsto \sum_{(c)} c_{(1)} \otimes \pi_\sigma(c_{(2)}).$$

The family $\mathscr{F} = \{f_{\sigma,\tau}\}_{\sigma,\tau \in G}$ is a cofactor set associated to Π.

Proposition 3.3. *Let* $\Pi : G \to Pic(C)$ *be a group homomorphism, and* $\mathscr{F} = \{f_{x,y}\}_{x,y \in G}$ *a cofactor set associated to* Π. *Let* $\alpha \in Z^2(Z(C),(kG)^*)$ *a normalized cocycle. Given* $\lambda, \mu \in G$ *and* $m \in M_{\lambda\mu}$, *we define*

$$g_{\lambda,\mu}(m) = \sum_{(m)} \langle \alpha_{\lambda,\mu}, 1^d(m_{(-1)})\rangle f_{\lambda,\mu}(m_{(0)}).$$

Then $\alpha \star \mathscr{F} = \{g_{x,y}\}_{x,y \in G}$ *is a cofactor set associated to* Π.

Proof. In order to prove that $\alpha \star \mathscr{F}$ is a cofactor set associated to Π, we need several previous facts.

Let $\mu, \lambda, \tau \in G$, and $f_{\mu\lambda,\tau} : M_{\mu\lambda\tau} \to M_{\mu\lambda} \square_C M_\tau$ the C-bicomodule isomorphism. For any $m \in M_{\mu\lambda\tau}$, we can set $f_{\mu\lambda,\tau}(m) = \sum_i m_i \otimes n_i$. Using that $f_{\mu\lambda,\tau}$ is a bicomodule map, from

$$\sum_i m_{i(-1)} \otimes m_{i(0)} \otimes n_i = \sum f_{\mu\lambda,\tau}(m)_{(-1)} \otimes f_{\mu\lambda,\tau}(m)_{(0)},$$

$$\sum f_{\mu\lambda,\tau}(m)_{(-1)} \otimes f_{\mu\lambda,\tau}(m)_{(0)} = \sum_{(m)} m_{(-1)} \otimes f_{\mu\lambda,\tau}(m_{(0)}),$$

we deduce the formula

$$\sum_i m_{i(-1)} \otimes m_{i(0)} \otimes n_i = \sum_{(m)} m_{(-1)} \otimes f_{\mu\lambda,\tau}(m_{(0)}). \tag{3}$$

Similarly, if we set $f_{\mu,\lambda\tau}(m) = \sum_j m'_j \otimes n'_j$ for any $m \in M_{\mu\lambda\tau}$, then

$$\sum_j m'_{j(-1)} \otimes m'_{j(0)} \otimes n'_j = \sum_{(m)} m_{(-1)} \otimes f_{\mu,\lambda\tau}(m_{(0)}). \tag{4}$$

Since $(f_{\mu,\tau}\Box 1)f_{\mu\lambda,\tau}=(1\Box f_{\lambda,\tau})f_{\mu,\lambda\tau}$, then

$$\sum_i f_{\mu,\lambda}(m_i)\otimes n_i=\sum_j m'_j\otimes f_{\lambda,\tau}(n'_j).$$

Applying $\rho^-\otimes 1$ and the fact that $f_{\mu,\lambda}$ is a bicomodule map,

$$\sum_i m_{i(-1)}\otimes f_{\mu,\lambda}(m_{i(0)})\otimes n_i=\sum_j m'_{j(-1)}\otimes m'_{j(0)}\otimes f_{\lambda,\tau}(n'_j).$$

Finally, applying $\Delta\otimes 1\otimes 1$, we obtain

$$\sum_i m_{i(-1)(1)}\otimes m_{i(-1)(2)}\otimes f_{\mu,\lambda}(m_{i(0)})\otimes n_i$$
$$=\sum_j m'_{j(-1)(1)}\otimes m'_{j(-1)(2)}\otimes m'_{j(0)}\otimes f_{\lambda,\tau}(n'_j). \qquad (5)$$

On the other hand, since $f_{\mu,\lambda\tau}(m)=\sum_j m'_j\otimes n'_j\in M_\mu\Box_C M_{\lambda\tau}$ we have that $\sum_j m'_{j(-1)}\otimes m'_{j(0)}\otimes n'_j\in C\Box_C M_\mu\Box_C M_{\lambda\tau}$. Then,

$$\sum_j m'_{j(-1)}\otimes m'_{j(0)(0)}\otimes m'_{j(0)(1)}\otimes n'_j=\sum_j m'_{j(-1)}\otimes m'_{j(0)}\otimes n'_{j(-1)}\otimes n'_{j(0)}. \qquad (6)$$

Now, we are ready to check that $\alpha\star\mathscr{F}$ is a cofactor set:

$$(g_{\lambda,\mu}\Box 1)g_{\lambda\mu,\tau}(m)=(g_{\lambda,\mu}\Box 1)\left(\sum_{(m)}\langle\alpha_{\mu\lambda,\tau},1^d(m_{(-1)})\rangle f_{\mu\lambda,\tau}(m_{(0)})\right)$$

$$=\sum\langle\alpha_{\mu\lambda,\tau},1^d(m_{i(-1)})\rangle g_{\mu,\lambda}(m_{i(0)})\otimes n_i \quad \text{by (3)}$$

$$=\sum\langle\alpha_{\mu\lambda,\tau},1^d(m_{i(-1)})\rangle\langle\alpha_{\mu,\lambda},1^d(m_{i(0)(-1)})\rangle f_{\mu,\lambda}(m_{i(0)(0)})\otimes n_i$$

$$=\sum\langle\alpha_{\mu\lambda,\tau},1^d(m_{i(-1)(1)})\rangle\langle\alpha_{\mu,\lambda},1^d(m_{i(-1)(2)})\rangle f_{\mu,\lambda}(m_{i(0)})\otimes n_i$$

