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Fuzzy random renewal process with queueing applications

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ABSTRACT

Using extension principle associated with a class of continuous Archimedean triangular norms, this paper studies a fuzzy random renewal process in which the interarrival times are assumed to be independent and identically distributed fuzzy random variables. Some limit theorems in chance measure and in expected value for the sum of fuzzy random variables are proved on the basis of the continuous Archimedean triangular norm based arithmetics. Furthermore, we discuss the fuzzy random renewal process based on the obtained limit theorems, and derive a fuzzy random elementary renewal theorem for the long-run expected renewal rate. The renewal theorem obtained in this paper can degenerate to the corresponding classical result in stochastic renewal process. Finally, two case studies of queueing systems are provided to illustrate the application of the fuzzy random elementary renewal theorem.

1. Introduction

It is widely assumed that in stochastic renewal processes, interarrival times and rewards are independent and identically distributed (i.i.d.) random variables, and stochastic renewal theory, based on probability theory, has been well developed (see [1,2]). In order to study the renewal process with imprecise and vague parameters, fuzzy renewal process was recently proposed and investigated by several researchers. For instance, Zhao and Liu [3] considered a fuzzy renewal process which is generated by a sequence of positive i.i.d. fuzzy variables, and obtained a fuzzy elementary renewal theorem and a fuzzy renewal reward theorem, respectively. Hong [4] discussed a renewal process in which interarrival times and rewards were depicted by *L-R* fuzzy variables under triangular norm (t-norm for short) based fuzzy operations.

In practical applications, owing to subjective judgement, imprecise human knowledge and perception, it is reasonable to suppose that statistic data also inherit uncertainty in the sense of vagueness. That is, vagueness should be integrated into techniques of statistical analysis, where the data is represented by fuzzy sets or fuzzy variables rather than real numbers or tuples of real numbers. Fuzzy random variable, which was introduced by Kwakernaak [5,6] is right the tool to study such hybrid uncertain phenomena where fuzziness and randomness exist at the same time. It was defined as a measurable function from a probability space to a collection of fuzzy numbers. Since then, its variants as well as extensions were developed by other researchers, e.g., Kruse and Meyer [7], Puri and Ralescu [8], Luhandjula [9], López-Diaz and Gil [10], and Liu and Liu [11].

Based on the concept of fuzzy random variable, some renewal processes in fuzzy random environment have been discussed in the literature. Hwang [12] investigated a renewal process in which the interarrival times are assumed as i.i.d. fuzzy random variables, and proved an almost sure convergence theorem with probability measure for the renewal rate. Modeling the rewards as i.i.d. fuzzy random variables, Popova and Wu [13] studied a fuzzy random renewal reward process

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and derived a theorem for the long-run average reward in the form of level-wise convergence with probability one. Zhao and Tang [14] discussed some properties of fuzzy random renewal processes, and obtained a Blackwell's renewal theorem and a Smith's key renewal theorem for fuzzy random interarrival times.

In the above researches on the fuzzy random renewal process, the operations of fuzzy realizations of fuzzy random variables which are fuzzy numbers or fuzzy variables, base the extension principle with the minimum t-norm. Nevertheless, several researches in the past two decades show that the classical extension principle is not always the optimal way to combine fuzzy numbers (e.g. image processing [15,16], measurement theory [17,18], fuzzy control [19,20], and other areas [21]). The operations associated with different kinds of t-norms may be required for the fuzzy numbers in different specific applications and situations. A more general extension principle makes use of a general t-norm operator. Such generalized extension principle yields different operators for fuzzy numbers or fuzzy variables, in accordance with different t-norms. Recently, a number of researches have been done with such t-norm based operators on fuzzy numbers (see Cooman [22], Deschrijver [23], Hong and Ro [24], Klement et al. [25], Yuan and Lee [26], and Guh et al. [27]), and fuzzy random variables (see Terán [28], Wang and Watada [29]).

Motivated by the above fact, in the present paper, we introduce the extension principle associated with a class of continuous Archimedean t-norms to the fuzzy random renewal process, and derive a fuzzy random elementary renewal theorem for the long-run expected renewal rate of the renewal process by using the expected value of fuzzy random variable. In contrast with [12–14] whose results hold only for minimum t-norm, the results obtained in this paper can be applied to more general situations for the class of continuous Archimedean t-norms, such as product t-norm, Dombi t-norm, and Yager t-norm. What's more, the present research not only derives the fuzzy random renewal theorems, but also discusses some critical properties and the nationality on the fuzzy random renewal process. Finally, an interesting fact is that the fuzzy random elementary renewal theorem derived in this paper can degenerate well to the classical elementary renewal theorem in stochastic process. The remainder of this paper is organized as follows. In Section 2, we recall some preliminaries on fuzzy variables and fuzzy random variables. Section 3 derives some useful limit theorems for the sum of fuzzy random variables. In Section 4, we discuss fuzzy random renewal process. Section 5 gives two queueing applications of the fuzzy random elementary renewal theorem. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

2.1. Fuzzy variables

Given a universe Γ, let Pos be a set function defined on the family of subsets of the universe, $\mathcal{P}(\Gamma)$. The set function Pos is said to be a possibility measure if it satisfies the following conditions

(P1) $\mathrm{Pos}(\emptyset) = 0$, and $\mathrm{Pos}(\Gamma) = 1$;
(P2) $\mathrm{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \mathrm{Pos}(A_i)$ for any subclass $\{A_i \mid i \in I\}$ of $\mathcal{P}(\Gamma)$, where I is an arbitrary index set.

The triplet $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Pos})$ is called a *possibility space*. Based on possibility measure, a self-dual set function Cr, called *credibility measure*, was defined as [30]:

$$\mathrm{Cr}(A) = \frac{1}{2}\left(1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)\right), \quad A \in \mathcal{P}(\Gamma) \tag{1}$$

where A^c is the complement of A.

A function $Y : \Gamma \to \Re$ is said to be a fuzzy variable defined on Γ (see [31]), and the possibility distribution μ_Y of Y is defined by $\mu_Y(t) = \mathrm{Pos}\{Y = t\}$, $t \in \Re$, which is the possibility of event $\{Y = t\}$. A fuzzy variable Y is said to be positive almost surely, if $\mathrm{Pos}\{Y \le 0\} = 0$.

Definition 1 (*[30]*). Let Y be a fuzzy variable. The expected value of Y is defined as

$$E[Y] = \int_0^{\infty} \mathrm{Cr}\{Y \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{Y \le r\}\mathrm{d}r \tag{2}$$

provided that one of the two integrals is finite.

Especially, for a positive fuzzy variable Y, $E[Y] = \int_0^{\infty} \mathrm{Cr}\{Y \ge r\}\mathrm{d}r$.

Example 1. Suppose $Y = (a, b, c)$ is a triangular fuzzy variable, whose possibility distribution is

$$\mu_Y(x) = \begin{cases} (x-a)/(b-a), & \text{if } a \le x \le b \\ (c-x)/(c-b), & \text{if } b \le x \le c \\ 0, & \text{otherwise.} \end{cases}$$

By (1) and (2), we can compute the expected value of Y as

$$E[Y] = \frac{a + 2b + c}{4}.$$

Now, we recall the concept of triangular norm (t-norm). A t-norm is a function $\top : [0, 1]^2 \to [0, 1]$ such that for any $x, y, z \in [0, 1]$ the following four axioms are satisfied [25]:

(T1) Commutativity: $\top(x, y) = \top(y, x)$.
(T2) Associativity: $\top(x, \top(y, z)) = \top(\top(x, y), z)$.
(T3) Monotonicity: $\top(x, y) \le \top(x, z)$ whenever $y \le z$.
(T4) Boundary condition: $\top(x, 1) = x$.

The associativity (T2) allows us to extend each t-norm $\top$ in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, \ldots, x_n) \in [0, 1]^n$

$$\top_{k=1}^{n} x_k = \top(\top_{k=1}^{n-1} x_k, x_n) = \top(x_1, x_2, \ldots, x_n).$$

A t-norm $\top$ is said to be Archimedean if $\top(x, x) < x$ for all $x \in (0, 1)$. It is easy to check that the minimum t-norm is not Archimedean.

Moreover, from Schweizer and Sklar [32], every continuous Archimedean t-norm $\top$ can be represented by a continuous and strictly decreasing function $f : [0, 1] \to [0, \infty]$ with $f(1) = 0$ and

$$\top(x_1, \ldots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n)) \tag{3}$$

for all $x_i \in (0, 1)$, $1 \le i \le n$, where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y), & \text{if } y \in [0, f(0)] \\ 0, & \text{if } y \in (f(0), \infty). \end{cases}$$

The function f is called the additive generator of $\top$.

