JOURNAL OF ALGEBRA 175, 451–477 (1995)

Minimal Injective and Flat Resolutions of Modules over Gorenstein Rings

Jinzhong Xu

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027

Communicated by Melvin Hochster

Received April 12, 1993

Over a commutative Noetherian ring R, the Bass invariants $\mu_i(p, M)$ were defined for any module M and any prime $p \in \operatorname{Spec} R$ by H. Bass (*Math. Z.* **82**, 1963, 8–28). In the first part of this paper, we study these numbers further. We are concerned primarily with the modules having a certain vanishing property of their Bass numbers. For instance, we show that R is Gorenstein if and only if $\mu_i(p, F) = 0$ whenever $\operatorname{ht}(p) \neq i$ for any flat module F. In the second part, we define the invariants $\pi_i(p, M)$ for any prime $p \in \operatorname{Spec} R$ and module M by a minimal flat resolution of M. As with the Bass invariants, we can characterize Gorenstein rings and modules by a vanishing property of these numbers. For instance, injective modules are just those modules M having $\pi_i(p, M) = 0$ for all prime p with $\operatorname{ht}(p) \neq i$ and all $i \geq 0$. Finally, we introduce strongly cotorsion modules and show that these modules M are just those modules having $\pi_i(p, M) = 0$ for all prime p with $\operatorname{ht}(p) > i$ and all $i \geq 0$. From this paper we will see that flat covers defined by E. Enochs (*Israel J. Math.* **39**, 1981, 189–209) behave in a manner dual to the behavior of injective envelopes.

1. INTRODUCTION

Let R be a commutative Noetherian ring, M be an R-module. Using the minimal injective resolution of M, H. Bass defined the ith invariant $\mu_i(p, M)$ for any prime $p \in \operatorname{Spec} R$ in [1]. These numbers were shown particularly interesting for the regular module R when R is Gorenstein. In general, it is noted that most nice properties of $\mu_i(p, M)$ depend on M being finitely generated. Here, we study the minimal injective resolutions of modules of finite flat dimension and do not assume them to be finitely generated. We are particularly interested in the vanishing property of the Bass numbers. For instance, Theorem 2.1 shows that R is Gorenstein if and only if any flat R-module F has $\mu_i(p, F) = 0$ for all prime p with $\operatorname{ht}(p) \neq i$ and all $i \geq 0$. In [18], a finitely generated module having the

0021-8693/95 $12.00

above property is called a Gorenstein module. On the other hand, in [3, 4], Enochs studied commutative Noetherian rings over which injective envelopes of flat modules are flat. It turns out that for such rings R is generically Gorenstein. That is, R_p is Gorenstein for any $p \in \mathrm{Ass}(R)$. These rings were also studied in[1, Proposition 6.1; 9, Theorem 2.4]. Here, it is shown that for any module M, $\mathrm{f.dim}_R E(M) \leq \mathrm{f.dim}_R M$ if and only if R is Gorenstein (Theorem 2.2). In Section 2, many other properties of modules over Gorenstein rings will be proved.

As a generalization of a projective cover of a module, Enochs defined flat covers of modules by commutative diagrams in [2]. In Auslander's terminology, a flat cover is a minimal right $\mathscr{F}$-approximation, where $\mathscr{F}$ stands for subcategory of flat modules (see [14]). Then in Section 3, we consider minimal flat resolutions by flat covers of modules. Note that over a commutative Noetherian ring R, every pure injective flat module F can be uniquely written in the form $F = \prod T_p$, where T_p is a completion of a free R_p-module with respect to p-adic topology [7, p. 183]. This result is similar to the Matlis theorem for injective modules. Using this fact, we can define the ith invariant $\pi_i(p, M)$ for $p \in \mathrm{Spec}\, R$ and a module M which admis a minimal flat resolution. By the vanishing property of these invariants, some interesting properties of modules over Gorenstein rings will be given in this section. For example, Theorem 3.2 shows that over a Gorenstein ring R, injective modules are just those modules E having $\pi_i(p, E) = 0$ for all prime p with $\mathrm{ht}(p) \neq i$ and all $i \geq 0$.

In Section 4, we introduce the strongly cotorsion modules. We prove the existence of minimal flat resolution for strongly cotorsion modules (which may have infinite injective dimension and infinite flat dimension) over a Gorenstein ring. We also show that over a Gorenstein ring R, strongly cotorsion modules are just those modules M having the numbers $\pi_i(p, M) = 0$ for all prime p with $\mathrm{ht}(p) > i$ and all $i \geq 0$. As a consequence, we show that any module over a n-Gorenstein ring admits a minimal flat resolution. This is a partial answer to an open problem posed by Enochs in [2]. That is, when does every module over a ring always admit a minimal flat resolution? At the end of Section 4, we determine all modules M having $\mu_i(p, M) = 0$ for all prime p with $\mathrm{ht}(p) > i$ and all $i \geq 0$. These modules are the duals of strongly cotorsion modules in a certain sense. We call them strongly torsion free because they are at least torsion free.

Throughout this paper, all rings R are commutative Noetherian with the identity, all modules are unitary. For any module X, $\mathrm{f.dim}_R X$ stands for the flat dimension of R-module X, $\mathrm{inj.dim}_R X$ stands for the injective dimension of X, $\mathrm{proj.dim}_R X$ stands for the projective dimension of X, $E(X)$ stands for its injective envelope, and $F(X)$ stands for its flat cover if it exists. All other notation is standard. For instance, $\mathrm{ht}(p)$ means the height of p, Dim R means the Krull dimension of R, and R_p means the localization of R at a prime p.

2. MINIMAL INJECTIVE RESOLUTIONS OF FLAT MODULES

Let R be a commutative Noetherian ring and M an R-module. A minimal injective resolution of M is an exact sequenec

$$0 \to M \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots \to E_i \xrightarrow{d_i} \cdots$$

such that for each $i \geq 0$, E_i is an injective envelope of $\ker(d_i)$. It is well known that each E_i has a unique decomposition $E_i = \oplus E(R/p)$, $p \in$ Spe R [8]. If $\mu_i(p, M)$ denotes the ith Bass number, it can be written in the form

$$E_i = \oplus_{p \in \operatorname{Spec} R} \mu_i(p, M) E(R/p).$$

DEFINITION 2.1. A commutative Noetherian ring is called Gorenstein if $\text{inj.dim}_{R_m} R_m$ is finite for any maximal ideal m.

H. Bass established many characterizations of Gorenstein rings in his article [1]. For convenience, we quote one result of his fundamental theorem as a lemma.

LEMMA 2.1. *A commutative Noetherian ring R is Gorenstein if and only if it admits a minimal injective resolution as*

$$0 \to R \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots \to E_i \xrightarrow{d_i} \cdots$$

such that $E_i = \oplus_{\text{ht}(p)=i} E(R/p)$. *Namely,* $\mu_i(p, R) = \delta_{i\,\text{ht}(p)}$.

We will see that there are several generalizations of this classical characterization of Gorenstein rings. First we need a preliminary result which will be used later.

PROPOSITION 2.1. *Let R be a commutative Noetherian ring. Then the following are equivalent.*

(1) *R is Gorenstein.*
(2) $\text{f.dim}_R E(R/m) = \text{ht}(m)$ *for any maximal ideal m.*
(3) $\text{f.dim}_R E(R/m) < \infty$ *for any maximal ideal m.*
(4) $\text{f.dim}_R E(R/p) = \text{ht}(p)$ *for any $p \in$* Spec R.
(5) $\text{f.dim}_R E(R/p) < \infty$ *for any $p \in$* Spec R.

Proof. (1) ⇒ (2). Note that $\text{f.dim}_R E(R/m) = \text{f.dim}_{R_m} E(R/m)$ by the isomorphism $E(R/m)_m \cong E(R/m)$ [8]. Also note that $\text{ht}(m) = \text{ht}(m_m)$. Then we may assume that (R, m) is a local Gorenstein ring. For any finitely generated R-module M, we have the natural isomorphism [5, VI, Proposition 5.3],

$$\text{Hom}_R\big(\text{Ext}^i_R(M, R), E(R/m)\big) \cong \text{Tor}^r_i\big(M, \text{Hom}_R(R, E(R/m))\big).$$

It follows that $\text{f.dim}_R E(R/m) = \text{inj.dim}_R R = \text{ht}(m)$.