by comodule property

$$=\sum\langle\alpha_{\mu\lambda,\tau},1^d(m_{i(-1)(2)})\rangle\langle\alpha_{\mu,\lambda},1^d(m_{i(-1)(1)})\rangle f_{\mu,\lambda}(m_{i(0)})\otimes n_i$$

since 1^d is cocentral

$$=\sum\langle\alpha_{\mu,\lambda\tau},1^d(m_{i(-1)(2)})\rangle\langle\alpha_{\lambda,\tau},\Phi_{M_{\mu^{-1}}}1^d(m_{i(-1)(1)})\rangle f_{\mu,\lambda}(m_{i(0)})\otimes n_i$$

by cocycle condition

$$=\sum\langle\alpha_{\mu,\lambda\tau},1^d(m'_{j(-1)(2)})\rangle\langle\alpha_{\lambda,\tau},\Phi_{M_{\mu^{-1}}}1^d(m'_{j(-1)(1)})\rangle m'_{j(0)}\otimes f_{\lambda,\tau}(n'_j) \quad \text{by (5)}$$

$$=\sum\langle\alpha_{\mu,\lambda\tau},1^d(m'_{j(-1)})\rangle\langle\alpha_{\lambda,\tau},\Phi_{M_{\mu^{-1}}}1^d(m'_{j(0)(-1)})\rangle m'_{j(0)(0)}\otimes f_{\lambda,\tau}(n'_j)$$

by comodule property

$$=\sum\langle\alpha_{\mu,\lambda\tau},1^d(m'_{j(-1)})\rangle\langle\alpha_{\lambda,\tau},1^d(m'_{j(0)(1)})\rangle\otimes m'_{j(0)(0)}\otimes f_{\lambda,\tau}(n'_j)$$

by definition of $\Phi_{M_{\mu^{-1}}}$

$$= \sum \langle \alpha_{\mu,\lambda\tau}, 1^d(m'_{j(-1)})\rangle\langle \alpha_{\lambda,\tau}, 1^d(n'_{j(-1)})\rangle m'_{j(0)} \otimes f_{\lambda,\tau}(n'_{j(0)}) \quad \text{by (6)}$$

$$= \sum \langle \alpha_{\mu,\lambda\tau}, 1^d(m'_{j(-1)})\rangle m'_{j(0)} \otimes g_{\lambda,\tau}(n'_j)$$

$$= (1 \square g_{\lambda,\tau}) \left(\sum \langle \alpha_{\mu,\lambda\tau}, 1^d(m'_{j(-1)})\rangle f_{\mu,\lambda\tau}(m'_{j(0)}) \right)$$

$$= (1 \square g_{\lambda,\tau}) g_{\mu,\lambda\tau}(m).$$

Let $\omega : M_e \to C$ be the isomorphism given together the cofactor set $\mathscr{F} = \{f_{\lambda,\mu}\}_{\lambda,\mu \in G}$. Then, $(1 \square \omega) f_{\lambda,e}(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$. Now, using this and the normalized cocycle condition of α, we have

$$\begin{aligned}(1 \square \omega) g_{\lambda,e}(m) &= (1 \square \omega) \left(\sum_{(m)} \langle \alpha_{\lambda,e}, 1^d(m_{(-1)})\rangle f_{\lambda,e}(m_{(0)}) \right) \\ &= \sum_{(m)} \langle \varepsilon, m_{(-1)}\rangle (1 \square \omega) f_{\lambda,e}(m_{(0)}) \\ &= (1 \square \omega) f_{\lambda,e}(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}.\end{aligned}$$

Similarly, $(\omega \square 1) f_{e,\lambda} = \rho^-$. □

Proposition 3.4. *Let $\Pi : G \to Pic(C)$ be a group homomorphism and $\mathscr{F} = \{f_{\lambda,\mu}\}_{\lambda,\mu \in G}$, $\mathscr{G} = \{g_{\lambda,\mu}\}_{\lambda,\mu \in G}$ two cofactor sets. Then there is a unique $\alpha \in Z^2(Z(C), (kG)^*)$ such that*

$$f_{\lambda,\mu}(m) = \sum_{(m)} \langle \alpha_{\lambda,\mu}, 1^d(m_{(-1)})\rangle g_{\lambda,\mu}(m_{(0)})$$

for all $m \in M_{\lambda\mu}$.

Proof. Given $\lambda, \mu \in G$, consider the C-bicomodule map $g^{-1}_{\lambda,\mu} f_{\lambda,\mu} : M_{\lambda\mu} \to M_{\lambda\mu}$. By Proposition 2.3, there exists a unique linear map $\alpha_{\lambda,\mu} \in U(Z(C)^*)$ such that

$$g^{-1}_{\lambda,\mu} f_{\lambda,\mu}(m) = \sum_{(m)} \langle \alpha_{\lambda,\mu}, 1^d(m_{(-1)})\rangle m_{(0)} \quad \forall m \in M_{\lambda\mu}.$$

Then

$$f_{\lambda,\mu}(m) = \sum_{(m)} \langle \alpha_{\lambda,\mu}, 1^d(m_{(-1)})\rangle g_{\lambda,\mu}(m_{(0)}) \quad \forall m \in M_{\lambda\mu}.$$

We set $g_{\lambda,\mu\tau}(m_{(0)}) = \sum_i m_i \otimes n_i$. Since $g_{\lambda,\mu\tau}$ is a C-bicomodule map,

$$\begin{aligned}\sum m_{(0)(-1)} \otimes g_{\lambda,\mu\tau}(m_{(0)(0)}) &= \sum g_{\lambda,\mu\tau}(m_{(0)})_{(-1)} \otimes g_{\lambda,\mu\tau}(m_{(0)})_{(0)} \\ &= \sum m_{i(-1)} \otimes m_{i(0)} \otimes n_i.\end{aligned}$$

Applying $(1 \square 1 \square g_{\mu,\tau})$ yields

$$\sum m_{(0)(-1)} \otimes (1 \square g_{\mu,\tau}) g_{\lambda,\mu\tau}(m_{(0)(0)}) = \sum m_{i(-1)} \otimes m_{i(0)} \otimes g_{\mu,\tau}(n_i). \tag{7}$$

A similar equality follows for $g_{\lambda\mu,\tau}(m_{(0)}) = \sum_j m'_j \otimes n'_j$.