Example 2. Some continuous Archimedean t-norms with additive generators are listed as follows:

(1) Yager t-norm ($\lambda \in (0, \infty)$):

$$\top_{\lambda}^{Y}(x, y) = \max\left\{1 - \sqrt[\lambda]{(1-x)^{\lambda} + (1-y)^{\lambda}}, 0\right\}$$

with additive generator $f_{\lambda}^{Y}(x) = (1-x)^{\lambda}$.

(2) Dombi t-norm ($\lambda \in (0, \infty)$):

$$\top_{\lambda}^{D}(x, y) = \frac{1}{1 + \sqrt[\lambda]{(\frac{1-x}{x})^{\lambda} + (\frac{1-y}{y})^{\lambda}}}$$

with additive generator $f_{\lambda}^{D}(x) = ((1-x)/x)^{\lambda}$.

(3) Product t-norm: $\top^{P}(x, y) = xy$ with additive generator $f^{P} = -\log$.

For fuzzy variables Y_k, $1 \le k \le m$ with possibility distributions μ_k, $1 \le k \le m$, and a function $g : \Re^m \to \Re$, the possibility distribution of $g(Y_1, Y_2, \ldots, Y_m)$ is determined by the possibility distributions $\mu_1, \mu_2, \ldots, \mu_m$ via the following generalized extension principle

$$\mu_{g(Y_1, Y_2, \ldots, Y_m)}(x) = \sup_{x_1, x_2, \ldots x_m \in \Re} \left\{ \top_{k=1}^{m} \mu_k(x_k) \mid x = g(x_1, x_2, \ldots, x_m) \right\}, \tag{4}$$

where $\top$ can be any general triangular norm. For a general triangular norm, by extension principle (4), the possibility distribution of arithmetic mean $(Y_1 + \cdots + Y_n)/n$ is

$$\mu_{\frac{1}{n}(Y_1+\cdots+Y_n)}(z) = \sup_{x_1+\cdots+x_n = nz} \top(\mu_{Y_1}(x_1), \ldots, \mu_{Y_n}(x_n)). \tag{5}$$

Furthermore, for a continuous Archimedean t-norm with additive generator f, from (3) and (5), the possibility distribution of $(Y_1 + \cdots + Y_n)/n$ can be determined by

$$\mu_{\frac{1}{n}(Y_1+\cdots+Y_n)}(r) = f^{[-1]}\left(\inf_{x_1+\cdots+x_n = nr} \sum_{k=1}^{n} f(\mu_{Y_k}(x_k)) \right). \tag{6}$$

2.2. Fuzzy random variables

We assume that $(\Omega, \Sigma, \Pr)$ is a probability space, and $\mathcal{F}_v$ is a collection of fuzzy variables defined on a possibility space $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Pos})$.

Definition 2 (*[11]*). A fuzzy random variable is a map $\xi : \Omega \to \mathcal{F}_v$ such that for any Borel subset B of $\Re$, the function $\xi^*(B)(\omega) = \mathrm{Pos}\{\gamma \in \Gamma \mid \xi(\omega)(\gamma) \in B\}$ is measurable with respect to ω.

Example 3. Let V be a random variable defined on probability space $(\Omega, \Sigma, \Pr)$. Define that for every $\omega \in \Omega, \xi(\omega) = (V(\omega) - 2, V(\omega) + 2, V(\omega) + 6)$ which is a triangular fuzzy variable defined on some possibility space $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Pos})$. Then, ξ is a (triangular) fuzzy random variable.

The measurability criteria of fuzzy random variable can be found in [33]. Furthermore, in order to measure a fuzzy random event, the mean chance measure is defined as follows. The readers who intend to learn more properties of the mean chance may refer to [34–36]

Definition 3 (*[34]*). Let ξ be a fuzzy random variable, and B a Borel subset of $\Re$. The mean chance of an event $\xi \in B$ is defined as

$$\mathrm{Ch}\{\xi \in B\} = \int_{\Omega} \mathrm{Cr}\{\xi(\omega) \in B\} \Pr(\mathrm{d}\omega). \tag{7}$$

Example 4. For the triangular fuzzy random variable ξ defined in Example 3, suppose V is a discrete random variable, which takes values $V_1 = 3$ with probability 0.4, and $V_2 = 6$ with probability 0.6. Try to find the mean chance of fuzzy random event $\{\xi \le 9\}$.

Recall that fuzzy random variable ξ takes fuzzy variables $\xi(V_1) = (1, 5, 9)$ with probability 0.2, and $\xi(V_2) = (4, 8, 12)$ with probability 0.8, by the definition, we can compute that $\mathrm{Cr}\{\xi(V_1) \le 9\} = 1$, with probability 0.4, and $\mathrm{Cr}\{\xi(V_2) \le 9\} = 0.625$, with probability 0.6. From (7), we can calculate $\mathrm{Ch}\{\xi \le 9\} = 1 \times 0.4 + 0.625 \times 0.6 = 0.775$.

Definition 4 (*[11]*). Let ξ be a fuzzy random variable defined on a probability space $(\Omega, \Sigma, \Pr)$. The expected value of ξ is defined as

$$E[\xi] = \int_{\Omega} \left[\int_0^{\infty} \mathrm{Cr}\{\, \xi(\omega) \ge r \,\}\, \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\, \xi(\omega) \le r \,\}\, \mathrm{d}r \right] \Pr(\mathrm{d}\omega). \tag{8}$$

Example 5. Consider the triangular fuzzy random variable ξ defined in Examples 3 and 5. Let's calculate the expected value of ξ.

From the distribution of random variable V, we know the fuzzy random variable ξ takes fuzzy variables $\xi(V_1) = (1, 5, 9)$ with probability 0.4, and $\xi(V_2) = (4, 8, 12)$ with probability 0.6. Next, we need to compute the expected values of fuzzy variables $\xi(V_1)$ and $\xi(V_2)$, respectively. That is $E[\xi(V_1)] = \frac{1+2\times 5+9}{4} = 5$, with probability 0.4, and $E[\xi(V_2)] = \frac{4+2\times 8+12}{4} = 8$, with probability 0.6. Finally, by the definition (8), the expected value of ξ is $E[\xi] = E[\xi(V_1)] \times 0.4 + E[\xi(V_2)] \times 0.6 = 6.8$.

Based on the mean chance of a fuzzy random event, it has been proved that (8) is equivalent to the following form (see [34]):

$$E[\xi] = \int_0^{\infty} \mathrm{Ch}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Ch}\{\xi \le r\}\mathrm{d}r. \tag{9}$$

Definition 5 (*[11]*). The fuzzy random variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be i.i.d. if and only if

$$\Big(\mathrm{Pos}\{\xi_k(\omega) \in B_1\}, \mathrm{Pos}\{\xi_k(\omega) \in B_2\}, \ldots, \mathrm{Pos}\{\xi_k(\omega) \in B_m\}\Big), \quad k = 1, 2, \ldots, n$$

are i.i.d. random vectors for any positive integer m and Borel sets $B_1, B_2, \ldots, B_m$ of $\Re$.

Proposition 1 (*[11]*). *If $\xi_1, \xi_2, \ldots, \xi_n$ are i.i.d. fuzzy random variables, then $E[\xi_k(\omega)], k = 1, 2, \ldots, n$ are i.i.d. random variables.*

For a sequence of fuzzy random variables, we have the following convergence mode.

Definition 6 (*[37]*). A sequence $\{\xi_n\}$ of fuzzy random variables is said to converge in chance Ch to a fuzzy random variable ξ, denoted by $\xi_n \xrightarrow{\mathrm{Ch}} \xi$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Ch}\{|\xi_n - \xi| \ge \varepsilon\} = 0.$$

3. Some limit theorems

In this paper, our discussion will be conducted under the following conditions:

A1. The operations of fuzzy variables are determined by the generalized extension principle (4), and $\top$ denotes any continuous Archimedean t-norm with an additive generator f.

A2. Π is a nonnegative real-valued function with $\Pi(0) = 1$, and Π is nonincreasing on $\Re^+$, nondecreasing on $\Re^-$.

In condition A1, the generalized extension principle provides any continuous Archimedean t-norm operator for fuzzy variables. As to condition A2, the function Π is called a possibility function which is used to represent the possibility distribution of a fuzzy variable. Through possibility function Π, we can construct convex possibility distributions such as triangular and norm distributions of fuzzy variables, where such convexity is critical to our desired results.