(2) ⇒ (3). This is obviously true.

(3) ⇒ (1). We only need to show that R_m is Gorenstein for any maximal ideal m. Since $\text{f.dim}_R E(R/m)$ is finite, so is $\text{f.dim}_{R_m} E(R/m)_m$. We may simply consider the local case (R, m) with $\text{f.dim}_R E(R/m)$ finite. Therefore, the above natural isomorphism implies that $\text{inj.dim}_{R_m} R_m$ is finite and then that R_m is Gorenstein.

Next, note that $E(R/p)_q \neq 0$ if and only if $p \subset q$ for any two prime ideals p, q. Then we can finish the implications (1) ⇒ (4) ⇒ (5) ⇒ (1) similarly. ∎

Now we begin to investigate the minimal injective resolutions of flat modules.

THEOREM 2.1. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein.*

(2) *For any flat R-module F, the minimal injective resolution*

$$0 \to F \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

is such that $E_i = \oplus \mu_i(p, F) E(R/p)$ *and* $\mu_i(p, F) = 0$ *if* $\text{ht}(p) \neq i$.

(3) *A module F is flat if and only if its minimal injective resolution is as in* (2).

Remark. As the referee suggested, by applying the results of [12] we may have a shorter proof for Theorem 2.1. Here, for consistence in the methods used later, we would like to give the following proof.

Proof. (2) ⇒ (1). For the regular module R, we have the minimal injective resolution by the assumption

$$0 \to R \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

such that if $E(R/p) \subset E_i$, then $\text{ht}(p) = i$. For any maximal ideal m, taking localization at m, we get that $(E_i)_m = 0$ for $i > \text{ht}(m)$. It follows that $\text{inj.dim}_{R_m} R_m$ is finite. Namely, R_m is Gorenstein. Hence so is R.

(1) $\Rightarrow$ (2). By Lemma 2.1, there is a special minimal injective resolution of the regular module R, denoted by $\mathscr{E}(R)$. For any flat module $F \cong F \otimes_R R$, taking the tensor product $F \otimes \mathscr{E}(R)$, we have an injective resolution of F

$$0 \to F \otimes R \to F \otimes E_0 \to F \otimes E_1 \to \cdots \to F \otimes E_i \to \cdots,$$

where $E_i = \oplus_{\mathrm{ht}(p)=i} E(R/p)$, and $F \otimes E_i$ is injective. It is not hard to see that $F \otimes E(R/p)$ is a direct sum of copies of $E(R/p)$. Therefore, $E(R/p) \subset F \otimes E_i$ only if $\mathrm{ht}(p) = i$. Finally, since any minimal injective resolution of F is a direct summand of $F \otimes \mathscr{E}(R)$, the conclusion follows.

(2) $\Rightarrow$ (3). Suppose F admits such a minimal injective resolution,

$$0 \to F \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

such that $E(R/p) \subset E_i$ only if $\mathrm{ht}(p) = i$. We have to show that F is flat. Assume $E_i \neq 0$. Then by Proposition 2.1, we have that $\mathrm{f.dim}_R E(R/p) = i = \mathrm{f.dim}_R E_i$. For any maximal ideal m, taking the localization at m, we get

$$0 \to F_m \to (E_0)_m \to (E_1)_m \to \cdots \to (E_s)_m \to \cdots .$$

Note that $(E_i)_m = 0$ for $i > \mathrm{ht}(m)$ and if $(E_i)_m \neq 0$, $\mathrm{f.dim}_{R_m}(E_i)_m = i$. So we have the exact sequence

$$0 \to F_m \to G_0 \to G_1 \to \cdots \to G_s \to 0.$$

Here, $s \leq \mathrm{ht}(m)$, $\mathrm{f.dim}_{R_m} G_i = i$.

Break this long exact sequence into short exact sequences as

$$0 \to K_1 \to G_{s-1} \to G_s \to 0$$
$$0 \to K_2 \to G_{s-2} \to K_1 \to 0$$
$$\cdots$$
$$0 \to K_{s-1} \to G_1 \to K_{s-2} \to 0$$
$$0 \to F_m \to G_0 \to K_{s-1} \to 0.$$

Now it is easy to see that $\mathrm{f.dim}\, K_1 = s - 1$, $\mathrm{f.dim}\, K_2 = s - 2, \ldots, \mathrm{f.dim}\, K_{s-1} = s - (s-1) = 1$, and then $\mathrm{f.dim}\, F_m = 0$. This means that F_m is flat for any maximal ideal m. Therefore, F is a flat R-module.

(3) $\Rightarrow$ (2). This is trivially true. ∎

By the above theorem, we have the following interesting consequence.

PROPOSITION 2.2. *Let (R, m) be local Gorenstein and F have finite flat dimension. Then F is flat if and only if every maximal R-sequence $\{\mu_1, \ldots, \mu_d\}$ is also an F-sequence.*

Proof. We only need to show the sufficiency. Consider the minimal injective resolution of F,

$$0 \to F \to E_0 \to E_1 \to \cdots \to E_i \to \cdots .$$

By Theorem 2.1, we have to show that $E(R/p) \subset E_i$ only if $\mathrm{ht}(p) = i$. First of all, we claim that if $E(R/p) \subset E_i$, then $\mathrm{ht}(p) \geq i$. Suppose $E(R/p)$ is contained in E_i and $\mathrm{ht}(p) < i$. Considering the localization of the resolution at the prime p, we have a minimal injective resolution of F_p as R_p-module [1]. Since R_p is Gorenstein and F_p has finite flat dimension, we have $\mathrm{inj.dim}_{R_p} F_p \leq \mathrm{Dim}\, R_p = \mathrm{ht}(p) < i$ [13, Corollary 5.6]. This implies that $(E_i)_p = 0$. But this is a contradiction because $E(R/p)$ is contained in E_i.

No we show that $\mathrm{ht}(p) \leq i$ if $E(R/p) \subset E_i$. Consider at $E_0 = \oplus E(R/p)$. It is not hard to see that $\mathrm{ht}(p) = 0$ by the hypothesis. Then assume that under the hypotheses, for $0 \leq i \leq s$, it is true that $E(R/p) \subset E_i$ only if $\mathrm{ht}(p) = i$ for any local Gorenstein ring. Now we consider the case $(s + 1)$. Suppose $E(R/p) \subset E_{s+1}$. Since $\mathrm{ht}(p) \geq s + 1$, there is a non-zero divisor $u \in p$ on both R and F. Using the functor $\mathrm{Hom}_R(R/uR, *)$, we have the minimal injective resolution of F/uF,

$$0 \to F/uF \to \mathrm{Hom}_R(R/uR, E_1) \to \cdots \to \mathrm{Hom}_R(R/uR, E_{s+1}) \to \cdots .$$

It is easy to see that all conditions are preserved by R/uR and F/uF. That is, $\bar{R} = R/uR$ is Gorenstein and $\bar{F} = F/uF$ has finite flat dimension as an $\bar{R}$-module. By the inductive hypothesis, we know that $E(\bar{R}/\bar{q}) \subset \bar{E}_s = \mathrm{Hom}_R(\bar{R}, E_{s+1})$ only if $\mathrm{ht}(\bar{q}) = s$. It follows that $\mathrm{ht}(p/(u) = s$ and then $\mathrm{ht}(p) = s + 1$ since $E(\bar{R}/\bar{p}) \subset \mathrm{Hom}_R(\bar{R}, E_{s+1})$. ∎

As a generalization of Theorem 2.1, we have the following.

THEOREM 2.2. *Let R be Gorenstein, M be an R-module. Then the following are equivalent.*

(1) $\mathrm{f.dim}_R M = s < \infty$.

(2) *M admits a minimal injective resolution as*

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

such that $E(R/p) \subset E_i$ only if $i \leq \mathrm{ht}(p) \leq i + s$ for $i \geq 0$ and s is the smallest among such integers. In other words, $\mu_i(p, M) \neq 0$ only if $i \leq \mathrm{ht}(p) \leq i + s$.