Again from Proposition 2.3, we may find a unique $\beta \in U(Z(C)^*)$ such that

$$(1 \square f_{\mu,\tau}) f_{\lambda,\mu\tau}(m) = \sum_{(m)} \langle \beta, 1^d(m_{(-1)}) \rangle (1 \square g_{\mu,\tau}) g_{\lambda,\mu\tau}(m_{(0)}).$$

Now,

$$\begin{aligned}
(1 \square f_{\mu,\tau}) f_{\lambda,\mu\tau}(m) &= (1 \square f_{\mu,\tau}) \left(\sum_{(m)} \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)}) \rangle g_{\lambda,\mu\tau}(m_{(0)}) \right) \\
&= \sum \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\mu,\tau}, 1^d(n_{i(-1)}) \rangle m_i \otimes g_{\mu,\tau}(n_{i(0)}) \\
&= \sum \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\mu,\tau}, 1^d(m_{i(1)}) \rangle m_{i(0)} \otimes g_{\mu,\tau}(n_i) \\
&\quad \text{since } \sum_i m_i \otimes n_i \in M_\lambda \square_C M_{\mu\tau} \\
&= \sum \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\mu,\tau}, \Phi_{M_{\lambda^{-1}}} 1^d(m_{i(-1)}) \rangle m_{i(0)} \otimes g_{\mu,\tau}(n_i) \\
&= \sum \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\mu,\tau}, \Phi_{M_{\lambda^{-1}}} 1^d(m_{(0)(-1)}) \rangle (1 \square g_{\mu,\tau}) g_{\mu,\lambda\tau}(m_{(0)(0)}) \\
&\quad \text{by (7)} \\
&= \sum \langle \alpha_{\lambda,\mu\tau}, 1^d(m_{(-1)(1)}) \rangle \langle \alpha_{\mu,\tau}, \Phi_{M_{\lambda^{-1}}} 1^d(m_{(-1)(2)}) \rangle (1 \square g_{\mu,\tau}) g_{\mu,\lambda\tau}(m_{(0)}) \\
&\quad \text{by comodule property} \\
&= \sum \langle \alpha_{\lambda,\mu\tau} \cdot (\alpha_{\mu,\tau} \Phi_{M_{\lambda^{-1}}}), 1^d(m_{(-1)}) \rangle (1 \square g_{\mu,\tau}) g_{\mu,\lambda\tau}(m_{(0)}).
\end{aligned}$$

The uniqueness of β yields, $\beta = \alpha_{\lambda,\mu\tau} \cdot (\alpha_{\mu,\tau} \Phi_{M_{\lambda^{-1}}})$. On the other hand,

$$\begin{aligned}
(f_{\lambda,\mu} \square 1) f_{\lambda\mu,\tau}(m) &= \sum_{(m)} \langle \alpha_{\lambda\mu,\tau}, 1^d(m_{(-1)}) \rangle (f_{\lambda,\mu} \square 1) g_{\lambda\mu,\tau}(m_{(0)}) \\
&= \sum \langle \alpha_{\lambda\mu,\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\lambda,\mu}, 1^d(m'_{j(-1)}) \rangle g_{\lambda,\mu}(m'_{j(0)}) \otimes n'_j \\
&= \sum \langle \alpha_{\lambda\mu,\tau}, 1^d(m_{(-1)}) \rangle \langle \alpha_{\lambda,\mu}, 1^d(m'_{(0)(-1)}) \rangle (g_{\lambda,\mu} \square 1) g_{\lambda\mu,\tau}(m_{(0)(0)}) \\
&= \sum \langle \alpha_{\lambda\mu,\tau}, 1^d(m_{(-1)(1)}) \rangle \langle \alpha_{\lambda,\mu}, 1^d(m_{(-1)(2)}) \rangle (g_{\lambda,\mu} \square 1) g_{\lambda\mu,\tau}(m_{(0)}) \\
&= \sum \langle \alpha_{\lambda\mu,\tau} \cdot \alpha_{\lambda,\mu}, 1^d(m_{(-1)}) \rangle (g_{\lambda,\mu} \square 1) g_{\lambda\mu,\tau}(m_{(0)}).
\end{aligned}$$

From the uniqueness of β it is deduced that

$$\beta = \alpha_{\lambda,\mu\tau} \cdot (\alpha_{\mu,\tau} \Phi_{M_{\lambda^{-1}}}) = \alpha_{\lambda\mu,\tau} \cdot \alpha_{\lambda,\mu},$$

which is just the cocycle condition for α defined as $\alpha(d) = \sum_{x,y \in G} \alpha_{x,y}(d) p_x \otimes p_y$ for all $d \in Z(C)$ □

Proposition 3.5. *In the conditions of the above proposition, the map* $\alpha \in B^2(Z(C), (kG)^*)$ *if and only if there is a C-bicolinear isomorphism of graded coalgebras from* $C\langle \mathscr{F}, \Pi, G \rangle$ *into* $C\langle \alpha \bigstar \mathscr{F}, \Pi, G \rangle$.