Suppose that $\{Y_n\}$ is a sequence of fuzzy variables with the same possibility distribution Π, from [24, Lemma 5] we know the possibility distribution $\mu_{Y_1+\cdots+Y_n}$ is also nonincreasing on $\Re^+$, and nondecreasing on $\Re^-$.

Denote Ξ as the support of possibility function Π, i.e., the closure of subset $\{t \in \Re \mid \Pi(t) > 0\}$ of $\Re$. For possibility distribution Π and Archimedean t-norm $\top$ with additive generater f, since $f : [0, 1] \to [0, \infty]$ is continuous and strictly decreasing, we know $f \circ \Pi : \Re \to [0, \infty]$ is nonincreasing on $\Re^-$, nondecreasing on $\Re^+$ with $f \circ \Pi(0) = 0$, and $f \circ \Pi(x) = f(0)$ for any $x \notin \Xi$.

Now, we consider the convex hull of the composition function $f \circ \Pi$ on Ξ, denoted $\mathrm{co}(f \circ \Pi)$, which is defined as

$$\mathrm{co}(f \circ \Pi)(z) = \inf \left\{ \sum_{k=1}^{n} \lambda_k (f \circ \Pi)(x_k) \right\} \quad (z \in \Xi) \tag{10}$$

where the infimum is taken over all representations of z as a (finite) convex combination $\sum_{k=1}^{n} \lambda_k x_k$ of points of Ξ. From the knowledge of convex analysis (see Tiel [38]), we know $\mathrm{co}(f \circ \Pi)$ is the largest convex function $h(x)$ such that $h(x) \le f \circ \Pi(x)$, for $x \in \Xi$.

Example 6. Let Π be the possibility distribution of triangular fuzzy variable $(-1, 0, 3)$, $\top^Y_\lambda(x, y)$ is a Yager t-norm with additive generator $f^Y_\lambda(x) = (1 - x)^\lambda$ where $\lambda \ge 1$. Find the convex hull of $f^Y_\lambda \circ \Pi$ on $\Xi = [-1, 3]$.

Since

$$\Pi(x) = \begin{cases} x + 1, & \text{if } x \in [-1, 0] \\ 1 - \dfrac{x}{3}, & \text{if } x \in [0, 3] \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$f^Y_\lambda \circ \Pi(x) = \begin{cases} (-x)^\lambda, & \text{if } x \in [-3, 0] \\ \left(\dfrac{x}{3}\right)^\lambda, & \text{if } x \in [0, 6] \\ 1, & \text{otherwise,} \end{cases}$$

which is a convex function on $[-1, 3]$. Thus, we have $\mathrm{co}(f^Y_\lambda \circ \Pi) = f^Y_\lambda \circ \Pi$ on $[-1, 3]$.

Before discussing the limit theorems for fuzzy random variables, we first give the following basic result on fuzzy variables.

Lemma 1. *Let Y_k, $k = 1, 2, \ldots$ be a sequence of fuzzy variables with identical possibility distribution Π, and $S_n = Y_1 + \cdots + Y_n$. If $\mathrm{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$, then*

$$\lim_{n\to\infty} \mu_{\frac{1}{n}S_n}(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof can be divided into the following three cases:

Case 1. $z = 0$. We have

$$\mu_{\frac{1}{n}S_n}(0) = \sup_{x_1+\cdots+x_n=0} \top(\mu_{Y_1}(x_1), \ldots, \mu_{Y_n}(x_n)) \ge \top(\Pi(0), \ldots, \Pi(0)) = 1.$$

Case 2. $z \ne 0$, $z \notin \Xi$. For any $\{x_1, x_2, \ldots, x_n\}$ with $x_1 + \cdots + x_n = nz$, there must be two points x_i and x_j, $1 \le i, j \le n$ such that $x_i \le z$ and $x_j \ge z$, which implies $\top(\Pi(x_i), \Pi(x_j)) \le \Pi(z)$. Therefore $\top(\Pi(x_1), \ldots, \Pi(x_n)) \le \Pi(z)$. It follows that

$$\mu_{\frac{1}{n}S_n}(z) = \sup_{x_1+\cdots+x_n=nz} \top(\Pi(x_1), \ldots, \Pi(x_n)) \le \Pi(z) = 0.$$

Case 3. $z \neq 0, z \in \Xi$. In this case, we note that if $x_k \notin \Xi$ for some k, then $f^{[-1]}\left(\sum_{k=1}^{n} f \circ \Pi(x_k)\right) = 0$. Therefore, by (3) and (5), we have

$$\mu_{\frac{1}{n}S_n}(z) = f^{[-1]}\left(\inf_{x_1+\cdots+x_n=nz} \sum_{k=1}^{n} f(\mu_{Y_k}(x_k))\right)$$
$$= f^{[-1]}\left(\inf_{\substack{x_1+\cdots+x_n=nz \\ x_k\in\Xi, 1\leq k\leq n}} \sum_{k=1}^{n} f \circ \Pi(x_k)\right).$$

By (10), we obtain

$$\inf_{\substack{x_1+\cdots+x_n=nz \\ x_k\in\Xi, 1\leq k\leq n}} \frac{1}{n}\sum_{k=1}^{n} f \circ \Pi(x_k) \geq \mathrm{co}(f \circ \Pi)(z),$$

or, equivalently,

$$\inf_{\substack{x_1+\cdots+x_n=nz \\ x_k\in\Xi, 1\leq k\leq n}} \sum_{k=1}^{n} f \circ \Pi(x_k) \geq n \cdot \mathrm{co}(f \circ \Pi)(z).$$

Since $f^{[-1]}$ is nonincreasing, we can deduce

$$\mu_{\frac{1}{n}S_n}(z) \leq f^{[-1]}(n \cdot \mathrm{co}(f \circ \Pi)(z)).$$

Noting that $\mathrm{co}(f \circ \Pi)(r) > 0$ for any nonzero $r \in \Xi$, we have

$$\mu_{\frac{1}{n}S_n}(z) \leq f^{[-1]}(n \cdot \mathrm{co}(f \circ \Pi)(z)) \to 0 \quad (n \to \infty).$$

Combining the above three cases proves the Lemma. □

Theorem 1. *Assume $\{\xi_k\}$ is a sequence of i.i.d. fuzzy random variables with $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ for almost every $\omega \in \Omega$, where U_k, $k = 1, 2, \ldots$, are random variables with finite expected values. If $\mathrm{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$, then we have*

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \xrightarrow{\mathrm{Ch}} E[U_1]. \tag{11}$$

Proof. Denoting $X_k = \xi_k - U_k$ and $S_n^* = X_1 + \cdots + X_n$, we know that for almost every $\omega \in \Omega$, $X_k(\omega) = \xi_k(\omega) - U_k(\omega)$ and $S_n^*(\omega) = X_1(\omega) + \cdots + X_n(\omega)$ are fuzzy variables, and possibility distributions $\mu_{X_k(\omega)}(x) = \Pi(x)$, $k = 1, 2, \ldots$. We first prove the following limit is valid,

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\left|\frac{1}{n}S_n^*\right| \geq \varepsilon\right\} = 0. \tag{12}$$

For any $\varepsilon > 0$ and almost every $\omega \in \Omega$, by Lemma 1, we have

$$\mathrm{Cr}\left\{\left|\frac{1}{n}S_n^*(\omega)\right| \geq \varepsilon\right\} \leq \mathrm{Pos}\left\{\frac{1}{n}S_n^*(\omega) \geq \varepsilon\right\} \vee \mathrm{Pos}\left\{\frac{1}{n}S_n^*(\omega) \leq -\varepsilon\right\}$$
$$= \mu_{\frac{1}{n}S_n^*(\omega)}(\varepsilon) \vee \mu_{\frac{1}{n}S_n^*(\omega)}(-\varepsilon) \to 0\ (n \to \infty). \tag{13}$$

Using dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\left|\frac{1}{n}S_n^*\right| \geq \varepsilon\right\} = \int_{\Omega} \lim_{n\to\infty} \mathrm{Cr}\left\{\left|\frac{1}{n}S_n^*(\omega)\right| \geq \varepsilon\right\} \mathrm{Pr}(\mathrm{d}\omega) = 0.$$

Therefore, (12) is valid.