Proof. (2) ⇒ (1). As before, taking the localization at any maximal ideal m, we have the minimal injective resolution

$$0 \to M_m \to (E_0)_m \to (E_1)_m \to \cdots \to (E_i)_m \to \cdots .$$

Then we know that $(E_i)_m = 0$ when $i > \text{ht}(m)$ and $i \leq \text{f.dim}_{R_m}(E_i)_m \leq i + s$ if $(E_i)_m \neq 0$. That is, we have

$$0 \to M_m \to G_0 \to G_1 \to \cdots \to G_t \to 0$$

such that $i \leq \text{f.dim}\, G_i \leq i + s$.

We now break this into short exact sequences

$$\begin{gathered} 0 \to K_1 \to G_{t-1} \to G_t \to 0 \\ 0 \to K_2 \to G_{t-2} \to K_1 \to 0 \\ \cdots \\ 0 \to K_{t-1} \to G_1 \to K_{t-2} \to 0 \\ 0 \to M_m \to G_0 \to K_{t-1} \to 0. \end{gathered}$$

Suppose that $\text{f.dim}\, M_m = u > s$. This will imply that $\text{f.dim}\, K_{t-1} \geq u + 1$, and then $\text{f.dim}\, G_t \geq u + t > s + t$. This is a contradiction. Therefore, we have that $\text{f.dim}_{R_m} M_m \leq s$ for any maximal ideal m and that $\text{f.dim}_R M \leq s$. On the other hand, suppose $\text{f.dim}\, M = u < s$, by our proof that (1) implies (2) we see that we can construct a minimal injective resolution of M such that $E(R/p) \subset E_i$ only if $i \leq \text{ht}(p) \leq i + u$ for any $i \geq 0$. This contradicts our choice of s.

(1) $\Rightarrow$ (2). We use the induction on the dimension of M. By Theorem 2.1, we know that it is true for $\text{f.dim}_R M = 0$. Supose it is true for all modules with flat dimension less than n and suppose that $\text{f.dim}_R M = n$. Let us construct the desired minimal injective resolution of M.

As usual, we consider the exact sequence

$$0 \to N \to F \to M \to 0$$

with F flat and $\text{f.dim}_R N = n - 1$. By the induction hypothesis, we have the desired minimal injective resolutions for both F and N as

$$\begin{gathered} 0 \to F \to E_0 \to E_1 \to \cdots \to E_i \to \cdots \\ 0 \to N \to G_0 \to G_1 \to \cdots \to G_i \to \cdots \end{gathered}$$

such that $E(R/p) \subset E_i$ only if $\text{ht}(p) = i$ and $E(R/p) \subset G_i$ only if $i \leq \text{ht}(p) \leq i + (n - 1)$.

Consider the pushout diagram,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N & \to & F & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & G_0 & \to & H & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & L_0 & = & L_0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

For the exact sequence $0 \to F \to H \to L_0 \to 0$, we can construct the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & H & \to & L_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_0 & \to & W_0 & \to & G_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_0 & \to & X_0 & \to & L_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here $W_0 = E_0 \oplus G_1$, it is injective. Then using the resolutions of K_0 and L_1, we get an injective resolution of H,

$$0 \to H \to W_0 \to W_1 \to \cdots \to W_i \to \cdots .$$

Here, $W_i = E_i \oplus G_{i+1}$. Therefore, $i \leq \text{f.dim}_R W_i \leq (i+1) + (n-1) = i + n$. By Proposition 2.1, we know that $E(R/p) \subset W_i$ only if $i \leq \text{ht}(p) \leq i + n$.

Now, consider the following pushout diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & G_0 & \to & H & \to & M & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & G_0 & \to & W_0 & \to & Z & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & X_0 & = & X_0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

It is easy to see that $Z = W_0/G_0$ is injective and $0 \le \text{f.dim}_R Z \le \text{f.dim}_R G_1 \le 1 + (n-1) = n$. Then, pasting the exact sequence $0 \to M \to Z \to X_0 \to 0$ and the resolution of X_0 together, we have that

$$0 \to M \to Z_0 \to Z_1 \to \cdots \to Z_i \to \cdots$$

is such that Z_i is injective and $i \le \text{f.dim}_R Z_i \le i + n$. Note that the ith term of any minimal injective resolution of M is a direct summand of Z_i. Therefore, by Proposition 2.1, $E(R/p)$ is in the ith term only if $i \le \text{ht}(p) \le i + n$. By the first part of the proof, we also can assert that n is the smallest among such integers. Otherwise, we deduce that $\text{f.dim}_R M < n$. ∎

Now we are ready to give some other characterizations of Gorenstein rings.

THEOREM 2.3. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein*

(2) *For any finitely generated module M,* $\text{f.dimE}(M) \le \text{f.dim}_R M$

(3) *For any module M,* $\text{f.dim}_R E(M) \le \text{f.dim}_R M$.

Proof. (1) ⇒ 3. If $\text{f.dim}_R M = \infty$, this is trivially true. If $\text{dim}_R M = s < \infty$, by Theorem 2.2, we know that $\text{f.dim}_R E(M) = \text{f.dim}_R E_0 \le s = \text{f.dim}_R M$.

(3) ⇒ (2). This is trivial.

(2) ⇒ (1). For any maximal ideal m, consider a maximal R-sequence $\{\mu_1, \dots, \mu_t\}$ in m. Then $M = R/(\mu_1, \dots, \mu_t)$ has finite flat dimension. By the assumption, $\text{f.dim}_R E(M) \le \text{f.dim}_R M$. On the other hand, since M is m-primary, $R/m \subset M$. This implies that $E(R/m) \subset E(M)$ and that it also has finite flat dimension because it is a direct summand of $E(M)$. Hence R is Gorenstein by Proposition 2.1. ∎

We may ask when $\text{f.dim}_R E(M) = \text{f.dim}_R M$ for any module M. It turns out that this condition is more restrictive.

THEOREM 2.4. *Let R be commutative Noetherian, then the following are equivalent.*

(1) *R is Gorenstein with* Dim $R \le 1$.

(2) *For any module M with finite flat dimension,* $\text{f.dim}_R E(M) = \text{f.dim}_R M$.

Proof. (1) ⇒ (2). If $\text{f.dim}_R M < \infty$, we know that $\text{f.dim}_R M = 0$ or 1 [13, Corollary 5.6]. Suppose $\text{f.dim}_R M = 1$, consider the exact sequence

$$0 \to M \to E(M) \to X \to o.$$

Since $\text{f.dim}_R X < \infty$, $\text{f.dim}_R X \leq 1$. But then it follows that $\text{f.dim}_R E(M) = 1$. Otherwise, we have $\text{f.dim}_R X = 2$, a contradiction.

(2) $\Rightarrow$ (1). First, by Theorem 2.3, R is Gorenstein. Suppose Dim $R > 0$. For any maximal ideal m, choose a maximal R-sequence in m, $\{\mu_1, \dots, \mu_t\}$. This gives us that $\text{f.dim}_R(\mu_1, \dots, \mu_t) = t - 1$. We claim that $t = 1$.

Since $E(R)$ is flat and $I = (\mu_1, \dots, \mu_t) \in R$, $E(I)$ is a direct summand of $E(R)$. This means that $E(I)$ is flat. But then by the assumption, $\text{f.dim}_R E(I) = \text{f.dim}_R I$. This implies that I is flat and $t - 1 = 0$. Namely, $t = 1$. Finally, $\text{ht}(m) = \text{ht}(I) = 1$ for any maximal ideal m. It follows that R is Gorenstein with Krull dimension one. ∎

COROLLARY 2.1. *Let (R, m) be commutative Noetherian. Then the following are equivalent.*

(1) *(R, m) is regular with* Dim $R \leq 1$.

(2) $\text{f.dim}_R M = \text{f.dim}_R E(M)$ *for all R-modules M.*

Proof. (1) $\Rightarrow$ (2). This is obvious. For (2) $\Rightarrow$ (1), by Theorem 2.4, (R, m) is Gorenstein with Dim $R \leq 1$. Note that $\text{f.dim}_R R/m = \text{f.dim}_R E(R/m) \leq 1$ by the assumption. It follows that R is regular because gl.dim $R = \text{f.dim}_R R/m$. ∎

THEOREM 2.5. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein with* Dim $R \leq n + 1$.