Proof. Suppose that $\alpha \in B^2(Z(C),(kG)^*)$, then there is a convolution invertible map $\beta : Z(C) \to (kG)^*$ $(\beta(d) = \sum_{g\in G} \beta_g(d)\ p_g \forall d \in Z(C))$ such that

$$\langle \alpha_{\lambda,\mu}, d\rangle = \sum \langle \alpha_{\lambda\mu}, c_{(1)}\rangle \langle \alpha_\lambda^{-1}, c_{(2)}\rangle \langle \alpha_\mu^{-1}, \Phi_{\lambda^{-1}}(c_{(3)})\rangle \quad \forall d \in Z(C).$$

Let $m \in M_\sigma$, we define $\eta_\sigma : C\langle \mathscr{F}, \Pi, G\rangle \to C\langle \alpha \bigstar \mathscr{F}, \Pi, G\rangle$ as

$$\eta_\sigma(m) = \sum_{(m)} \langle \beta_\sigma^{-1}, 1^d(m_{(-1)})\rangle m_{(0)}.$$

We check that $\eta = \bigoplus_{\sigma\in G} \eta_\sigma$ is a C-bicolinear isomorphism of graded coalgebras. It is not difficult to see that it is C-bicolinear.

$$\Delta(\eta(m)) = \Delta(\eta_\sigma(m)) = \Delta\left(\sum_{(m)} \langle \beta_\sigma^{-1}, 1^d(m_{(-1)})\rangle m_{(0)}\right)$$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_\sigma^{-1}, 1^d(m_{(-1)})\rangle \langle \alpha_{a,b}, 1^d(m_{(0)(-1)})\rangle f_{a,b}(m_{(0)(0)})$$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_\sigma^{-1}, 1^d(m^a_{i(-1)(1)})\rangle \langle \alpha_{a,b}, 1^d(m^a_{(-1)(2)})\rangle m^a_{i(0)} \otimes n^b_i$$

where we have set $f_{a,b}(m) = \sum_i m_i^a \otimes n_i^b$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_\sigma^{-1}, 1^d(m^a_{i(-1)(1)})\rangle \langle \beta_\sigma, 1^d(m^a_{i(-1)(2)})\rangle \langle \beta_a^{-1}, 1^d(m^a_{i(-1)(3)})\rangle$$
$$\langle \beta_b^{-1}, \Phi_{M_{a^{-1}}} 1^d(m^a_{i(-1)(4)})\rangle m^a_{i(0)} \otimes n^b_i$$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_a^{-1}, 1^d(m^a_{i(-1)(1)})\rangle \langle \beta_b^{-1}, \Phi_{M_{a^{-1}}} 1^d(m^a_{i(-1)(2)})\rangle m^a_{i(0)} \otimes n^b_i$$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_a^{-1}, 1^d(m^a_{i(-1)})\rangle \langle \beta_b^{-1}, \Phi_{M_{a^{-1}}} 1^d(m^a_{i(0)(-1)})\rangle m^a_{i(0)(0)} \otimes n^b_i$$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_a^{-1}, 1^d(m^a_{i(-1)})\rangle \langle \beta_b^{-1}, 1^d(m^a_{i(0)(1)})\rangle m^a_{i(0)(0)} \otimes n^b_i$$

by definition of $\Phi_{M_{a^{-1}}}$

$$= \sum_{(m)} \sum_{ab=\sigma} \langle \beta_a^{-1}, 1^d(m^a_{i(-1)})\rangle \langle \beta_b^{-1}, 1^d(n^b_{i(-1)})\rangle m^a_{i(0)} \otimes n^b_{i(0)}$$

since $\sum_i m^a_{i(0)} \otimes n^b_i \in M_a \square_C M_b$

$$= \sum_{ab=\sigma} (\eta_a \otimes \eta_b)\left(\sum_i m_i^a \otimes n_i^b\right) = \sum_{ab=\sigma} (\eta_a \otimes \eta_b) f_{a,b}(m)$$

$$= (\eta \otimes \eta)\Delta(m).$$

Assume that $\sigma \neq e$, then $\varepsilon(m) = 0 = \varepsilon\eta(m)$. If $\sigma = e$, then,

$$\varepsilon\eta(m) = \varepsilon_C\omega\eta(m) = \varepsilon_C\omega\left(\sum_{(m)}\langle\beta_e^{-1}, 1^d(m_{(-1)})\rangle m_{(0)}\right) = \varepsilon_C\omega(m) = \varepsilon(m),$$

since $\beta_e^{-1} = \varepsilon_C$.

Conversely, assume that $\eta : C\langle\mathscr{F}, \Pi, G\rangle \to C\langle\alpha \bigstar \mathscr{F}, \Pi, G\rangle$ is a C-bicolinear isomorphism of graded coalgebras. Then $\eta = \bigoplus_{\sigma \in G} \eta_\sigma$ where η_σ is a C-bicolinear isomorphism of M_σ. By Proposition 2.3 there is a unique $\beta_\sigma \in U(Z(C)^*)$ such that

$$\eta_\sigma^{-1}(m) = \sum_{(m)} \langle\beta_\sigma, 1^d(m_{(-1)})\rangle m_{(0)} \quad \forall m \in M_\sigma.$$