Furthermore, since ξ_k, $k = 1, 2, \ldots$ are i.i.d. fuzzy random variables, by Proposition 1, we have $E[\xi_k(\omega)]$, $k = 1, 2, \ldots$ are i.i.d. random variables. Noting that $X_k(\omega)$, $k = 1, 2, \ldots, n$ have identical possibility distribution Π for almost every $\omega \in \Omega$, they own the same expected value e, i.e., $E[X_k(\omega)] \equiv e$ for any $k = 1, 2, \ldots, n$ and almost every $\omega \in \Omega$, where e is a real number. Therefore, we have

$$E[\xi_k(\omega)] = e + U_k(\omega)$$

for any $k = 1, 2, \ldots, n$ and almost every $\omega \in \Omega$, which implies U_k, $k = 1, 2, \ldots$ are i.i.d. random variables. By the strong law of large numbers for random variables, we get

$$\Pr\left\{\omega \in \Omega \mid \frac{1}{n}\sum_{k=1}^{n} U_k(w) \to E[U_1]\right\} = 1.$$

Combining with (13), we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Cr}\left\{\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k(\omega) - E[U_1]\right| \geq \varepsilon\right\} &= \lim_{n\to\infty} \mathrm{Cr}\left\{\left|\frac{1}{n}S_n^*(\omega) + \frac{1}{n}\sum_{k=1}^{n} U_k(\omega) - E[U_1]\right| \geq \varepsilon\right\} \\ &\leq \lim_{n\to\infty}\left[\mathrm{Cr}\left\{\left|\frac{1}{n}S_n^*(\omega)\right| \geq \frac{\varepsilon}{2}\right\} + \mathrm{Cr}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} U_k(\omega) - E[U_1]\right| \geq \frac{\varepsilon}{2}\right\}\right] = 0\end{aligned}$$

for almost every $\omega \in \Omega$.

It follows from dominated convergence theorem that

$$\lim_{n\to\infty} \mathrm{Ch}\left\{\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k - E[U_1]\right| \geq \varepsilon\right\} = \int_{\Omega}\lim_{n\to\infty} \mathrm{Cr}\left\{\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k(\omega) - E[U_1]\right| \geq \varepsilon\right\}\Pr(\mathrm{d}\omega) = 0.$$

The proof of the theorem is complete. □

Theorem 2. *Assume $\{\xi_k\}$ is a sequence of i.i.d. fuzzy random variables with $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ for almost every $\omega \in \Omega$, where U_k, $k = 1, 2, \ldots$, are random variables with finite expected values. If $\mathrm{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$, and $E[\xi_1] < \infty$, then we have*

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{k=1}^{n}\xi_k\right] = E[U_1]. \tag{14}$$

Proof. Denote $X_k = \xi_k - U_k$, $S_n^* = X_1 + \cdots + X_n$. Since $\{\xi_n\}$ is a sequence of i.i.d. fuzzy random variables, from the proof of Theorem 1, we have U_k, $k = 1, 2, \ldots$, are i.i.d. random variables with finite expected values. Therefore,

$$E\left[\frac{1}{n}\sum_{k=1}^{n}\xi_k\right] - E[U_1] = E\left[\frac{1}{n}\sum_{k=1}^{n}\xi_k\right] - E\left[\frac{1}{n}\sum_{k=1}^{n}U_k\right] = E\left[\frac{1}{n}S_n^*\right].$$

Hence, (14) is equivalent to

$$\lim_{n\to\infty} E\left[\frac{1}{n}S_n^*\right] = 0.$$

We note that

$$E\left[\frac{1}{n}S_n^*\right] = \int_0^{\infty}\mathrm{Ch}\left\{\frac{1}{n}S_n^* \geq r\right\}\mathrm{d}r - \int_{-\infty}^{0}\mathrm{Ch}\left\{\frac{1}{n}S_n^* \leq r\right\}\mathrm{d}r,$$

to prove the theorem, it suffices to prove

$$\lim_{n\to\infty}\int_0^{\infty}\mathrm{Ch}\left\{\frac{1}{n}S_n^* \geq r\right\}\mathrm{d}r = 0 \tag{15}$$

and

$$\lim_{n\to\infty}\int_{-\infty}^{0}\mathrm{Ch}\left\{\frac{1}{n}S_n^* \leq r\right\}\mathrm{d}r = 0. \tag{16}$$

Since $\mathrm{co}(f \circ \Pi)(x) > 0$, for all nonzero $x \in \Xi$, given any $r > 0$ and almost every $\omega \in \Omega$, by Lemma 1, we have

$$\mathrm{Cr}\left\{\frac{1}{n}S_n^*(\omega) \geq r\right\} \leq \mathrm{Pos}\left\{\frac{1}{n}S_n^*(\omega) \geq r\right\} = \sup_{t\geq r}\mu_{\frac{1}{n}S_n^*(\omega)}(t) = \mu_{\frac{1}{n}S_n^*(\omega)}(r) \to 0$$

as $n \to \infty$. By dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\mathrm{Ch}\left\{\frac{1}{n}S_n^* \geq r\right\} = \int_{\Omega}\lim_{n\to\infty}\mathrm{Cr}\left\{\frac{1}{n}S_n^*(\omega) \geq r\right\}\Pr(\mathrm{d}\omega) = 0.$$

Noting that for almost every $\omega \in \Omega$, fuzzy variables $X_k(\omega)$, $k = 1, 2, \ldots$ have identical possibility distribution Π, then we can deduce

$$\mathrm{Cr}\left\{\frac{1}{n}S_n^*(\omega) \geq r\right\} \leq \mu_{\frac{1}{n}S_n^*(\omega)}(r) = \sup_{(x_1+x_2+\cdots+x_n)/n=r} \top_{k=1}^{n}\Pi(x_k) \leq \Pi(r) = \mu_{X_1(\omega)}(r) \tag{17}$$

for any $r > 0$ and $n = 1, 2, \ldots$.

From $\mu_{X_1(\omega)}(r) = \mathrm{Pos}\{X_1(\omega) \geq r\} < 1$, we obtain

$$\mu_{X_1(\omega)}(r) = 2\mathrm{Cr}\{X_1(\omega) \geq r\}$$

for any $r > 0$. Hence, (17) implies

$$\mathrm{Cr}\left\{\frac{1}{n}S_n^*(\omega) \geq r\right\} \leq 2\mathrm{Cr}\{X_1(\omega) \geq r\}, \quad n = 1, 2, \ldots.$$

Integrating with respect to ω on the above inequality, we obtain

$$\mathrm{Ch}\left\{\frac{1}{n}S_n^* \geq r\right\} \leq 2\mathrm{Ch}\{X_1 \geq r\}, \quad n = 1, 2, \ldots.$$

Since $E[X_1] = E[\xi_1 - U_1] = E[\xi_1] - E[U_1] < \infty$, applying Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^{\infty} \mathrm{Ch}\left\{\frac{1}{n}S_n \geq r\right\} \mathrm{d}r = \int_0^{\infty} \lim_{n\to\infty} \mathrm{Ch}\left\{\frac{1}{n}S_n \geq r\right\} \mathrm{d}r = 0,$$

which implies (15) is valid.

With the similar method, we can prove (16) is valid. The proof of the theorem is complete. □

Remark 1. From Lemma 1, Theorems 1 and 2, we see that a critical convexity condition is that $\mathrm{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$, which is also indispensable to the main results of the next section. This convexity condition is determined completely by the composition of the possibility function Π and the additive function of the chosen t-norm. The following example gives a situation that it is invalid.

Example 7. Consider the following possibility function Π,

$$\Pi(x) = \begin{cases} x+1, & \text{if } x \in [-1, 0] \\ \mathrm{e}^{-x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < -1, \end{cases}$$

and Yager t-norm $T_1^Y(x, y)$ with additive generator $f_1^Y(x) = 1 - x$. Find $\mathrm{co}(f_1^Y \circ \Pi)(x)$ on Ξ.

Since

$$f_1^Y \circ \Pi(x) = \begin{cases} -x, & \text{if } x \in [-1, 0] \\ 1-\mathrm{e}^{-x}, & \text{if } x \in [0, \infty) \\ 1, & \text{if } x < -1, \end{cases}$$

we can find that the convex hull of $f_1^Y \circ \Pi$ on $\Xi = [-1, \infty)$ is

$$\mathrm{co}(f_1^Y \circ \Pi)(x) = \begin{cases} -x, & \text{if } x \in [-1, 0] \\ 0, & \text{if } x \in [0, \infty). \end{cases}$$

4. Fuzzy random renewal process

Let ξ_n, $n = 1, 2, \ldots$ be a sequence of fuzzy random variables defined on the probability space $(\Omega, \Sigma, \Pr)$. For each n, we denote ξ_n the interarrival time between the $(n-1)$th and nth event. Define

$$S_0 = 0, \qquad S_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1.$$

It is clear that S_n is the time when the nth renewal occurs. Let $N(t)$ denote the total number of the events that have occurred by time t. Then we have

$$N(t) = \max\{n \mid S_n \leq t\}.$$

For any $\omega \in \Omega$, $N(t)(\omega) = \max\{n \mid S_n(\omega) \le t\}$ is a nonnegative integer-valued fuzzy variable on the possibility space $(\Gamma, \mathcal{P}(\Gamma), \text{Pos})$, and furthermore, $N(t)(\omega)(\gamma)$ is a nonnegative real integer for any $\gamma \in \Gamma$. We call $N(t)$ a fuzzy random renewal variable, and the process $\{N(t), t > 0\}$ a fuzzy random renewal process.