(2) $0 \leq \text{f.dim}_R M - \text{f.dim}_R E(M) \leq n$ *for all R-modules with finite flat dimension.*

Proof. (2) $\Rightarrow$ (1). Since the hypothesis implies that $\text{f.dim}_R E(M) \leq \text{f.dim}_R M$, by Theorem 2.3, R is Gorenstein. Suppose Dim $R > 0$ and let $\{\mu_1, \dots, \mu_s\}$ be a maximal R-sequence in a maximal ideal m. As before, $\text{f.dim}_R(\mu_1, \dots, \mu_s) = s - 1$ and $E(\mu_1, \dots, \mu_s)$ is a direct summand of $E(R)$. Hence, $\text{f.dim}_R(\mu_1, \dots, \mu_s) - \text{f.dim}_R E(\mu_1, \dots, \mu_s) \leq n$ implies that $s - 1 \leq n$ and then that Dim $R \leq n + 1$.

(1) $\Rightarrow$ (2). By Theorem 2.3, we have that $\text{f.dim}_R M - \text{f.dim}_R E(M) \geq 0$. If $\text{f.dim}_R M = n + 1$, then for some maximal ideal m, $\text{f.dim}\, R_m M_m = n + 1$. It is not hard to see that $\text{f.dim}_{R_m} E(M_m) = n + 1$ and then that $\text{f.dim}_R E(M) = n + 1$. If $\text{f.dim}_R M \leq n$, then obviously $\text{f.dim}_R M - \text{f.dim}_R E(M) \leq n$. ∎

COROLLARY 2.2. *Let R be commutative Noetherian. Then the following are equivalent*:

(1) *R is regular with* Dim $R \leq n + 1$.

(2) $0 < \text{f.dim}_R M - \text{f.dim}_R E(M) \leq n$ *for all R-modules M.*

(*Here, the right half inequality means that* $\text{f.dim}_R M \leq \text{f.dim}_R E(M) + n$.)

We have seen that over a Gorenstein ring R flat modules M have the vanishing property $\mu_i(p, M) = 0$ for all prime p with $\text{ht}(p) \neq i$. We will be back to investigate the modules M having $\mu_i(p, M) = 0$ for all prime p with $\text{ht}(p) > i$ in Section 4.

3. MINIMAL FLAT RESOLUTIONS OF INJECTIVE MODULES

DEFINITION 3.1. Let M be an R-module, F is a flat module. According to Enochs' terminology in [2], a homomorphism $\phi: F \to M$ is called a flat cover of M if (1) for any homomorphism $\phi': G \to M$ with G flat, there is a homomorphism $f: G \to F$ such that $\phi' = \phi f$ and (2) if $\phi = \phi f$ for some endomorphism of F, then f is an automorphism of F.

Remark. If in the definition, (1) is satisfied, then we call ϕ a flat precover. Enochs proved that if M admits a flat precover, then it also admits a flat cover and that flat covers are unique up to isomorphism [2, Theorem 3.1]. If $\mathscr{F}$ denotes all flat modules, in Auslander's terminology [14], a flat cover of M is a minimal right $\mathscr{F}$-approximation. We use $F(M)$ to denote the flat cover of M.

A minimal flat resolution of M is an exact sequence

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to M \to 0$$

such that for each i, F_i is a flat cover of $\text{im}(d_i)$. It is easy to see that for any flat module G, applying the functor $\text{Hom}_R(G, *)$ to the minimal flat resolution of M, we still have an exact sequence. Actually, if M has an exact sequence

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to M \to 0$$

such that F_i is flat and $\text{Hom}_R(G, *)$ makes it still exact for any flat module G, then M admits a minimal flat resolution [2, 5]. The above resolution is called a $\mathscr{F}$-resolution of M.

In this section, we are primarily concerned with minimal flat resolutions of modules. We will see that there are results similar to those we got in the last section.

Let R be a commutative Noetherian ring, F a pure injective flat module (or, equivalently, cotorsion flat), then by [7, Theorem p. 183], $F = \prod T_p$. Here, $p \in \text{Spec } R$, T_p is the completion of a free R_p-module with respect to p-adic topology. Now, suppose M has a minimal flat resolution, F_i is its

ith term, then for $i \geq 1$, F_i is pure injective flat [7, p. 183] and then $F_i = \prod T_p$. For $i = 0$, we take the pure injective envelope $PE(F_0)$ of F_0. Then we have $PE(F_0) = \prod T_p$ [6, p. 352]. If M is cotorsion, that is, $\operatorname{Ext}^1_R(F,M) = 0$ for any flat module, then F_0 is cotorsion and so $PE(F_0) = F_0$.

Similarly as in [1, 6], we define the ith invariant for any M which admits a minimal flat resolution as follows:

DEFINITION 3.2. Suppose M admits a minimal flat resolution

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to M \to 0.$$

For $i \geq 1$, $p \in \operatorname{Spec} R$, $\pi_i(p, M)$ is the cardinality of a base of a free R_p-module whose completion is T_p where $F_i = \prod T_p$ and each T_p is the completion of a free R_p-module. For $i = 0$, we define $\pi_0(p, M)$ as above but using the pure injective envelope $PE(F_0)$ of F_0.

It is easy to see that $\pi_i(p, M)$ are well defined and homologically independent. But we do not know when they are finite. We will study the calculations of these numbers further in a subsequent paper. Here, we are particularly interested in their vanishing property.

THEOREM 3.1. *Let R be commutative Noetherian, then the following are equivalent*:

(1) *R is Gorenstein.*

(2) *For any injective module E, $\pi_i(p, E) = 0$ whenever* $\operatorname{ht}(p) \neq i$ *for* $i \geq 0$.

Proof. (2) $\Rightarrow$ (1). For any maximal ideal m, we have the minimal flat resolution of $E = E(R/m)$.

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to E(R/m) \to 0.$$

Note that, by the assumption, $T_p \subset F_i = \prod T_p$ only if $\operatorname{ht}(p) = i$. Since R_m is flat, using th functor $\operatorname{Hom}_R(R_m, *)$, we have an exact sequence

$$\cdots \to \operatorname{Hom}_R(R_m, F_i) \to \cdots \to \operatorname{Hom}(R_m, F_0) \to \operatorname{Hom}_R(R_m, E(R/m)) \to 0.$$

Note that $\operatorname{Hom}_R(R_m, E(R/m)) \cong E(R/m)$ and for any $i \geq 0$, $\operatorname{Hom}_R(R_m, F_i) = \operatorname{Hom}_R(R_m, \prod T_p) \cong \prod \operatorname{Hom}_R(R_m, T_p)$.

Here T_p can be taken as $T_p = \mathrm{Hom}_R(E(R/p), E(R/p)^{(X)})$ for some set X [6, p. 353]. For each T_p, we have that $\mathrm{Hom}_R(R_m, T_p) =$

$$\mathrm{Hom}_R\big(R_m, \mathrm{Hom}_R\big(E(R/p), E(R/p)^{(X)}\big)\big)$$
$$\cong \mathrm{Hom}_R\big(R_m \otimes_R E(R/p), E(R/p)^{(X)}\big).$$

Since $R_m \otimes E(R/p)$ is injective, it follows that $\mathrm{Hom}_R(R_m \otimes E(R/p), E(R/p)^{(X)})$ is flat because $E(R/p)^{(X)}$ is injective. But, on the other hand, $R_m \otimes_R E(R/p) \cong E(R/p)_m \neq 0$ if and only if $p \subset m$. This cannot happen for $i > \mathrm{ht}(m)$ because of $\mathrm{ht}(p) = i$ by our assumption. Therefore, $E(R/m)$ has finite flat dimension. It follows that R is Gorenstein by Proposition 2.1.