Let $\sigma, \tau \in G$ be fixed, and $m \in M_{\sigma\tau}$. Given $a, b \in G$, we set $f_{a,b}(m) = \sum_i m_i^a \otimes n_i^b \in M_a \,\square_C\, N_b$. Then,

$$(\eta^{-1} \otimes \eta^{-1})\Delta(m) = (\eta^{-1} \otimes \eta^{-1}) \left(\sum_{ab=\sigma\tau} \langle\alpha_{a,b}, 1^d(m_{(-1)})\rangle f_{a,b}(m_{(0)})\right)$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b}, 1^d(m_{(-1)})\rangle\langle\beta_a, 1^d(m^a_{i(0)(-1)})\rangle\langle\beta_b, 1^d(n^b_{i(-1)})\rangle m^a_{i(0)(0)} \otimes n^b_{i(0)}$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b}, 1^d(m^a_{i(-1)})\rangle\langle\beta_a, 1^d(m^a_{i(0)(-1)})\rangle\langle\beta_b, 1^d(m^a_{i(1)})\rangle m^a_{i(0)(0)} \otimes n^b_i$$

$$\text{since } \sum_i m^a_{i(0)} \otimes n^b_i \in M_a \,\square_C\, N_b$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b}, 1^d(m^a_{i(-3)})\rangle\langle\beta_a, 1^d(m^a_{i(-2)})\rangle\langle\beta_b, \Phi_{M_{a^{-1}}} 1^d(m^a_{i(-1)})\rangle m^a_{i(0)} \otimes n^b_i$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b}, 1^d(m^a_{i(-1)(1)})\rangle\langle\beta_a, 1^d(m^a_{i(-1)(2)})\rangle\langle\beta_b, \Phi_{M_{a^{-1}}} 1^d(m^a_{i(-1)(1)})\rangle m^a_{i(0)} \otimes n^b_i$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b} \cdot \beta_a \cdot (\beta_b\Phi_{M_{a^{-1}}}), 1^d(m^a_{i(-1)})\rangle m^a_{i(0)} \otimes n^b_i$$

$$= \sum_{ab=\sigma\tau} \langle\alpha_{a,b} \cdot \beta_a \cdot (\beta_b\Phi_{M_{a^{-1}}}), 1^d(m_{(-1)})\rangle f_{a,b}(m_{(0)}).$$

On the other hand,

$$\Delta\eta^{-1}(m) = \sum_{ab=gh}\sum_{(m)} \langle\beta_{gh}, 1^d(m_{(-1)})\rangle f_{a,b}(m_{(0)}).$$

Since η is a coalgebra map, $f_{\sigma,\tau}$ an isomorphism, and $\beta_{\sigma\tau}$ is unique, we obtain that $\beta_{\sigma\tau} = \alpha_{\sigma,\tau} \cdot \alpha_\sigma \cdot (\beta_\tau\Phi_{M_{\sigma^{-1}}})$. From this, $\alpha_{\sigma,\tau} = \beta_{\sigma\tau} \cdot \beta_\sigma^{-1} \cdot (\beta_\tau^{-1}\Phi_{M_{\sigma^{-1}}})$ and thus $\alpha \in B^2(Z(C), (kG)^*)$. $\square$

Let $\mathscr{P} = \{C\langle\mathscr{F}, \Pi, G\rangle\}_{\mathscr{F} \in \mathbf{F}_s(\Pi)}$ be the set of divisorially graded coalgebras. We denote by $\mathscr{C}(C, \Pi)$ the set of isomorphism classes of divisorially graded coalgebras. $D, D' \in \mathscr{C}(C, \Pi)$ are isomorphic if there exists a C-bicolinear isomorphism of graded coalgebras from D into D'.

Theorem 3.6. *Let* $\mathscr{F} \in \mathbf{F}_s(\Pi)$. *The map from* $H^2(Z(C),(kG)^*)$ *into* $\mathscr{C}(C,\Pi)$, $[\alpha] \mapsto C\langle \alpha \bigstar \mathscr{F}, \Pi, G\rangle$ *is bijective.*

Proof. Follows from Propositions 3.3–3.5 □

4. The strong Picard group: applications

Besides Morita–Takeuchi theory, another Morita theory for coalgebras was developed by I-Peng Lin [10]. For two coalgebras C, D, this theory studies the equivalences from $\mathscr{M}^C$ into $\mathscr{M}^D$ which arise from an equivalence from ${}_{C^*}\mathscr{M}$ into ${}_{D^*}\mathscr{M}$ where $\mathscr{M}^C$ and $\mathscr{M}^D$ are considered as full subcategories of ${}_{C^*}\mathscr{M}$ and ${}_{D^*}\mathscr{M}$, respectively, via rational modules. A right C-comodule M is called an *ingenerator* if it is a finitely cogenerated injective cogenerator. It is said that C is *strongly equivalent* to D if $\mathscr{M}^C$ is equivalent to $\mathscr{M}^D$ via inverse equivalences $f:\mathscr{M}^C \to \mathscr{M}^D$ and $g:\mathscr{M}^D \to \mathscr{M}^C$, $f(C)$ is an ingenerator in $\mathscr{M}^D$ and $g(D)$ is an ingenerator in $\mathscr{M}^C$. If both coalgebras have finite dimensional coradical, then strongly equivalent is the same as equivalent, see [10, p. 322]. The following is the theorem characterizing strong equivalences, [10, Theorem 5]:

Theorem 4.1. *Let C and D be coalgebras. If* $\mathscr{M}^C$ *is strongly equivalent to* $\mathscr{M}^D$ *via* $f:\mathscr{M}^C \to \mathscr{M}^D$ *and* $g:\mathscr{M}^D \to \mathscr{M}^C$, *then there are ingenerators* $P \in \mathscr{M}^C$ *and* $Q \in \mathscr{M}^D$ *such that* ${}_{C^*}\mathscr{M}$ *is equivalent to* ${}_{D^*}\mathscr{M}$ *via*

$$F(-) = {}_{D^*}P^*_{C^*} \otimes_{C^*} - : {}_{C^*}\mathscr{M} \to {}_{D^*}\mathscr{M},$$

$$G(-) = {}_{C^*}Q^*_{D^*} \otimes_{D^*} - : {}_{D^*}\mathscr{M} \to {}_{C^*}\mathscr{M}.$$

Moreover, f *and* g *are naturally isomorphic to* F *and* G, *respectively.*

A subgroup of $Pic(C)$ can be defined by considering invertible bicomodules which are ingenerators.