In addition, for any given $\omega \in \Omega$ and integer n, $S_n(\omega) = \xi_1(\omega) + \cdots + \xi_n(\omega)$ is also a fuzzy variable, and for any $t > 0$ we have the following equivalent events:

$$N(t)(\omega) < n \Longleftrightarrow S_n(\omega) > t,$$
$$N(t)(\omega) \ge n \Longleftrightarrow S_n(\omega) \le t,$$
$$N(t)(\omega) = n \Longleftrightarrow S_n(\omega) \le t < S_{n+1}(\omega).$$

Under the conditions A1, A2, in this section, we assume that the t-norm $\top$ with additive generator f and possibility function Π satisfy the following condition.

A3. $\text{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$, and there exits a positive real number a such that $\Pi(-a) = 0$.

What's more, we assume that the interarrival times $\{\xi_n\}$ between two events are almost positive fuzzy random variables, that is for each n, and almost every $\omega \in \Omega$, fuzzy variable $\xi_n(\omega)$ is positive almost surely. Therefore, in the following, we use fuzzy random variables ξ_k with $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ to characterize the fuzzy random interarrival times, where each random variable $U_k \ge a$ almost surely.

Theorem 3. *Assume $\{\xi_n\}$ is a sequence of fuzzy random interarrival times with $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ for almost every $\omega \in \Omega$, where random variables $U_k \ge a$ almost surely, and $N(t)$ is the fuzzy random renewal variable. Then we have*

$$\frac{1}{N(t)} \xrightarrow{\text{Ch}} 0.$$

Proof. Denoting $X_k = \xi_k - U_k$, $S_n^* = X_1 + \cdots + X_n$, we have

$$\begin{aligned} \Pi(n) &= \top\big(\Pi(0), \ldots, \Pi(0), \Pi(n)\big) \\ &\le \sup_{x_1+x_2+\cdots+x_n=n} \top_{k=1}^{n} \mu_{X_k(\omega)}(x_k) \\ &= \mu_{S_n^*(\omega)}(n) = \mu_{\frac{1}{n}S_n^*(\omega)}(1). \end{aligned}$$

By Lemma 1, we have $\lim_{t\to\infty} \Pi(t) = 0$.

Moreover, for any $\varepsilon > 0$, we let M be the smallest integer such that $M > 1/\varepsilon$. Since for any $\omega \in \Omega$ and $t > 0$,

$$1/N(t)(\omega) \ge \varepsilon \Longleftrightarrow N(t)(\omega) < M,$$

we have

$$\begin{aligned} N(t)(\omega) < M &\Longleftrightarrow S_M(\omega) > t \\ &\Longleftrightarrow \frac{S_M(\omega)}{M} - \frac{1}{M}\sum_{k=1}^{M} U_k(\omega) > \frac{t}{M} - \frac{1}{M}\sum_{k=1}^{M} U_k(\omega) \\ &\Longleftrightarrow \frac{S_M^*(\omega)}{M} > \frac{t}{M} - \frac{1}{M}\sum_{k=1}^{M} U_k(\omega). \end{aligned}$$

Without losing any generality, we let $t > \sum_{k=1}^{M} U_k(\omega)$. Therefore,

$$\begin{aligned} \text{Ch}\left\{\frac{1}{N(t)} \ge \varepsilon\right\} &= \int_\Omega \text{Cr}\left\{\frac{1}{N(t)(\omega)} \ge \varepsilon\right\} \Pr(\mathrm{d}\omega) \\ &= \int_\Omega \text{Cr}\left\{\frac{S_M^*(\omega)}{M} > \frac{t}{M} - \frac{1}{M}\sum_{k=1}^{M} U_k(\omega)\right\} \Pr(\mathrm{d}\omega) \\ &\le \int_\Omega \text{Pos}\left\{\frac{S_M^*(\omega)}{M} > \frac{t}{M} - \frac{1}{M}\sum_{k=1}^{M} U_k(\omega)\right\} \Pr(\mathrm{d}\omega) \\ &\le \int_\Omega \Pi\left(\frac{1}{M} \cdot \left(t - \sum_{k=1}^{M} U_k(\omega)\right)\right) \Pr(\mathrm{d}\omega). \end{aligned}$$

Noting that $\lim_{t\to\infty} \Pi(t) = 0$, it follows from dominated convergence theorem that

$$\lim_{t\to\infty} \text{Ch}\left\{\frac{1}{N(t)} \ge \varepsilon\right\} = \int_\Omega \lim_{t\to\infty} \text{Cr}\left\{\frac{1}{N(t)(\omega)} \ge \varepsilon\right\} \Pr(\mathrm{d}\omega) = 0.$$

The proof is complete. □

Remark 2. Theorem 3 is critical, since it implies the rationality of the fuzzy random renewal process modeled in this paper. From the result of Theorem 3, we can see that as in the stochastic renewal process, the total number of renewals that occurs in the fuzzy random process is infinite, which implies each interarrival time should be finite.

Theorem 4. *Suppose that* $\{\xi_k\}$ *is a sequence of i.i.d. fuzzy random interarrival times with* $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ *for almost every* $\omega \in \Omega$*, where each* U_k *is a random variable with finite expected value such that* $U_k \geq a$ *almost surely, and* $N(t)$ *is the fuzzy random renewal variable. Then we have*

$$\frac{S_{N(t)}}{N(t)} \xrightarrow{\text{Ch}} E[U_1].$$

Proof. First of all, we shall prove $S_{N(t)(\omega)}(\omega)/N(t)(\omega) \xrightarrow{\text{Cr}} E[U_1]$ for almost every $\omega \in \Omega$. Since for sequences of fuzzy variables, convergence in credibility is equivalent to convergence almost uniformly, it suffices to prove

$$\Pr\left\{\omega \in \Omega \mid \frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} \xrightarrow{a.u.} E[U_1]\right\} = 1.$$

On one hand, from the proof of Theorem 1, we know that for almost every $\omega \in \Omega$, $S_n(\omega)/n \xrightarrow{\text{Cr}} E[U_1]$, which is equivalent to $S_n(\omega)/n \xrightarrow{a.u.} E[U_1]$. Thus, for any $\delta > 0$, there exists $A \in \mathcal{P}(\Gamma)$ such that $\text{Cr}(A) < \delta/2$, and $\{S_n(\omega)/n\}$ converges uniformly to $E[U_1]$ on $\Gamma \setminus A$. Hence, for any $\varepsilon > 0$, there exists a positive integer M such that for every $\gamma \in \Gamma \setminus A$,

$$\left|\frac{S_n(\omega)(\gamma)}{n} - E[U_1]\right| < \varepsilon$$

whenever $n \geq M$.

On the other hand, from the proof of Theorem 3, we get $1/N(t)(\omega) \xrightarrow{\text{Cr}} 0$, which is equivalent to $1/N(t)(\omega) \xrightarrow{a.u.} 0$. Therefore, for any $\delta > 0$, there exists $B \in \mathcal{P}(\Gamma)$ such that $\text{Cr}(B) < \delta/2$, and $\{1/N(t)(\omega)\}$ converges uniformly to zero on $\Gamma \setminus B$. Thus, for the above positive integer M, there exists a positive real number t_M such that for every $\gamma \in \Gamma \setminus B$,

$$\frac{1}{N(t)(\omega)(\gamma)} \leq \frac{1}{M}$$

whenever $t \geq t_M$.