(1) $\Rightarrow$ (2). In order to construct the desired minimal flat resolution for any injective module E, consider the minimal injective resolution of R by Lemma 2.1.

$$0 \to R \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots \to E_i \xrightarrow{d_i} \cdots$$

such that $E_i = \oplus_{\mathrm{ht}(p)=i} E(R/p)$. Using the functor $\mathrm{Hom}_R(*, E)$, we have that

$$\cdots \to \mathrm{Hom}_R(E_i, E) \to \cdots \to \mathrm{Hom}_R(E_0, E) \to \mathrm{Hom}_R(R, E) \to 0.$$

Note that $\mathrm{Hom}_R(R, E) = E, \mathrm{Hom}_R(E_i, E) \cong \prod \mathrm{Hom}_R(E(R/p), E)$ is flat. Easily, we have that

$$\mathrm{Hom}_R(E(R/p), E) = \mathrm{Hom}_R\big(E(R/p) \otimes_R R_p, E\big)$$
$$\cong \mathrm{Hom}_R\big(E(R/p), \mathrm{Hom}_R(R_p, E)\big).$$

Since $\mathrm{Hom}_R(R_p, E)$ is R_p injective, hence $\mathrm{Hom}_R(R_p, E) = \oplus E(R/q)$, $q \subset p$. Set $\mathrm{Hom}_R(R_p, E) = A \oplus B$, where $A = \oplus_{q=p} E(R/q)$, $B = \oplus_{q \neq p} E(R/q)$. Note that for $q \subset p, q \neq p$, $\mathrm{Hom}_R(E(R/p), E(R/q)) = 0$, then $\mathrm{Hom}_R(E(R/p), \prod E(R/q)) = 0$ for all $q \subset p, q \neq p$. Therefore, $\mathrm{Hom}_R(E(R/p), B) = 0$ because B is a direct summand of $\prod E(R/q)$, $q \neq p$. Consequently, $\mathrm{Hom}_R(E(R/p), E) = \mathrm{Hom}_R(E(R/p), A) = \mathrm{Hom}_R(E(R/p), E(R/p)^{(X)}) = T_p$ for some set X, and then $\mathrm{Hom}_R(E_i, E) \cong \prod T_p$ with $\mathrm{ht}(p) = i$. Now it is easy to see that all ith terms F_i of the minimal flat resolution of E are direct summands of $\prod T_p$ with $\mathrm{ht}(p) = i$, correspondingly. Also, $F_i = \prod T_p$ with $\mathrm{ht}(p) = i$ by Theorem of [7]. ∎

From the above argument and the fact f.dim $E(R/p) = \mathrm{ht}(p)$, we have that

COROLLARY 3.1. *If R is Gorenstein, then for any $p \in \mathrm{Spec}\, R$,* $\pi_{\mathrm{ht}(p)}(p, E(R/p)) = 1$.

We have the following which is dual to Proposition 2.1.

PROPOSITION 3.1. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein.*

(2) $\text{inj.dim}_R T_m = \text{ht}(m)$ *for any maximal ideal m.*

(3) $\text{inj.dim}_R T_m < \infty$ *for any maximal ideal m.*

(4) $\text{inj.dim}_R T_p = \text{ht}(p)$ *for any prime* $p \in \text{Spec } R$.

(5) $\text{inj.dim}_R T_p < \infty$ *for any prime* $p \in \text{Spec } R$.

Proof. (1) ⇒ (4). We note that $T_p = \text{Hom}_R(E(R/p), E(R/p)^{(X)})$ and $\text{f.dim}_R E(R/p) = \text{ht}(p)$. This implies that $\text{inj.dim}_R T_p \le \text{ht}(p)$. But on the other hand, it is easy to see that $\hat{R}_p = \text{Hom}_R(E(R/p), E(R/p))$ is a direct summand of T_p and $\text{inj.dim}_{\hat{R}_p} \hat{R}_p = \text{inj.dim}_{R_p} R_p = \text{ht}(p)$. Therefore, $\text{inj.dim}_R T_p = \text{ht}(p)$.

(4) ⇒ (5). This is trivial.

(5) ⇒ (1). Note that R_p is a direct summand of T_p. It follows that $\text{inj.dim}_{R_p} R_p < \infty$ for any $p \in \text{Spec } R$. Namely, R is Gorenstein. The other implications can be established similarly. ∎

LEMMA 3.1. *Let R be commutative Noetherian and let F be flat. If the pure injective envelope of F,* $PE(F) = \prod T_p$ *with* $\text{ht}(p) = 0$, *then* $F = PE(F)$ *is pure injective.*

Proof. This is by Theorem 2.1 of [6]. ∎

THEOREM 3.2. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein.*

(2) *An R-module X is injective if and only if* $\pi_i(p, X) = 0$ *for all prime p with* $\text{ht}(p) \ne i$ *and all* $i \ge 0$.

Proof. By Theorem 3.1, we only need to show that if R is Gorenstein and X has $\pi_i(p, X) = 0$ for all prime p with $\text{ht}(p) \ne i$ and all $i \ge 0$, then X is injective. By the previous lemma, for all $i \ge 0$, if F_i is not zero, then $F_i = \prod T_p$ with $\text{ht}(p) = i$.

Let C be an injective cogenerator of R-modules. In order to prove that X is injective, we only need to show that $\text{Hom}_R(X, C) = X^*$ is flat. By Theorem 2.1, X^* is flat if and only if $\mu_i(p, X^*) = 0$ for all prime p with $\text{ht}(p) \ne i$ and all $i \ge 0$.

For any $i \ge 0$, $\text{Hom}_R(F_i, C)$ is injective and has flat dimension less than or equal to i by Proposition 3.1. Then, $F_i^* \cong \oplus E(R/q)$ with $\text{ht}(q) \le i$. We claim that $\mathrm{ht}(q) \geq i$. Otherwise, p is not contained in q for all p for which the T_p of F_i is non-zero.

For any q, we have that

$$\mathrm{Hom}_R(E(R/q), F_i^*) \cong \mathrm{Hom}_R\big(E(R/q) \otimes \prod T_p, C\big).$$

Note that $E(R/q) \otimes \prod T_p = 0$ [6, Proof of Lemma 1.3]. But easily the left side is not zero because $E(R/q)$ is a direct summand of F_i^*. This is a contradiction. ∎

By constructing a minimal flat resolution for a module M of finite injective dimension, we will see the vanishing property of the numbers $\pi_i(p, M)$. This gives us a generalization of Theorem 3.1.

THEOREM 3.3. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein.*

(2) *If M has $\mathrm{inj.dim}_R M = s < \infty$, then M admits a minimal flat resolution and for any $i \geq 1$, $\pi_i(p, M) \neq 0$ only if $i \leq \mathrm{ht}(p) \leq i + s$. For $s = 0$, $\mathrm{inj.dim}_R F_0 \leq \mathrm{inj.dim}_R M = s$ (F_0 may not have the form $\prod T_p$).*

Proof. (2) ⇒ (1). Consider the case $s = 0$ and M is injective. Since $\mathrm{inj.dim}_R F_0 \leq \mathrm{f.dim}_R M$, $F_0 = F(M)$ also injective. Then F_0 is flat and injective. It also has the form $F_0 = \prod T_p$ with $\mathrm{ht}(p) = 0$. Now, for $i \geq 0$, $F_i = \prod T_p$ with $\mathrm{ht}(p) = i$ for any injective module M. Now, by Theorem 3.1, R is Gorenstein.

Before we prove the implication (1) ⇒ (2), we need a lemma which is useful for constructing flat covers.

LEMMA 3.2. *Suppose $0 \to K \to Y \to M \to 0$ is exact. If both K and M have flat covers, and K is cotorsion (that is, $\mathrm{Ext}^1_R(G, K) = 0$ for any flat module G), then we have a diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_1 & \to & X & \to & L_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G & \to & W & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & Y & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that $W = G \oplus F \to Y$ is a flat precover of Y and such that X is also cotorsion. Here, $F \to M \to 0$ is a flat cover of M, $G \to K \to 0$ is a flat cover of K.

Proof. Since K is cotorsion, there is a homomorphism $F \to Y$ making the lower right corner triangle commutative. Therefore we can construct homomorphisms naturally and get the desired diagram. ∎

Now, let us continue the proof of implication (1) ⇒ (2). We are proceeding by induction on the injective dimension of M. Suppose it is true for all modules with injective dimension less than $n + 1$. We consider the case $\text{inj.dim}_R M = n + 1$.

