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Tail Fields of Partially Exchangeable Arrays

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We give an elementary, direct proof that if an array of random variables $\{(X_{ij}, \alpha, \xi_i, \eta_j);\ i, j \in \mathbb{N}\}$ is separately exchangeable, then $\mathbf{X} = \{X_{ij};\ i, j \in \mathbb{N}\}$ and $\{(\alpha, \xi_i, \eta_j);\ i, j \in \mathbb{N}\}$ are conditionally independent given the shell σ-field $\mathscr{S}^{\mathbf{X}}$ of $\mathbf{X}$. We show further that if $(\mathbf{X}, \mathbf{Y}) = \{(X_{ij}, Y_{ij});\ i, j \in \mathbb{N}\}$ is separately exchangeable, then $\mathbf{X}$ and $\mathscr{S}^{\mathbf{X},\mathbf{Y}}$ are conditionally independent given $\mathscr{S}^{\mathbf{X}}$. © 1989 Academic Press, Inc.

We consider *separately exchangeable* arrays $\{X_{ij};\ i, j \in \mathbb{N}\}$, that is, arrays such that

$$\{X_{ij};\ i, j \in \mathbb{N}\} \sim \{X_{\pi(i),\sigma(j)};\ i, j \in \mathbb{N}\}$$

for any permutations π and σ of $\mathbb{N}$. (*Joint exchangeability* is the same condition with $\pi = \sigma$.)

We will use $\mathbf{X}$ to denote the entire array $\{X_{ij};\ i, j \in \mathbb{N}\}$, and similarly $\boldsymbol{\xi}$ or $\boldsymbol{\eta}$ for a sequence $\{\xi_i;\ i \in \mathbb{N}\}$ or $\{\eta_j;\ j \in \mathbb{N}\}$.

A number of authors ([A1], [K], and in a special case, [Ly]) have offered proofs that if the random variables α, ξ_i, η_j, ζ_{ij}, $i, j \in \mathbb{N}$ are i.i.d., and f is a measurable function, then the separately exchangeable array $X_{ij} = f(\alpha, \xi_i, \eta_j, \zeta_{ij})$, $i, j \in \mathbb{N}$, is conditionally independent (c.i.) of α, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ given the *shell* σ-field

$$\mathscr{S}^{\mathbf{X}} = \bigcap_n \{X_{ij};\ i \vee j > n\}.$$

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We feel that the most simple and direct proof of this result is that contained in our unpublished paper [H1, Corollary 5.10]. That proof was for arrays with an arbitrary finite number of indices, and was bound up with an unfamiliar model for separately exchangeable arrays. Here we give the proof disentangled from these complications, and slightly cleaned up.

Our result is the following. Superficially, it appears more general than that quoted above, but using Aldous' Representation Theorem ([A1], Theorem 1.4) and padding the representation of **X**, one can see that they are actually equivalent.

THEOREM. *If* $\{(X_{ij}, \alpha, \xi_i, \eta_j);\ i, j \in \mathbb{N}\}$ *is separately exchangeable, then* **X** *and* $(\alpha, \boldsymbol{\xi}, \boldsymbol{\eta})$ *are c.i. given* $\mathscr{S}^{\mathbf{X}}$.

The proof of the theorem is a generalization of the proof of Lemma 1 of [H2], which essentially proves this theorem when each of **X**, $\boldsymbol{\xi}$, and $\boldsymbol{\eta}$ is i.i.d. (and hence $\mathscr{S}^{\mathbf{X}}$ is trivial). In fact, the proof given here is just that proof, fortified with the most basic facts about tail and shell σ-fields.

REVIEW OF CONDITIONAL INDEPENDENCE

Before proving our main theorem, we need to recall some elementary facts about conditional independence. Recall first that a family of σ-fields, $\mathscr{G}_i$, $i \in I$, is conditionally independent given a σ-field $\mathscr{F}$ iff for any distinct $i_1, ..., i_n \in I$, and $G_{i_j} \in \mathscr{G}_{i_j}$, $j = 1, ..., n$,

$$P(G_{i_1}, ..., G_{i_n} \mid \mathscr{F}) = P(G_{i_1} \mid \mathscr{F}) \cdots P(G_{i_n} \mid \mathscr{F}).$$

Random variables X_i, $i \in I$ are c.i. given $\mathscr{F}$ if $\sigma(X_i)$, $i \in I$, are c.i. given $\mathscr{F}$. We say that X_i, $i \in I$, are c.i.i.d. given $\mathscr{F}$ if they are also conditionally identically distributed given $\mathscr{F}$; that is, $P(X_i \in \cdot \mid \mathscr{F})$ is independent of i.