Combining the above, we obtain that for any $\delta > 0$, there exists $A \cup B \in \mathcal{P}(\Gamma)$ such that $\text{Cr}\{A \cup B\} \leq \text{Cr}\{A\} + \text{Cr}\{B\} < \delta$, and $\left\{S_{N(t)(\omega)}(\omega)/N(t)(\omega)\right\}$ converges uniformly to $E[U_1]$ on $\Gamma \setminus (A \cup B)$. In fact, for any $\varepsilon > 0$, there exists a positive real number t_M such that for all $\gamma \in \Gamma \setminus (A \cup B)$, we have

$$N(t)(\omega)(\gamma) \geq M, \quad \text{and} \quad \left|\frac{S_{N(t)(\omega)(\gamma)}(\omega)(\gamma)}{N(t)(\omega)(\gamma)} - E[U_1]\right| < \varepsilon$$

whenever $t \geq t_M$. Therefore, for almost every $\omega \in \Omega$,

$$\frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} \xrightarrow{a.u.} E[U_1],$$

which is equivalent to

$$\Pr\left\{\omega \in \Omega \mid \frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} \xrightarrow{\text{Cr}} E[U_1]\right\} = 1.$$

Furthermore, applying dominated convergence theorem, we have

$$\lim_{t\to\infty} \text{Ch}\left\{\left|\frac{S_{N(t)}}{N(t)} - E[U_1]\right| \geq \varepsilon\right\} = \int_\Omega \lim_{t\to\infty} \text{Cr}\left\{\left|\frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} - E[U_1]\right| \geq \varepsilon\right\} \Pr(\mathrm{d}\omega) = 0,$$

the desired result is valid. □

Theorem 5. *Assume* $\{\xi_k\}$ *is a sequence of i.i.d. fuzzy random interarrival times with* $\mu_{\xi_k(\omega)}(x) = \Pi(x - U_k(\omega))$ *for almost every* $\omega \in \Omega$*, where each* U_k *is a random variable with finite expected value such that* $U_k \geq a$ *almost surely, and* $N(t)$ *is the fuzzy random renewal variable. Then we have*

$$\frac{N(t)}{t} \xrightarrow{\text{Ch}} \frac{1}{E[U_1]}.$$

Proof. First, it follows from Theorem 4 that for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{S_{N(t)}}{N(t)} - E[U_1]\right| \ge \varepsilon\right\} = 0. \tag{18}$$

We now prove

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{S_{N(t)+1}}{N(t)} - E[U_1]\right| \ge \varepsilon\right\} = 0. \tag{19}$$

In fact, for any $\omega \in \Omega$, we have

$$\begin{aligned}\left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)} - E[U_1]\right| &= \left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1}\cdot\frac{N(t)(\omega)+1}{N(t)(\omega)} - E[U_1]\right| \\ &= \left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1}\cdot\frac{N(t)(\omega)+1}{N(t)(\omega)} - \frac{N(t)(\omega)+1}{N(t)(\omega)}\cdot E[U_1] + \frac{N(t)(\omega)+1}{N(t)(\omega)}\cdot E[U_1] - E[U_1]\right| \\ &\le \frac{1}{N(t)(\omega)}\left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1} - E[U_1]\right| + \left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1} - E[U_1]\right| + \frac{E[U_1]}{N(t)(\omega)}.\end{aligned}$$

Without any loss of generality, letting $0 < \varepsilon < 1$, we can obtain

$$\begin{aligned}&\mathrm{Ch}\left\{\left|\frac{S_{N(t)+1}}{N(t)} - E[U_1]\right| \ge \varepsilon\right\} = \int_\Omega \mathrm{Cr}\left\{\left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)} - E[U_1]\right| \ge \varepsilon\right\}\Pr(\mathrm{d}\omega) \\ &\le \int_\Omega \left[\mathrm{Cr}\left\{\frac{1}{N(t)(\omega)}\left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1} - E[U_1]\right| + \left|\frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)+1} - E[U_1]\right| + \frac{E[U_1]}{N(t)(\omega)} \ge \varepsilon\right\}\right]\Pr(\mathrm{d}\omega) \\ &\le \mathrm{Ch}\left\{\frac{1}{N(t)} \ge \frac{\varepsilon}{3}\right\} + 2\mathrm{Ch}\left\{\left|\frac{S_{N(t)+1}}{N(t)+1} - E[U_1]\right| \ge \frac{\varepsilon}{3}\right\} + \mathrm{Ch}\left\{\frac{1}{N(t)} \ge \frac{\varepsilon}{3E[U_1]}\right\}.\end{aligned}$$

From (18) and Theorem 3, we can obtain (19).

Again, without losing any generality, taking $0 < \varepsilon < 1/E[U_1]$, for any $\omega \in \Omega$, the following inequality

$$\left|\frac{N(t)(\omega)}{S_{N(t)(\omega)}(\omega)} - \frac{1}{E[U_1]}\right| \ge \varepsilon$$

implies

$$\left|\frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} - E[U_1]\right| \ge \frac{E^2[U_1]\varepsilon}{1+E[U_1]\varepsilon}.$$

Combining (18), we have that

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{N(t)}{S_{N(t)}} - \frac{1}{E[U_1]}\right| \ge \varepsilon\right\} &= \lim_{t\to\infty}\int_\Omega \mathrm{Cr}\left\{\left|\frac{N(t)(\omega)}{S_{N(t)(\omega)}(\omega)} - \frac{1}{E[U_1]}\right| \ge \varepsilon\right\}\Pr(\mathrm{d}\omega) \\ &\le \lim_{t\to\infty}\int_\Omega \mathrm{Cr}\left\{\left|\frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} - E[U_1]\right| \ge \frac{E^2[U_1]\varepsilon}{1+E[U_1]\varepsilon}\right\}\Pr(\mathrm{d}\omega) \\ &= \lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{S_{N(t)}}{N(t)} - E[U_1]\right| \ge \frac{E^2[U_1]\varepsilon}{1+E[U_1]\varepsilon}\right\} = 0. \end{aligned}\tag{20}$$

Using the similar method, we can prove

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{N(t)}{S_{N(t)+1}} - \frac{1}{E[U_1]}\right| \ge \varepsilon\right\} = 0. \tag{21}$$

Next, by the definitions of $N(t)$, $S_{N(t)}$ and $S_{N(t)+1}$, we know that for any $\omega \in \Omega$,

$$S_{N(t)(\omega)}(\omega) \le t < S_{N(t)(\omega)+1}(\omega),$$

which implies

$$\frac{S_{N(t)(\omega)}(\omega)}{N(t)(\omega)} \le \frac{t}{N(t)(\omega)} < \frac{S_{N(t)(\omega)+1}(\omega)}{N(t)(\omega)}. \tag{22}$$

Equivalently, we have

$$\frac{N(t)(\omega)}{S_{N(t)(\omega)+1}(\omega)} < \frac{N(t)(\omega)}{t} \leq \frac{N(t)(\omega)}{S_{N(t)(\omega)}(\omega)}.$$

Therefore, for any $\varepsilon > 0$, one has

$$\begin{aligned} \mathrm{Ch}\left\{\left|\frac{N(t)}{t} - \frac{1}{E[U_1]}\right| \geq \varepsilon\right\} &= \int_\Omega \mathrm{Cr}\left(\left\{\frac{N(t)(\omega)}{t} \geq \frac{1}{E[U_1]} + \varepsilon\right\} \bigcup \left\{\frac{N(t)(\omega)}{t} \leq \frac{1}{E[U_1]} - \varepsilon\right\}\right) \mathrm{Pr}(\mathrm{d}\omega) \\ &\leq \int_\Omega \left[\mathrm{Cr}\left\{\frac{N(t)(\omega)}{t} \geq \frac{1}{E[U_1]} + \varepsilon\right\} + \mathrm{Cr}\left\{\frac{N(t)(\omega)}{t} \leq \frac{1}{E[U_1]} - \varepsilon\right\}\right] \mathrm{Pr}(\mathrm{d}\omega) \\ &\leq \mathrm{Ch}\left\{\frac{N(t)(\omega)}{S_{N(t)(\omega)}(\omega)} \geq \frac{1}{E[U_1]} + \varepsilon\right\} + \mathrm{Ch}\left\{\frac{N(t)(\omega)}{S_{N(t)(\omega)+1}(\omega)} \leq \frac{1}{E[U_1]} - \varepsilon\right\}. \end{aligned}$$

By (20) and (21), the above inequality implies

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\left|\frac{N(t)}{t} - \frac{1}{E[U_1]}\right| \geq \varepsilon\right\} = 0.$$