As standard, choose an exact sequence

$$0 \to M \to E \to N \to 0.$$

Here, E is injective and $\text{inj.dim}_R N = n$.

By the inductive assumption, we have the desired minimal flat resolutions for both E and N.

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to E \to 0$$

$$\cdots \to G_i \xrightarrow{d_i'} G_{i-1} \to \cdots \to G_0 \to N \to 0$$

with $F_i = \prod T_p$, $\text{ht}(p) = i$ for $i \geq 0$, $G_i = \prod T_p$, $i \leq \text{ht}(p) \leq i + n$ for $i \geq 1$ and $\text{f.dim}_R G_0 \leq \text{f.dim}_R M$.

Consider the following pullback diagram of $E \to N$ and $G_0 \to N$,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_0 & = & K_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & Z & \to & G_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & E & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then, we consider the exact sequence

$$0 \to K_0 \to Z \to N \to 0.$$

Using the flat covers of E and K_0, and noting that K_0 is cotorsion, by Lemma 3.2, we can construct the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1 & \to & X_0 & \to & L_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G_1 & \to & W_0 & \to & F_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_0 & \to & Z & \to & E & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here, $W_0 = G_1 \oplus F_0$. Then, using the flat precover $W_0 \to Z$, we have the following pullback diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & X_0 & = & X_0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & H_0 & \to & W_0 & \to & G_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & M & \to & Z & \to & G_0 & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Now, H_0 is flat and $H_0 \to M$ is a flat precover of M. Further, $\text{inj.dim}_R H_0 \leq \text{inj.dim}_R G_0 + 1 \leq n + 1$. Next, in order to construct a flat precover of X_0, we consider the following diagram by Lemma 3.2.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_2 & \to & X_1 & \to & L_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G_2 & \to & W_1 & \to & F_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1 & \to & X_0 & \to & L_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here, $W_1 = G_2 \oplus F_1$, $G_2 \to K_1$ and $F_1 \to L_0$ are flat covers, and K_1 is cotorsion. Also, $W_1 \to X_0$ is a flat precover. But then $W_1 = \prod T_p \oplus \prod T_q = \prod T'_p$, $1 \leq \text{ht}(p') \leq 2 + n$.

In general, for $i \geq 1$, we have that $0 \to X_i \to W_i \to X_{i-1} \to 0$. Here, $W_i = G_{i+1} \oplus F_i = \prod T_p \to X_{i-1}$ is a flat precover, $i \leq \mathrm{ht}(p) \leq (i+1) + n = i + (n+1)$. Therefore, pasting them together, we have an exact sequence,

$$\cdots \to W_i \xrightarrow{d_i} W_{i-1} \to \cdots \to H_0 \to M \to 0$$

such that (1) $W_i = \prod T_p$ with $i \leq \mathrm{ht}(p) \leq i + (n+1)$, $i \geq 1$; (2) $\mathrm{inj.dim}_R H_0 \leq \mathrm{inj.dim}_R M$; (3) $\mathrm{Hom}_R(G, *)$ makes it exact for any flat module G. Therefore, M has a minimal flat resolution such that each ith term is a direct summand of W_i and then our conclusion follows. ▮

In [4], Enochs studied the rings for which every injective module has a flat cover which is also injective. Equivalently, these are the rings for which every flat module has an injective envelope which is also flat. Now we have the following

THEOREM 3.4. *Let R be commutative Noetherian. Then the following are equivalent.*

(1) *R is Gorenstein.*

(2) *For any module M with* $\mathrm{inj.dim}_R M = s < \infty$, *then* $\mathrm{inj.dim}_R F(M) \leq \mathrm{inj.dim}_R M = s$.

Proof. (1) ⇒ (2). It follows by Theorem 3.3.

(2) ⇒ (1). For any maximal ideal m, let $\{\mu_1, \ldots, \mu_n\}$ be a maximal R-sequence in m. Then, $M = R/(\mu_1, \ldots, \mu_n)$ has finite length and has finite flat dimension. This implies that $M^v = \mathrm{Hom}_R(M, E(R/m))$ has finite injective dimension. By the assumption, its flat cover F has finite injective dimension. That is, we have that

$$0 \to K \to F \to M^v \to 0.$$

Now taking the duals again, we have that

$$0 \to M^{vv} \to F^v \to K^v \to 0.$$

Note that $M \subset M^{vv}$ and F^v has finite flat dimension. On the other hand, since M is m-primary, $R/m \subset M$ and then $E(R/m) \subset E(M) \subset F^v$. This implies that $E(R/m)$ has finite flat dimension. By Proposition 2.1, R is Gorenstein. ▮

4. STRONGLY COTORSION MODULES

Recall that in the previous sections we have discussed the minimal injective resolutions of modules which have finite flat dimension and minimal flat resolutions of modules finite injective dimension. In this section we will discuss the minimal flat resolutions of strongly cotorsion modules which may have infinite flat dimension and infinite injective dimension.

DEFINITION 4.1. A module G is called strongly cotorsion if $\operatorname{Ext}_R^1(X, G) = 0$ for any X of finite flat dimension.

Remark. (1) Any strongly cotorsion module is cotorsion and any injective module is strongly cotorsion. (2) If R is n-Gorenstein, that is, R is Gorenstein with $\operatorname{Dim} R = n$, then strongly cotorsion modules are just the so-called Gorenstein injective modules in [17].

We will prove the existence of minimal flat resolutions of strongly cotorsion modules M and characterize them by a vanishing property of the numbers $\pi_i(p, M)$. In particular, we can apply these results to n-Gorenstein rings. For instance, any module over a n-Gorenstein ring admits a minimal flat resolution.

PROPOSITION 4.1. *Let R be Gorenstein and let G be strongly cotorsion. If G has finite flat dimension, then it is injective.*

Proof. By Theorem 2.3, the injective envelope of G, $E(G)$, has finite flat dimension because G has finite flat dimension. Consider the standard exact sequence

$$0 \to G \to E(G) \to X \to 0.$$

Easily, X also has finite flat dimension. Now, using the functor $\operatorname{Hom}_R(*, G)$, by the fact that $\operatorname{Ext}_R^1(X, G) = 0$, we have the exact sequence

$$0 \to \operatorname{Hom}_R(X, G) \to \operatorname{Hom}_R(E, G) \to \operatorname{Hom}_R(G, G) \to 0.$$

It follows that $0 \to G \to E(G) \to X \to 0$ is split, and then G is injectve. ∎

LEMMA 4.1. *Let R be Gorenstein and let G be strongly cotorsion. Then there is an exact sequene*

$$0 \to K \to E \to G \to 0$$

such that E is injective and K is also strongly cotorsion.

Proof. First of all, we show that G is a surjective image of an injective module. Consider the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & P & \to & G & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & E(P) & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here, $P \to E(P)$ is the injective envelope of P, and P is a projective module. By Theorem 2.3, $E(P)$ has finite dimension, hence so does X. By an argument similar to that in the above proposition, we know that $0 \to G \to D \to X \to 0$ is split. Then it follows that G is a surjective image of $E(P)$.

By Theorm 2.1 of [2], G admits an injective cover φ: $E \to G$, which is surjective because G is a surjective image of an injective module. In other words, we have the exact sequence

$$0 \to K \to E \to G \to 0.$$

By a property of the injective cover, we know that $\mathrm{Ext}_R^1(W, K) = 0$ for any injective module W (similar to Lemma 2.2 of [7]). We claim that K is also strongly cotorsion.

For any X with finite flat dimension, we have the exact sequence by using $\mathrm{Hom}_R(X, *)$:

$$0 = \mathrm{Ext}_R^1(X, G) \to \mathrm{Ext}_R^2(X, K) \to \mathrm{Ext}_R^2(X, E) = 0.$$

This means that $\mathrm{Ext}_R^2(X, K) = 0$ for all modules X with finite flat dimension.