$1(F)$ denotes the indicator function of the set F.

LEMMA 1. (1) $\mathscr{G}$ *and* $\mathscr{H}$ *are c.i. given* $\mathscr{F}$ *iff for any* $G \in \mathscr{G}$, $P(G \mid \mathscr{F}, \mathscr{H}) = P(G \mid \mathscr{F})$.

(2) *If* $\mathscr{F}_0 \subseteq \mathscr{F} \subseteq \mathscr{G}$, *and* $\mathscr{G}$ *and* $\mathscr{H}$ *are c.i. given* $\mathscr{F}_0$, *then* $\mathscr{G}$ *and* $\mathscr{H}$ *are c.i. given* $\mathscr{F}$.

(3) *If* $\mathscr{G}$ *and* $\mathscr{H}$ *are c.i. given* $\mathscr{E} \vee \mathscr{F}$, *and* $\mathscr{G}$ *and* $\mathscr{F}$ *are c.i. given* $\mathscr{E}$, *then* $\mathscr{G}$ *and* $\mathscr{F} \vee \mathscr{H}$ *are c.i. given* $\mathscr{E}$.

Proof. (1) is Theorem 25.3A on page 351 of [Lo]. For (2), let $H \in \mathscr{H}$. Then, by (1),

$$P(H \mid \mathscr{F}, \mathscr{G}) = P(H \mid \mathscr{F}_0),$$

since $\mathscr{F}_0$, $\mathscr{F} \subseteq \mathscr{G}$. Since $\mathscr{F}_0 \subseteq \mathscr{F} \subseteq \mathscr{G}$, it follows that $P(H \mid \mathscr{F}) = P(H \mid \mathscr{F}_0)$. The result follows by (1).

To prove (3), let $G \in \mathscr{G}$. Then

$$\begin{aligned} P(G \mid \mathscr{E}, \mathscr{F}, \mathscr{H}) &= P(G \mid \mathscr{E}, \mathscr{F}) \\ &= P(G \mid \mathscr{E}). \end{aligned}$$

The first equality follows by the first hypothesis and (1), the second equality by the second hypothesis and (1). The result now follows by (1).

Conditioning on Shell Fields

If Y_i, $i \in \mathbb{N}$ are exchangeable, we define the tail σ-field $\mathscr{T}^{\mathbf{Y}}$ of $\mathbf{Y}$ in the usual way, by

$$\mathscr{T}^{\mathbf{Y}} = \bigcap_n \sigma\{Y_i; i > n\}.$$

LEMMA 2. *Let $\{Y_i; i \in \mathbb{N}\}$ be an exchangeable sequence and $\{X_{ij}; i, j \in \mathbb{N}\}$ be a separately exchangeable array.*

(1) *Y_i, $i \in \mathbb{N}$ are c.i.i.d. given $\mathscr{T}^{\mathbf{Y}}$.*

(2) *X_{ij}, $i, j \in \mathbb{N}$ are c.i. given $\mathscr{S}^{\mathbf{X}}$.*

Proof. (1) follows from Proposition (6.4) and Lemma (6.5) in [A2], and (2) is Proposition (14.7) in [A2].

LEMMA 3. *If the sequence (Y_i, A), $i \in \mathbb{N}$ is exchangeable, then $\mathbf{Y}$ and A are c.i. given $\mathscr{T}^{\mathbf{Y}}$.*

Proof. This result follows from (3.8) and (2.16) in [A2].

Proof of Theorem. First apply Lemma 3 to $Y_i = (X_{ij}, \xi_i; j \in \mathbb{N})$, $i \in \mathbb{N}$, and $A = (\alpha, \eta_j; j \in \mathbb{N})$ to get $\mathbf{Y}$ and A c.i. given $\mathscr{T}^{\mathbf{Y}}$. By Lemma 1 (2), $(\mathbf{X}, \boldsymbol{\xi})$ and $(\alpha, \boldsymbol{\eta})$ are c.i. given $\mathscr{S}^{\mathbf{X}} \vee \sigma(\boldsymbol{\xi})$. By Lemma 3 again, $\mathbf{X}$ and $\boldsymbol{\xi}$ are c.i. given the tail of $Z_j = (X_{ij}; i \in \mathbb{N})$, hence given $\mathscr{S}^{\mathbf{X}}$, by Lemma 1 (2) again. The result now follows by Lemma 1 (3), with $\mathscr{G} = \sigma(\mathbf{X})$, $\mathscr{H} = \sigma(\alpha, \boldsymbol{\eta})$, $\mathscr{F} = \sigma(\boldsymbol{\xi})$, and $\mathscr{E} = \mathscr{S}^{\mathbf{X}}$.