The proof of the theorem is complete. □

Lemma 2. *Suppose ξ_k, $k = 1, 2, \ldots$ are fuzzy random interarrival times with $\mu_{\xi_k(\omega)} = \Pi(x - U_k(\omega))$ for almost every $\omega \in \Omega$, where U_k are random variables, and $N(t)$ is the fuzzy random renewal variable. If $U_k \geq a$ almost surely, then for any real number $t, r > 0$, we have*

$$\mathrm{Ch}\left\{\frac{N(t)}{t} \geq r\right\} \leq \int_\Omega \mathrm{Pos}\left\{\frac{1}{\xi_1(\omega) - (U_1(\omega) - a)} \geq r\right\} \mathrm{Pr}(\mathrm{d}\omega).$$

Proof. We denote $X_k = \xi_k - U_k$, $S_n^* = X_1 + \cdots + X_n$. Since for every $\omega \in \Omega$, $X_k(\omega)$ are identically distributed fuzzy variables, we have for any given real number $t, r > 0$ and almost every $\omega \in \Omega$,

$$\begin{aligned} \mathrm{Pos}\left\{\frac{N(t)(\omega)}{t} \geq r\right\} &= \mathrm{Pos}\left\{N(t)(\omega) \geq rt\right\} \\ &= \mathrm{Pos}\left\{N(t)(\omega) \geq M\right\} = \mathrm{Pos}\left\{S_M(\omega) \leq t\right\} \\ &= \mathrm{Pos}\left\{S_M(\omega) - \sum_{k=1}^M U_k(\omega) \leq t - \sum_{k=1}^M U_k(\omega)\right\} \\ &= \mathrm{Pos}\left\{S_M^*(\omega) \leq t - \sum_{k=1}^M U_k(\omega)\right\} \\ &= \sup_{\sum_{k=1}^M X_k(\omega) \leq t - \sum_{k=1}^M U_k(\omega)} \top_{k=1}^M \mu_{X_k(\omega)}(x_k) \\ &\leq \mathrm{Pos}\left\{X_1(\omega) \leq \frac{t}{M} - \sum_{k=1}^M \frac{U_k(\omega)}{M}\right\}, \end{aligned}$$

where M is the smallest integer such that $M \geq rt$. Furthermore, since $U_k(\omega) \geq a$ almost surely for any k, the following inequalities

$$\begin{aligned} \mathrm{Pos}\left\{\frac{N(t)(\omega)}{t} \geq r\right\} &\leq \mathrm{Pos}\left\{X_1(\omega) \leq \frac{1}{r} - a\right\} \\ &\leq \mathrm{Pos}\left\{\xi_1(\omega) - (U_1(\omega) - a) \leq \frac{1}{r}\right\} \\ &= \mathrm{Pos}\left\{\frac{1}{\xi_1(\omega) - (U_1(\omega) - a)} \geq r\right\} \end{aligned}$$

hold for almost every $\omega \in \Omega$. Thus,

$$\mathrm{Cr}\left\{\frac{N(t)(\omega)}{t} \geq r\right\} \leq \mathrm{Pos}\left\{\frac{1}{\xi_1(\omega) - (U_1(\omega) - a)} \geq r\right\}.$$

Integrating with respect to ω on the above inequality, we have

$$\mathrm{Ch}\left\{\frac{N(t)}{t}\geq r\right\}\leq\int_{\Omega}\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a)}\geq r\right\}\mathrm{Pr}(\mathrm{d}\omega).$$

The proof of the lemma is complete. □

Theorem 6 (*Fuzzy Random Elementary Renewal Theorem*). *Assume $\{\xi_k\}$ is a sequence of i.i.d. fuzzy random interarrival times with $\mu_{\xi_k(\omega)}(x)=\Pi(x-U_k(\omega))$ for almost every $\omega\in\Omega$, where each U_k is a random variable with finite expected value such that $U_k\geq a+h$ with $h>0$ almost surely, and $N(t)$ is the fuzzy random renewal variable. Then we have*

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{E[U_1]}. \tag{23}$$

Furthermore, if $E[\xi_1]$ is finite, then we have

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{\lim\limits_{n}E[S_n/n]}. \tag{24}$$

Proof. First of all, we are to prove

$$\int_0^\infty\left[\int_\Omega\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{Pr}(\mathrm{d}\omega)\right]\mathrm{d}r<\infty. \tag{25}$$

Since $U_k(\omega)\geq a+h$ holds almost surely for each k, we have for almost every $\omega\in\Omega$,

$$\mathrm{Pos}\{\xi_1(\omega)-(U_1(\omega)-a-h)\leq h\}=0$$

which is equivalent to

$$\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq\frac{1}{h}\right\}=0.$$

Therefore, the following inequality

$$\int_0^\infty\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{d}r=\int_0^{1/h}\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{d}r\leq\frac{1}{h}$$

holds for almost $\omega\in\Omega$. Furthermore, by Fubini's Theorem, we can obtain

$$\begin{aligned}&\int_0^\infty\left[\int_\Omega\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{Pr}(\mathrm{d}\omega)\right]\mathrm{d}r\\&\quad=\int_\Omega\left[\int_0^\infty\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{d}r\right]\mathrm{Pr}(\mathrm{d}\omega)\leq\frac{1}{h}.\end{aligned}$$

This proves (25).

Since $N(t)(\omega)$ are nonnegative fuzzy variables for any $\omega\in\Omega$, we have

$$\frac{E[N(t)]}{t}=\int_\Omega\left[\int_0^\infty\mathrm{Cr}\left\{\frac{N(t)(\omega)}{t}\geq r\right\}\mathrm{d}r\right]\mathrm{Pr}(\mathrm{d}\omega)=\int_0^\infty\mathrm{Ch}\left\{\frac{N(t)}{t}\geq r\right\}\mathrm{d}r.$$

According to Theorem 5, we know

$$\frac{N(t)}{t}\xrightarrow{\mathrm{Ch}}\frac{1}{E[U_1]}.$$

Therefore, from Liu et al. [37, Theorem 4], we have

$$\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{N(t)}{t}\geq r\right\}=\mathrm{Ch}\left\{\frac{1}{E[U_1]}\geq r\right\}$$

for almost every $r\in\Re$. Furthermore, by Lemma 2, for any $t,r>0$, one has

$$\mathrm{Ch}\left\{\frac{N(t)}{t}\geq r\right\}\leq\int_\Omega\mathrm{Pos}\left\{\frac{1}{\xi_1(\omega)-(U_1(\omega)-a-h)}\geq r\right\}\mathrm{Pr}(\mathrm{d}\omega).$$

It follows from Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\int_0^{\infty}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{N(t)}{t}\geq r\right\}\mathrm{d}r=\frac{1}{E[U_1]}.$$

Moreover, if $E[\xi]$ is finite, we can deduce directly from Theorem 2 that

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{E[U_1]}=\frac{1}{\lim_n E\left[S_n/n\right]}.$$

The proof of the theorem is complete. □

Remark 3. If $\{\xi_k\}$ in Theorem 6 degenerates to a sequence of random variables, then both of results (23) and (24) can degenerate to that of the stochastic elementary renewal theorem. On one hand, for any given $\omega\in\Omega$, fuzzy variables $\xi_k(\omega)$ degenerate to crisp numbers. Hence the possibility distribution of $\xi_k(\omega)$ degenerates to

$$\mu_{\xi_k(\omega)}(r)=\Pi(r-U_k(\omega))=\begin{cases}1, & \text{if } r=U_k(\omega)\\ 0, & \text{otherwise,}\end{cases}$$

which implies $\xi_k=U_k$ for $k=1,2,\ldots$. Therefore, $\{\xi_k\}$ is a sequence of i.i.d. random variables with the same finite expected value $E[\xi_1]=E[U_1]$ and the result (23) of Theorem 6 degenerates to

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{E[\xi_1]},$$

which is the right result of stochastic elementary renewal theorem (see [2]). On the other hand, since it is clear that

$$\frac{1}{\lim_n E\left[S_n/n\right]}=\frac{1}{\lim_n E\left[(\xi_1+\xi_2+\cdots+\xi_n)/n\right]}=\frac{1}{E[\xi_1]}$$

for i.i.d. random variables $\xi_1,\xi_2,\ldots$. Therefore, (24) also degenerates to the result of stochastic elementary renewal theorem.

Furthermore, if fuzzy random interarrival times are symmetric, then we can obtain the following interesting result.