Definition 4.2. The strong Picard group of C, denoted by $Pic^s(C)$, is defined as the subset consisting of $[M] \in Pic(C)$ such that M and M^{-1} are ingenerators as right C-comodules.

We prove in the following lemma that the definition does not depend of the right or left side.

Lemma 4.3. *Let* M *be a right* C-*comodule*.

(i) M *is finitely cogenerated if and only if* M^* *is finitely generated.*

(ii) *Assume that* M *is an injective cogenerator. Then*, M *is an ingenerator if and only if* M^* *is a progenerator.*

(iii) *If* M *is an invertible* (C,C)-*bicomodule*, *then* M_C *is an ingenerator if and only if* ${}_CM$ *is an ingenerator.*

Proof. (i) If M is finitely cogenerated, it is clear that M^* is finitely generated. Conversely, suppose that M^* is finitely generated, then there is a finite dimensional vector space W and a surjective C^*-module map $g: W \otimes C^* \to M^*$. Let $\{e_i\}_{i=1}^n$ be a basis for W and $\{e_i^*\}_{i=1}^n$ its dual basis in W^*. We define $\bar{g}: M \to W^* \otimes C$ as

$$\bar{g}(m) = \sum_{i=1}^{n} \sum_{(m)} e_i^* \otimes \langle g(e_i \otimes \varepsilon), m_{(0)} \rangle m_{(1)}.$$

It is easy to check that $\bar{g}$ is a C-comodule map. We claim that $\bar{g}^* = g$.

Let $c^* \in C^*$, $m \in M$,

$$\begin{aligned} \langle \bar{g}^*(e_j \otimes c^*), m \rangle &= \sum_{i=1}^{n} \sum_{(m)} \langle e_i^*, e_j \rangle \langle g(e_i \otimes \varepsilon), m_{(0)} \rangle \langle c^*, m_{(1)} \rangle \\ &= \langle g(e_j \otimes \varepsilon) c^*, m \rangle \\ &= \langle g(e_j \otimes c^*), m \rangle. \end{aligned}$$

It follows that $\bar{g}$ is injective and hence M is finitely cogenerated.

(ii) It was remarked in [10, p. 319] that if M is an ingenerator, then M^* is a progenerator. The converse straightforward follows from (i).

(iii) Since M is an invertible (C, C)-bicomodule, by Takeuchi [17, Theorem 3.5], M_C is a quasi-finite injective cogenerator if and only if ${}_CM$ is so. If ${}_CM$ is an ingenerator, then M^* is a progenerator as left C^*-module. Since M^* is an invertible (C^*, C^*)-bimodule (Theorem 4.1), from classical Morita theory, M^* is a progenerator as right C^*-module. From (ii), M_C is an ingenerator. □

In view of Theorem 2.1(a), $Pic^s(C)$ represents the set of strong self-equivalences of $\mathcal{M}^C$. $Pic^s(C)$ is a subgroup of $Pic(C)$ since any strong equivalence maps ingenerators to ingenerators. If C has finite dimensional coradical, then $Pic^s(C) = Pic(C)$. Note that the image of ω in (2) lies in $Pic^s(C)$. In particular if C has the Aut-Pic property, then $Pic^s(C) = Pic(C) = Out(C)$. From Theorem 4.1 follows that:

Proposition 4.4. *The map* $(-)^*: Pic^s(C) \to Pic(C)$, $[M] \mapsto [M^*]$ *is injective. Hence* $Pic^s(C)$ *may be viewed as a subgroup of* $Pic(C^*)$.

We apply the theory of the Picard group to the study of strongly graded coalgebras. We give an hypothesis to characterize those graded coalgebras with strongly graded dual ring. We recall from [12] the notion of graded dual ring. Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a graded coalgebra. For any $\sigma \in G$ we put $R_\sigma = \{f \in C^* \mid f(C_\tau) = 0 \; for\ all\ \tau \neq \sigma\}$, (note that $R_\sigma \cong C_\sigma^*$ as vector spaces). We define $R = \sum_{\sigma \in G} R_\sigma = \bigoplus_{\sigma \in G} R_\sigma$. R is a

G-graded ring with the convolution product and unit ε. R is called the *graded dual ring* of the graded coalgebra C. A graded coalgebra having a strongly graded dual ring is necessarily strongly graded by Dăscălescu et al. [4, Corollary 2.2]. However, the converse is not true, see example below. The strong Picard group allows us to characterize those coalgebras having a strongly graded dual ring.

Proposition 4.5. *Let* $C=\bigoplus_{\sigma\in G} C_\sigma$ *be a strongly graded coalgebra and* $\Pi: G\to Pic(C_e)$, $\sigma\mapsto[C_\sigma]$ *the canonical group homomorphism. Suppose that* $Im(\Pi)\subseteq Pic^{s}(C_e)$.

(i) *The graded dual ring* $R=\bigoplus_{\sigma\in G} R_\sigma$ *is strongly graded with canonical group homomorphism* $\Pi^*=(-)^*\Pi: G\to Pic(R_e)$.