Now, for any Y with finite flat dimension, by Theorem 2.3, $E(Y)$ has finite flat dimension. Then, consider the standard exact sequence

$$0 \to Y \to E(Y) \to X \to 0.$$

Note that X also has finite flat dimension and then $\mathrm{Ext}_R^2(X, K) = 0$. But then, using $\mathrm{Hom}_R(*, K)$, we have

$$0 = \mathrm{Ext}_R^1(E(Y), K) \to \mathrm{Ext}_R^1(Y, K) \to \mathrm{Ext}_R^2(X, K) = 0.$$

This implies that $\mathrm{Ext}_R^1(Y, K) = 0$ for any Y with finite flat dimension. Namely, K is strongly cotorsion. ∎

So far, we do not know if every module over a Gorenstein ring has a flat cover, even more we do not know if it exists for any cotorsion module. But, over a Gorenstein ring, we can show that it exists for any strongly cotorsion module. In fact, any strongly cotorsion module admits a special minimal flat resolution.

THEOREM 4.1. *Let R be Gorenstein, M a module. Then the following are equivalent.*

(1) *M is strongly cotorsion.*

(2) *M admits a minimal flat resolution*

$$\cdots \to F_i \overset{d_i}{\to} F_{i-1} \to \cdots \to F_0 \to M \to 0$$

such that $F_i = \prod T_p$ *for* $\mathrm{ht}(p) \leq i$. *In other words,* $\pi_i(p, M) = 0$ *for any* p *with* $\mathrm{ht}(p) > i$.

Proof. (1) ⇒ (2). By the last lemma, we have an exact sequence

$$0 \to N \to E \to M \to 0$$

such that E is injective and N is also strongly cotorsion. If E is injective, by Theorem 3.1, it has a minimal flat resolution,

$$\cdots \to F_i \overset{d_i}{\to} F_{i-1} \to \cdots \to F_0 \to E \to 0$$

such that $F_i = \prod T_p$ with $\mathrm{ht}(p) = i$. For convenience, we break this into short exact sequences

$$0 \to K_0 \to F_0 \to E \to 0$$

$$0 \to K_1 \to F_1 \to k_0 \to 0$$

$$\cdots$$

$$0 \to K_{i+1} \to F_{i+1} \to K_i \to 0$$

Now, consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0 & = & K_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & H_0 & \to & F_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N & \to & E & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since H_0 is cotorsion, $0 \to H_0 \to F_0 \to M \to 0$ gives us a flat precover of M. By symmetry, N admits a flat cover $0 \to W_0 \to G_0 \to N \to 0$ such that $G_0 = \prod T_p$ with $\mathrm{ht}(p) = 0$.

Now, let us look at the exact sequence $0 \to K_0 \to H_0 \to N \to 0$. By Lemma 3.2, we have the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1 & \to & H_1 & \to & W_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_1 & \to & P_0 & \to & G_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_0 & \to & H_0 & \to & N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Hence we have a flat precover of H_0, $P_0 = F_1 \oplus G_0 = \prod T_p \to H_0$ with $\mathrm{ht}(p) \le 1$. Therefore, for a strongly cotorsion module M, we have the following two exact sequences

$$0 \to H_0 \to F_0 \to M \to 0$$

$$0 \to H_1 \to P_0 \to H_0 \to 0$$

giving flat precovers and such that $F_0 = \prod T_p$ with $\mathrm{ht}(p) = 0$, $P_0 = F_1 \oplus G_0 = \prod T_p$ with $\mathrm{ht}(p) \le 1$. Symmetrically, we get the same for N. In other words, we have the following two exact sequences

$$0 \to W_0 \to G_0 \to N \to 0$$

$$0 \to W_1 \to G_1 \to W_0 \to 0$$

such that they are flat covers and $G_0 = \prod T_p$ with $\mathrm{ht}(p) = 0$ and $G_1 = \prod T_p$ with $\mathrm{ht}(p) \le 1$.

Now, repeating the same procedure, we get the following diagram by looking at the exact sequence $0 \to K_1 \to H_1 \to W_0 \to 0$ and using Lemma 3.2.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_2 & \to & H_2 & \to & W_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_2 & \to & P_1 & \to & G_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_1 & \to & H_1 & \to & W_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here $P_1 = F_2 \oplus G_1 = \prod T_p$ with $\text{ht}(p) \le 2$, and $0 \to H_2 \to P_1 \to H_1 \to 0$ gives us a flat precover of H_1. By symmetry, W_1 also admits a similar exact sequence which gives us a flat cover of W_1 and is such that $F(W_1) = G_2 = \prod T_p$ with $\text{ht}(p) \le 2$. Therefore, we can continue this procedure and get exact sequences

$$0 \to H_0 \to Z_0 \to M \to 0$$
$$0 \to H_1 \to Z_1 \to H_0 \to 0$$
$$\cdots$$
$$0 \to H_{i+1} \to Z_{i+1} \to H_i \to 0$$
$$\cdots$$

such that $Z_{i+1} \to H_i$, $Z_0 \to M$ are flat precovers, $Z_i = \prod T_p$ with $\text{ht}(p) \le i$. It is easy to see that M admits a minimal flat resolution such that its ith term F_i^* is a direct summand of Z_i and so then $F_i^* = \prod T_p$ with $\text{ht}(p) \le i$.

(2) $\Rightarrow$ (1). Suppose M admits a minimal flat resolution whose ith term is $F_i = \prod T_p$ with $\text{ht}(p) \le i$. First. note that if $\text{inj.dim}_R T_p \le i$ when $\text{ht}(p) \le i$, then $\text{inj.dim}_R F_i \le i$. If X is a module with finite flat dimension s and K is cotorsion, then by induction, it is not hard to prove that $\text{Ext}_R^{s+1}(X, K) = 0$. Now, for any module X with finite flat dimension, we have to show that $\text{Ext}_R^1(X, M) = 0$. Consider the minimal flat resolution of M

$$\cdots \to F_{i+1} \to F_i \to \cdots \to F_0 \to M \to 0.$$

Here, for $i \ge 0$, $F_i = \prod T_p$ with $\text{ht}(p) \le i$. Then we have the following short exact sequences

$$0 \to K_{s-1} \to F_{s-1} \to K_{s-2} \to 0$$
$$0 \to K_{s-2} \to F_{s-2} \to K_{s-3} \to 0$$
$$\cdots$$
$$0 \to K_1 \to F_1 \to K_0 \to 0$$
$$0 \to K_0 \to F_0 \to M \to 0$$

Note that $\text{Ext}_R^{s+1}(X, K_{s-1}) = 0$ since that X has finite flat dimension s and K_{s-1} is cotorsion. Also note that $\text{Ext}_R^s(X, F_{s-1}) = 0$ because $\text{inj.dim}_R F_{s-1} \le s - 1$. Then, by applying $\text{Ext}(X, *)$ to the first exact sequence, we have that $\text{Ext}_R^s(X, K_{s-2}) = 0$. Repeating this procedure, we have $\text{Ext}_R^2(X, K_0) = 0$. Finally, applying $\text{Ext}_R(X, *)$ to the last exact sequence, we have

$$0 = \text{Ext}_R^1(X, F_0) \to \text{Ext}_R^1(X, M) \to \text{Ext}_R^2(X, K_0) = 0.$$

It follows that $\text{Ext}_R^1(X, M) = 0$ and that M is strongly cotorsion. ∎

COROLLARY 4.1. *Over a Gorenstein ring any flat cover of a strongly cotorsion module is an injective module.*

As an application of the previous result, we have the following

PROPOSITION 4.2. *Let R be Gorenstein and let G be strongly cotorsion. Then if G has finite injective dimension, it is injective.*

Proof. By Theorem 4.1, G admits a minimal flat resolution as

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to G \to 0$$

such that $F_i = \prod T_p$ for $\operatorname{ht}(p) \le i$.