Corollary 1. *Assume $\{\xi_k\}$ is a sequence of i.i.d. fuzzy random interarrival times with $\mu_{\xi_k(\omega)}(x)=\Pi(x-U_k(\omega))$ for almost every $\omega\in\Omega$, where Π is symmetric with respect to zero and each U_k is a random variable with finite expected value such that $U_k\geq a+h$ with $h>0$ almost surely, and $N(t)$ is the fuzzy random renewal variable. Then we have*

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{E[\xi_1]}.$$

Proof. Since Π is symmetric with respect to zero, we know for every $\omega\in\Omega$, $\mu_{\xi_1(\omega)}$ is symmetric with respect to $U_1(w)$, and we can calculate that

$$\int_0^{\infty}\mathrm{Cr}\{\xi_1(\omega)\geq r\}\mathrm{d}r=U_1(\omega).$$

Integrating with respect to ω on the above equality we have

$$E[\xi_1]=\int_{\Omega}\left[\int_0^{\infty}\mathrm{Cr}\{\xi_1(\omega)\geq r\}\mathrm{d}r\right]\mathrm{Pr}(\mathrm{d}\omega)=E[U_1].$$

It follows from Theorem 6 that

$$\lim_{t\to\infty}\frac{E[N(t)]}{t}=\frac{1}{E[\xi_1]}.$$

The proof of the lemma is complete. □

Remark 4. Corollary 1 has the same result with stochastic elementary renewal theorem (see [2]).

Table 1
Distributions of the service times.

Service i	Service time T_i (min)
1	$T_1 = (2, 3, 7)$
2	$T_2 = (3, 4, 8)$
3	$T_3 = (4, 5, 9)$
4	$T_4 = (5, 6, 10)$
5	$T_5 = (6, 7, 11)$
6	$T_6 = (7, 8, 12)$

5. Queueing applications

In order to explain further the application of the fuzzy random elementary renewal theorem (Theorem 6), in this section, we give two queueing examples on service systems under hybrid uncertainty of fuzziness and randomness. For more recent researches on queue management and related decision systems combining fuzzy information, one may refer to [39–43]

Example 8. Suppose that customers arrive at a single-service station in accordance with a renewal process. Upon arrival, a customer is immediately served if the server is idle, and the customer waits in line if the server is busy. We assume that the customers are served independently with each other and the service time ξ_k between the $(k-1)$th and kth customers being served in this single-service station is a positive triangular fuzzy random variable as

$$\xi_k(\omega) = \Big(U_k(\omega) - 2, U_k(\omega), U_k(\omega) + 4\Big) \text{ (minutes)}, \tag{26}$$

for $k = 1, 2, \ldots$, where each U_k is a uniform random variable in the interval [3,6], i.e., $U \sim \mathcal{U}(3, 6)$. Taking t-norm $\top$ as the product t-norm,, let us compute the long-term expected number of customers served by this single-service station.

From the assumptions, the total service time S_n for the first n customers can be calculated by $S_n = \xi_1 + \cdots + \xi_n$. The total number of customers that have been served by time t, denoted $N(t)$, can be given by

$$N(t) = \max\{n > 0 \mid 0 < S_n \le t\}.$$

From (26), the possibility function Π is exactly the distribution of a triangular fuzzy variable $(-2, 0, 4)$, that is

$$\Pi(x) = \begin{cases} (x+2)/2, & \text{if } x \in [-2, 0] \\ (4-x)/4, & \text{if } x \in [0, 4] \\ 0, & \text{otherwise.} \end{cases}$$

Noting that the additive function of product t-norm is $f^P = -\log$, we have

$$f^P \circ \Pi(x) = \begin{cases} -\log\left(\dfrac{x+2}{2}\right), & \text{if } x \in [-2, 0] \\ -\log\left(\dfrac{4-x}{4}\right), & \text{if } x \in [0, 4] \\ 1, & \text{otherwise,} \end{cases}$$

which is a convex function on $\Xi = [-2, 4]$. Hence, $\mathrm{co}(f^P \circ \Pi)(x) = f^P \circ \Pi(x) > 0$ for any nonzero $x \in [-2, 4]$. Therefore, by Theorem 6, we can calculate the long-term average expected number of customers served by this system is

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{E[U_1]} \approx 0.29.$$

That is, the expected number of customers served per minute in the long run is about 0.29.

Example 9. We now consider a multi-service system. Assume that there are 6 kinds of services provided by the system and the customers come for the service i at probability p_i, where $p_i = 1/6$, $i = 1, 2, \ldots, 6$, which is characterized by a discrete random variable U. The customers are served independently with each other and, the service time T_i (minutes) provided by service i in the station are assumed to be positive triangular fuzzy variables showed in Table 1.

Taking t-norm $\top$ as a Yager t-norm $\top^Y_\lambda$, $\lambda \ge 1$, we shall calculate the expected number of customers served for long run of this multi-service station.

Let ξ_k be the interarrival time between that the $(k-1)$th and kth customers taking services, $k = 1, 2, \ldots$. Under the above assumptions, we know the service asked by the kth customer is stochastic, which is characterized by the random variable U, while the service time provided by each service in the station is fuzzy, which is characterized by triangular fuzzy variable T_i, for $i = 1, 2, \ldots, 6$. Taking all the scenarios into account, the interarrival times $\{\xi_k\}$ can be considered as a sequence of i.i.d. fuzzy random variables from the probability space to $\{T_i, i = 1, 2, \ldots, 6\}$ with $\Pr\{\xi = T_i\} = p_i$. The distributions of the interarrival times $\xi_k, k = 1, 2, \ldots$ can be presented as follows:

$$\xi_k \sim \begin{pmatrix} T\dot{1} & T\dot{2} & T\dot{3} & T\dot{4} & T\dot{5} & T\dot{6} \\ p\dot{1} & p\dot{2} & p\dot{3} & p\dot{4} & p\dot{5} & p\dot{6} \end{pmatrix}.$$

Similar to Example 6, the total service time S_n for the first n customers is calculated by $S_n = \xi_1 + \cdots + \xi_n$, and the total number of customers that have been served by time t, is given by

$$N(t) = \max\{n > 0 \mid 0 < S_n \leq t\}.$$

By the distributions of fuzzy random interarrival times $\{\xi_k\}$, without losing any generality, we assign values of random variable U by $\widehat{U}_i = 2 + i$ for $i = 1, 2, \ldots, 6$, hence, each ξ_k can be rewritten as

$$\xi_k(\omega) = \Big(U(\omega) - 1, U(\omega), U(\omega) + 3\Big) \text{ (minutes)}.$$

Therefore, we can find the possibility function $\Pi = (-1, 0, 3)$. Furthermore, by Example 6, we know $\mathrm{co}(f_\lambda^Y \circ \Pi)(x) > 0$ for any nonzero $x \in [-1, 3]$.

As a consequence, by Theorem 6, we can calculate the average expected number of customers served by this system in the long run through the following formula

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\widehat{U}_1 p_1 + \widehat{U}_2 p_2 + \cdots + \widehat{U}_6 p_6} \approx 0.18.$$

That is, in this multi-service system, the expected number of customers served per minute in the long run is about 0.18.

6. Conclusions

Based on the extension principle with the class of continuous Archimedean t-norms and expected value of fuzzy random variable, this paper studied a fuzzy random renewal process and obtained the following major new results.

• Some limit theorems in chance measure and in expected value were proved for the sum of fuzzy random variables (Theorems 1–2).

• A fuzzy random elementary renewal theorem (Theorem 6) was derived for the long-run expected fuzzy random renewal rate, which generalizes the stochastic elementary renewal theorem to the fuzzy random environment. Some properties of fuzzy random renewal process were also discussed (Theorems 3–5), where Theorem 3 ensures the nationality of the fuzzy random renewal process.

• Two queueing applications were provided so as to illustrate the proposed elementary renewal theorem, which show that the results derived in this paper can be applied to general cases involving different Archimedean t-norms, e.g., Example 8 (with product t-norm) and Example 9 (with Yager t-norm).

There is much room for the improvement and development of the current research. For one thing, we note that the convexity condition that $\mathrm{co}(f \circ \Pi)(x) > 0$ for any nonzero $x \in \Xi$ is critical for the main results of this paper, which depends on the distributions of the fuzzy random interarrival times and the chosen continuous Archimedean t-norms. Observing the queueing examples, checking the convexity condition is an indispensable procedure in the practical applications. Therefore, a significant subject in our future work is to find the conditions of some specific distributions and chosen t-norms under which the convexity condition can be satisfied automatically. Furthermore, although the convergence results on fuzzy random renewal process derived in this paper hold for all continuous Archimedean t-norms, the convergence characteristics (such as convergence speed) with respect to different t-norms have not been compared yet. This interesting problem will also be discussed in a subsequent paper. Finally, based on our present results, some other renewal processes in the fuzzy random environment such as fuzzy random renewal reward process should be developed and, additional application researches such as equipment replacement policy can be conducted.

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