(ii) *Let* $\mathbf{F}_s(\Pi^*)$ *denote the set of factor set associated to* Π^*, *and* $\mathscr{C}(R_e,\Pi^*)$ *the set consisting of isomorphism classes of strongly graded rings* $R_e\langle\mathscr{F},\Pi^*,G\rangle$ *with* $\mathscr{F}\in\mathbf{F}_s(\Pi^*)$. *Then,* $\mathscr{C}(C_e,\Pi)\cong\mathscr{C}(R_e,\Pi^*)$ *as sets.*

Proof. (i) Given $\sigma,\tau\in G$, let $u_{\sigma,\tau}: C_{\sigma\tau}\to C_\sigma\Box_{C_e} C_\tau$ be the canonical cofactor set. Consider the dual map $u^*_{\sigma,\tau}:(C_\sigma\Box_{C_e} C_\tau)^*\cong R_\sigma\otimes_{R_e} R_\tau\to R_{\sigma\tau}$ (note that $[C_\sigma]\in Pic^{s}(C_e)$ by hypothesis). The family $\mathscr{U}^*=\{u^*_{x,y}\}_{x,y\in G}$ is a factor set associated to $\Pi^*: G\to Pic(R_e)$, $\sigma\mapsto R_\sigma=C^*_\sigma$ and $R\cong R_e\langle\mathscr{U}^*,\Pi,G\rangle$.

(ii) Let $\sigma\in G$ and $d^*\in Z(R_e)$. We define an action ${}^{\sigma}d^*=\Phi^*_{M_{\sigma^{-1}}}(d^*)$. This action is just the dual of the coaction $\rho: Z(C_e)\to(kG)^*\otimes Z(C_e), d\mapsto\sum_{\sigma\in G} p_\sigma\otimes\Phi_{C_{\sigma^{-1}}}(d)$ for all $d\in Z(C_e)$. In light of Corollary 2.4 we may identify $Z(C_e)^*$ with $Z(R_e)$. Let $d^*\in Z(R_e)$, $d\in Z(C_e)$ and $\tau\in G$. Then

$$\begin{aligned}
\langle\rho^*(\tau\otimes d^*),d\rangle &= \langle\tau\otimes d^*,\rho(d)\rangle\\
&= \sum_{\sigma\in G}\langle\tau\otimes d^*, p_\sigma\otimes\Phi_{C_{\sigma^{-1}}}(d)\rangle\\
&= \sum_{\sigma\in G}\langle\tau, p_\sigma\rangle\,\langle d^*,\Phi_{C_{\sigma^{-1}}}(d)\rangle\\
&= \langle d^*,\Phi_{C_{\tau^{-1}}}(d)\rangle\\
&= \langle\Phi^*_{C_{\tau^{-1}}}(d^*),d\rangle\\
&= \langle{}^{\tau}d^*,d\rangle.
\end{aligned}$$

If $\alpha\in H^2(Z(C_e),(kG)^*)$, then $\alpha^*: kG\otimes kG\to U(Z(R_e))$ defined as $\alpha^*(g,h)=\alpha_{g,h}$ is a cocycle. Moreover, the map $(-)^*: H^2(Z(C_e),(kG)^*)\to H^2(G,U(Z(R_e)))$, $\alpha\mapsto\alpha^*$ is a group isomorphism (see [20, Lemma 3.1]). By Theorem 3.6, $\mathscr{C}(C_e,\Pi)\cong H^2(Z(C_e),(kG)^*)$ and by Năstăsescu and Van Oystaeyen [15, Corollary I.3.18], $\mathscr{C}(R_e,\Pi^*)\cong H^2(G,U(Z(R_e)))$. Hence $\mathscr{C}(C_e,\Pi)\cong\mathscr{C}(R_e,\Pi^*)$. $\square$

Theorem 4.6. *Let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a graded coalgebra and $R=\bigoplus_{\sigma\in G} R_\sigma$ its graded dual ring. R is strongly graded if and only C is strongly graded and $Im(\Pi)\subseteq Pic^s(C_e)$. Equivalently, C is finitely cogenerated as right C_e-comodule.*

Proof. Suppose that $C=\bigoplus_{\sigma\in G} C_\sigma$ is a strongly graded coalgebra and let $\Pi: G\to Pic(C_e)$, $\sigma\mapsto[C_\sigma]$ be the canonical group homomorphism. By hypothesis, $Im(\Pi)$ lies in $Pic^s(C_e)$. From Proposition 4.5, the dual graded ring $R=\bigoplus_{\sigma\in G} R_\sigma$ is strongly graded.

Conversely, let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a graded coalgebra such that the graded dual ring $R=\bigoplus_{\sigma\in G} R_\sigma$ is strongly graded. From [5, Corollary 2.2], C is a strongly graded coalgebra. Then, all C_σ are invertible (C_e,C_e)-bicomodules. Since R is strongly graded, by Năstăsescu and Van Oystaeyen [15, Proposition I.3.6], $C_\sigma^*\cong R_\sigma$ is an invertible (R_e,R_e)-bimodule for all $\sigma\in G$. By Lemma 4.3, C_σ is an ingenerator and thus $Im(\Pi)\subseteq Pic^s(C_e)$.

Since $C=\bigoplus_{\sigma\in G} C_\sigma$ is a strongly graded coalgebra, the group G is necessarily finite (see [12, Corollary 6.4]). All C_σ are finitely cogenerated as right C_e-comodule if and only if C is finitely cogenerated as right C_e-comodule via the projection $\pi_e: C\to C_e$. □

This result improves [12, Proposition 6.2] where C_σ were asked to be of finite dimension for all $\sigma\in G$. If C_e is finite dimensional, from the foregoing theorem, we deduce that C is finite dimensional (note that G is finite). Thus we rediscover [4, Corollary 2.4].