Let C be any injective cogenerator for the category of R-modules. In order to show that G is injective, we only need to show that $G^* = \operatorname{Hom}_R(G, C)$ is flat. By Theorem 2.1, we have to check that $\mu_i(p, G^*) = 0$ whenever $\operatorname{ht}(p) \ne i$. But, taking the duals, we have the exact sequence

$$0 \to G^* \to F_0^* \to F_1^* \to \cdots \to F_i^* \to \cdots .$$

Note that $F_i^* = \operatorname{Hom}_R(F_i, C)$ is injective and $\text{f.dim}_R F_i^* \le i$ because $\text{inj.dim}_R F_i \le i$. Hence, $F_i^* = \oplus E(R/p)$ with $\operatorname{ht}(p) \le i$ by Proposition 2.1. Now, it is easy to see that $\mu_i(p, G^*) = 0$ if $\operatorname{ht}(p) > i$. On the other hand, G^* has finite flat dimension because G has finite injective dimension. By Theorem 2.2, $\mu_i(p, G^*) \ne 0$ only if $i \le \operatorname{ht}(p) \le i + \text{f.dim}_R G^*$. Thus it follows that $\mu_i(p.G^*) = 0$ whenever $\operatorname{ht}(p) = i$. Therefore, the conclusion follows by Theorem 2.1. ∎

Now, as in Theorem 3.3, we can construct minimal flat resolutions for modules which are not strongly cotorsion, but which have a finite resolution with strongly cotorsion modules.

THEOREM 4.2. *Let R be Gorenstein. If M is such that there is an exact sequence*

$$0 \to M \to G_0 \to G_1 \to \cdots \to G_t \to 0,$$

where each G_i is strongly cotorsion, then M has a minimal flat resolution

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0$$

such that for $i \ge 1$, $F_i = \prod T_p$ with $\operatorname{ht}(p) \le t + i$ and such that F_0 has finite injective dimension less than or equal to t.

Proof. The proof is similar to that of Theorem 3.3. ∎

Now, we apply the above result to n-Gorenstein rings.

COROLLARY 4.2. *If R is an n-Gorenstein ring and M is any R-module, then M admits a resolution with strongly cotorsion modules as in Theorem* 4.2. *Therefore, M has a minimal flat resolution.*

Proof. By Corollary 5.6 of [13], if M is a module M over a n-Gorenstein ring R, then $\operatorname{Ext}_R^{n+1}(X, M) = 0$ for any module X of finite flat dimension. Consider the partial injective resolution of M,

$$0 \to M \to E_0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to G \to 0.$$

here, E_i is injective. The above implies that G is strongly cotorsion. Therefore, the conclusion follows by Theorem 4.2. ∎

Remark. Regarding the existence of flat covers over more general rings, we have proved that every module over a commutative Noetherian ring of finite Krull dimension has a flat cover in [16]. But the general problem is still open.

Now let us go back to the first section. Recall that we have shown that over a Gorenstein ring R, flat modules are just those modules M having $\mu_i(p, M) = 0$ for all prime p with $\operatorname{ht}(p) \neq i$ and all $i \geq 0$. We are ready to determine all modules M having the vanishing property $\mu_i(p, M) = 0$ for all p with $\operatorname{ht}(p) > i$.

DEFINITION 4.2. A R-module M is called strongly torsion free if $\operatorname{Tor}_1^R(X, M) = 0$ for all modules X of fintie flat dimension.

Let C be an injective cogenerator of R-modules. It is easy to see that M is strongly torsion free if and only if the dual $M^* = \operatorname{Hom}_R(M, C)$ is strongly cotorsion.

THEOREM 4.3. *Let R be Gorenstein, M a R-module. Then the following are equivalent.*

(1) *M is strongly torsion free.*

(2) *$\mu_i(p, M) = 0$ for all prime p with $\operatorname{ht}(p) > i$ and all $i \geq 0$.*

Proof. Suppose M has the stated vanishing property. Then M admits a minimal injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_i \to \cdots, \cdots .$$

Here, $E_i = \oplus E(R/p)$ with $\operatorname{ht}(p) \leq i$. We are going to show that M^* is strongly cotorsion. Taking the duals, we have a $\mathcal{F}$-resolution of M^*, and each $F_i = \operatorname{Hom}_R(E_i, C)$. Since $E_i = \oplus E(R/p)$ with $\operatorname{ht}(p) \leq i$, it is easy to see that $F_i = \prod T_p$ with $\operatorname{ht}(p) \leq i$ for each i. Note that any minimal $\mathcal{F}$-resolution M^* is a direct summand of this dual resolution of M^*. We have proved that M^* is strongly cotorsion by Theorem 4.1. Hence, M is strongly torsion free.

Conversely, suppose M is strongly torsion free. Consider the exact sequence

$$0 \to M \to M^{**} \to N \to 0.$$

Since $M \to M^{**}$ is pure and M^{**} is strongly torsion free, N is also strongly torsion free.

Note that M^* is strongly cotorsion. It has a special minimal flat resolution such that each $F_i = \prod T_p$ with $\mathrm{ht}(p) \leq i$. Therefore, $\mathrm{Hom}_R(F_i, C) = \oplus E(R/p)$ with $\mathrm{ht}(p) \leq i$ because $\mathrm{inj.dim}_R F_i \leq i$. In other words, M^{**} has the desired minimal injective resolution. Now, by the argument dual to that for the implication (1) $\Rightarrow$ (2) of Theorem 4.1, we have the desired minimal injective resolution of M. ∎

Remark. In the above theorem, we consider the case $i = 0$; we see that M is a submodule of E_0 which is flat. Therefore, strongly torsion free modules are torsion free when R is a Gorenstein domain.

PROPOSITION 4.3. *Let R be Gorenstein. If M is strongly torsion free, then the following are equivalent.*

(1) *M has finite flat dimension.*

(2) *M has finite injective dimension.*

(3) *M is flat.*

Proof. (2) $\Rightarrow$ (1). Note that M^* is strongly cotorsion with finite injective dimension. By Proposition 4.2, it is injective and so then M is flat. The implication (3) $\Rightarrow$ (1) can be proved by taking the dual of M and using Proposition 4.1. ∎

ACKNOWLEDGMENTS

Professor Edgar Enochs first encouraged the author to work on the current subjects. The author expresses great gratitude to him for his inspiration. Also, the author thanks the referee for his very careful reading and good advice about revising the original manuscript.

REFERENCES

1. H. Bass, On the ubiquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.
2. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981), 189–209.
3. T. Cheatham and E. Enochs, Injective hulls of flat modules, *Comm. Algebra* **8**, No. 20 (1980), 1989–1998.

4. E. Enochs, Remarks on commutative Noetherian rings whose flat modules have flat injective envelopes, *Portugal. Math.* **45**, No. 2 (1985), 151–156.
5. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, New Jersey, 1965.
6. E. Enochs, Minimal pure injective resolutions of flat modules, *J. Algebra* **105** (1987), 351–364.
7. E. Enochs, Flat covers and flat cotorsion modules, *Proc. Amer. Math .Soc.* **92** (1984), 179–184.
8. E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.
9. T. Ishikawa, On injective modules and flat modules, *J. Math. Soc. Japan* **17** (1965), 291–296.
10. L. Gruson and C. U. Jensen, Dimensions reliees aux foncteurs $\varprojlim^{(i)}$, *in* "Lecture Notes in Mathematics," vol. 867, pp. 234–294, Springer-Verlag, New York/Berlin, 1981.
11. I. Kapalansky, "Commutative Rings," Univ. of Chicago Press, Chicago, 1974.
12. L. G. Chouinard II, On finite weak and injective dimension, *Proc. Amer. Math. Soc.* **60** (1976), 57–60.
13. H. Bass, Injective dimension in Noetherian rings, *Trans. Amer. Math. Soc.* **102** (1962), 18–29.
14. M. Auslander and R. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, *Mém. Soc. Math. France* **38** (1989), 15–37.
15. R. Belshoff, E. Enochs, and Jinzhong Xu, The existence of flat covers, *Proc. Amer. Math. Soc.* **122** (1994), 985–991.
16. Jinzhong Xu, The existence of flat covers over Noetherian rings of finite Krull dimension, *Proc. Amer. Math. Soc.* **123** (1995), 27–32.
17. E. Enochs, and O. Jenda, Gorenstein, injective and projective modules, submitted for publication.
18. R. Fossum, H. B. Foxby, P. Griffith, and I. Reiten, Minimal injective resolutions with applications to dualizing modules and Gorenstein modules, *Inst. Hautes Études Sci. Publ. Math.* **45** (1975), 193–315.