COROLLARY 1. *In the theorem, if $\mathbf{X}$ is i.i.d. then $\mathbf{X}$ and $(\alpha, \boldsymbol{\xi}, \boldsymbol{\eta})$ are independent. If $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are also i.i.d., then $\mathbf{X}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and α are independent.*

Proof. The first statement follows from the Theorem because if $\mathbf{X}$ is i.i.d., $\mathscr{S}^{\mathbf{X}}$ is trivial. If $\boldsymbol{\xi}$ is also i.i.d., then $\mathscr{T}^{\boldsymbol{\xi}}$ is trivial, hence, by Lemma 3,

$\boldsymbol{\xi}$ and $A = (\alpha, \boldsymbol{\eta})$ are independent. Similarly, $\boldsymbol{\eta}$ and α are independent. This proves the second statement.

The following generalization follows easily using Aldous' Representation Theorem (in fact it requires only part of the proof of that result). We give it by way of asking whether it has a direct proof like that we have given for our theorem. This corollary generalizes the theorem, because α, $\boldsymbol{\xi}$, and $\boldsymbol{\eta}$ are $\mathscr{S}^{(\alpha, \boldsymbol{\xi}, \boldsymbol{\eta})}$-measurable.

COROLLARY 2. *If* $\{(X_{ij}, Y_{ij}); i, j \in \mathbb{N}\}$ *is separately exchangeable then* $\mathbf{X}$ *and* $\mathscr{S}^{\mathbf{X},\mathbf{Y}}$ *are c.i. given* $\mathscr{S}^{\mathbf{X}}$.

Proof. By Lemma 2(2) X_{ij}, $i, j \in \mathbb{N}$ are c.i. given each of $\mathscr{S}^{\mathbf{X}}$ and $\mathscr{S}^{\mathbf{X},\mathbf{Y}}$, so it suffices to show that X_{11} and $\mathscr{S}^{\mathbf{X},\mathbf{Y}}$ are c.i. given $\mathscr{S}^{\mathbf{X}}$.

By the Representation Theorem, we may assume that there is a Borel function f and an i.i.d. family of random variables $\{\alpha, \xi_i, \eta_j, \xi_{ij}; i, j \in \mathbb{N}\}$, such that for each $i, j \in \mathbb{N}$, $(X_{ij}, Y_{ij}) = f(\alpha, \xi_i, \eta_j, \zeta_{ij})$. Then X_{11} and $\{(X_{ij}, Y_{ij}); i \vee j > 1\}$ are c.i. given α, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$. But, by the Theorem, X_{11} and α, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are c.i. given $\mathscr{S}^{\mathbf{X}}$. The result follows.

Our proofs generalize to prove the analogous results for exchangeable systems of processes with n indices, for any finite n (Corollary 5.10 in [H1]).

REFERENCES

[A1] ALDOUS, D. J. (1981). Representations for partially exchangeable arrays of random variables, *J. Multivariate Anal.* **11** 581–598.

[A2] ALDOUS, D. J. (1985). Exchangeability and related topics, in *École d'Été de Probabilités de Saint-Flour XIII*—1983 (P. L. Hennequin, Ed.), pp. 1–198. Lecture Notes in Mathematics, Vol. 1117, Springer-Verlag, Berlin.

[H1] HOOVER, D. N. (1981). Relations on probability spaces and arrays of random variables, preprint.

[H2] HOOVER, D. N. (1982). Row-column exchangeability and a generalized model for probability, in *Exchangeability in Probability and Statistics* (G. Koch and F. Spizzichino, Eds.), pp. 281–291, North-Holland, Amsterdam.

[K] KALLENBERG, O. On the presentation theorem for exchangeable arrays, *J. Multivariate Anal.* **30** 137–154.

[Lo] LOÈVE, M. (1955). *Probability Theory*, van Nostrand, New York.

[Ly] LYNCH, J. (1984). Canonical row-column exchangeable arrays, *J. Multivariate Anal.* **15** 135–140.

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