Corollary 4.7. *Let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a strongly graded coalgebra. If $Pic^s(C_e)=Pic(C_e)$, then $R=\bigoplus_{\sigma\in G} R_\sigma$ is a strongly graded ring.*

In particular, the graded dual ring of a strongly graded coalgebra having degree e component with finite dimensional coradical is strongly graded. Also, as C_e has the Aut-Pic property the dual graded ring is strongly graded since $Pic(C)=Pic^s(C)\cong Out(C)$. In this case we may say more on these coalgebras. These are precisely the graded crossed coproducts.

Theorem 4.8. *Let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a strongly graded coalgebra and suppose that $Im(\Pi)\subseteq Out(C_e)$. Then, C is a graded crossed coproduct.*

Proof. By Proposition 4.5(i), the dual graded ring $R=\bigoplus_{\sigma\in G} R_\sigma$ is strongly graded. By hypothesis, for each $\sigma\in G$, we may find an automorphism $f_\sigma: C_e\to C_e$ such that $C_\sigma\cong {}_{f_\sigma}C_{e1}$ as (C_e,C_e)-bicomodules. Denote this isomorphism by $\theta_\sigma: C_\sigma\to{}_{f_\sigma}C_{e1}$. The dual map $\theta_\sigma^*: {}_{f_\sigma^*}R_{e1}\to R_\sigma$ is an isomorphism of (R_e,R_e)-bimodules ($f_\sigma^*: R_e\to R_e$ is the dual of f_σ). We set $u_\sigma=\theta_\sigma^*(\varepsilon)$. Since θ_σ^* is a R_e-bimodule map, $R_\sigma=R_e\cdot u_\sigma=u_\sigma\cdot R_e$. By hypothesis, R is a strongly graded ring, so that $R_\sigma\cdot R_{\sigma^{-1}}=R_e$. Hence $u_\sigma\cdot R_e\cdot u_{\sigma^{-1}}=R_e$

and $u_{\sigma^{-1}} \cdot R_e \cdot u_\sigma = R_e$ for all $\sigma \in G$. Let $\phi, \psi \in R_e$ such that $u_\sigma \cdot \phi \cdot u_{\sigma^{-1}} = \varepsilon$ and $u_{\sigma^{-1}} \cdot \psi \cdot u_\sigma = \varepsilon$. We write $v_\sigma = \phi \cdot u_{\sigma^{-1}} = u_{\sigma^{-1}} \cdot \psi$, then for $c \in C_e$ we have that

$$\sum_{(c)} u_\sigma(\pi_\sigma(c_1)) v_\sigma(\pi_{\sigma^{-1}}(c_2)) = \sum_{(c)} v_\sigma(\pi_{\sigma^{-1}}(c_1)) u_\sigma(\pi_\sigma(c_2)) = \varepsilon(c),$$

which just means that C is a graded crossed coproduct. □

Corollary 4.9. *Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a graded coalgebra such that C_e has the Aut-Pic property. Then C is a graded crossed coproduct.*

From [3, Theorem 2.6], basic coalgebras have the Aut-Pic property. As a consequence of the foregoing theorem we get:

Corollary 4.10. *Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a graded coalgebra and suppose that C_e is either basic, pointed, or cocommutative. Then C is a graded crossed coproduct.*

Theorem 4.6 also solves the question proposed in [7, p. 408] in the case where the Hopf algebra is kG. The question pointed out there is the following: let H be a finite dimensional Hopf algebra and C/D an H^*-Galois coextension (see [7] for its definition). It was proved in [7, Proposition 1.5] that if C is finitely cogenerated as (left or right) D-comodule, then the dual extension C/D is H^*-Galois. Is C^*/D^* H-Galois if and only if C/D is H^*-Galois and C is finitely cogenerated as D-comodule?

It is known that $(kG)^*$-Galois coextensions are precisely the strongly graded coalgebras (see [5, Theorem 2.1]), and kG-Galois extensions are the strongly graded rings. Then, Theorem 4.6 gives an affirmative answer for the Hopf algebra kG. Using Lemma 4.3 we can solve the above question for any finite dimensional Hopf algebra.

Theorem 4.11. *Let H be a finite dimensional Hopf algebra, and C/D an H^*-Galois coextension. Then, C^*/D^* is H-Galois if and only if C/D is H^*-Galois and C is finitely cogenerated as D-comodule.*

Proof. Suppose that C^*/D^* is a Galois extension then, from [2, Theorem 1.2], C^* is finitely generated as D^*-module. By Lemma 4.3, C is finitely cogenerated as D-comodule. Finally, [7, Proposition 1.4] yields that C/D is H^*-Galois. The converse is just [7, Proposition 1.5]. □

We end this paper by giving an example of coalgebra such that its strong Picard group is a proper subgroup of the Picard group.

Example 4.12 (*Dăscălescu et al.* [5, *Example* 2.3]). Let X be an infinite set and $C = kX$ be the group-like coalgebra. Consider the family of simples comodules $S_x = kx$ for $x \in X$, and let $P = \bigoplus_{x \in X} S_x^{(n_x)}$ where $\{n_x\}_{x \in X}$ is a non-bounded set of natural numbers. P is a quasi-finite injective cogenerator and it has associated a Morita–Takeuchi context (C, D, P, Q, f, g) where $D = e_{-C}(P)$ and $Q = h_{-C}(P, C)$. Associated to this context there is a matrix coalgebra (see [4, Section 2])

$$E = \begin{pmatrix} C & P \\ Q & D \end{pmatrix},$$

which is a $\mathbb{Z}_2$-graded coalgebra by setting

$$E_0 = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}.$$

It was proved in [5, Example 2.3] that $E = E_0 \oplus E_1$ is a strongly graded coalgebra with non-strongly graded dual ring $R = R_0 \oplus R_1$. By Corollary 4.7, $Pic^s(E_0) \neq Pic(E_0)$. From Theorem 4.6, E_1 may not be finitely cogenerated as E_0-comodule.

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