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Existence and convergence of best proximity points

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Abstract

Consider a self map T defined on the union of two subsets A and B of a metric space and satisfying $T(A) \subseteq B$ and $T(B) \subseteq A$. We give some contraction type existence results for a best proximity point, that is, a point x such that $d(x, Tx) = \operatorname{dist}(A, B)$. We also give an algorithm to find a best proximity point for the map T in the setting of a uniformly convex Banach space.

Keywords: Best proximity point; Uniformly convex Banach space; Contraction; Strict convexity

1. Introduction

Let A and B be nonempty closed subsets of a complete metric space X. A generalized version of mappings $T : A \cup B \to X$ satisfying

$$T(A) \subseteq B \quad \text{and} \quad T(B) \subseteq A \tag{1.1}$$

were the subject of [2]. The results were motivated by the observation that if for some k in $(0, 1)$, the mapping T also satisfied,

$$d(Tx, Ty) \leqslant kd(x, y) \quad \text{for all } x \in A,\ y \in B, \tag{1.2}$$

then $A \cap B \neq \emptyset$ and so T has a unique fixed point in $A \cap B$.

In order to extend this to the case when $A \cap B = \emptyset$, we introduce a generalization of (1.2) which does not entail $A \cap B$ to be nonempty and ask, not for the existence of a fixed point of T, but for a best proximity point; that is, a point x in $A \cup B$ such that $d(x, Tx) = \operatorname{dist}(A, B)$.

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2. Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the context of our results.

Define

$$
\begin{aligned}
&P_A(x) = \{y \in X\colon\ d(x,y) = d(x,A)\};\\
&\operatorname{dist}(A,B) = \inf\{d(x,y)\colon\ x \in A,\ y \in B\};\\
&A_0 = \{x \in A\colon\ d(x,y') = \operatorname{dist}(A,B) \text{ for some } y' \in B\};\\
&B_0 = \{y \in B\colon\ d(x',y) = \operatorname{dist}(A,B) \text{ for some } x' \in A\}.
\end{aligned}
$$

There are some sufficient conditions which guarantee the nonemptiness of A_0 and B_0. One such simple condition is that A is compact and B is approximatively compact with respect to A (every sequence $\{x_n\}$ of B such that $d(y,x_n) \to d(y,B)$ for some y in A should have a convergent subsequence).

The following lemma gives another set of sufficient conditions in reflexive Banach spaces.

Lemma 2.1. [1] *Let X be a reflexive Banach space, let A be a nonempty closed, bounded and convex subset of X and let B be a nonempty closed, convex subset of X. Then A_0 and B_0 are nonempty and satisfy $P_B(A_0) \subseteq B_0$ and $P_A(B_0) \subseteq A_0$.*

Definition 2.2. A subset K of a metric space X is *boundedly compact* if each bounded sequence in K has a subsequence converging to a point in K.

Suppose X is a uniformly convex (and hence reflexive) Banach space with modulus of convexity δ. Then $\delta(\varepsilon) > 0$ for $\varepsilon > 0$, and $\delta(.)$ is strictly increasing. Moreover, if $x, y, p \in X$, $R > 0$, and $r \in [0, 2R]$,

$$
\left.\begin{array}{l}
\|x-p\| \leqslant R\\
\|y-p\| \leqslant R\\
\|x-y\| \geqslant r
\end{array}\right\}
\quad \Rightarrow \quad
\left\|\frac{x+y}{2} - p\right\| \leqslant \left(1 - \delta\left(\frac{r}{R}\right)\right)R.
$$

Definition 2.3. Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies:

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(2) For some $k \in (0,1)$ we have $d(Tx,Ty) \leqslant kd(x,y) + (1-k)\operatorname{dist}(A,B)$, for all $x \in A$, $y \in B$.

Note that (2) implies that T satisfies $d(Tx,Ty) \leqslant d(x,y)$, for all $x \in A$, $y \in B$, also (2) can be rewritten as $(d(Tx,Ty) - \operatorname{dist}(A,B)) \leqslant k(d(x,y) - \operatorname{dist}(A,B))$, for all $x \in A$, $y \in B$.

3. Main results

First we give a simple but very useful approximation result.

Proposition 3.1. *Let A and B be nonempty subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then starting with any x_0 in $A \cup B$ we have $d(x_n, Tx_n) \to \operatorname{dist}(A,B)$, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots.$*

Proof. Now

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant kd(x_{n-1}, x_n) + (1-k)\operatorname{dist}(A, B) \\ &\leqslant k\big(d(x_{n-1}, x_{n-2}) + (1-k)\operatorname{dist}(A, B)\big) + (1-k)\operatorname{dist}(A, B) \\ &= k^2 d(x_{n-1}, x_{n-2}) + \big(1-k^2\big)\operatorname{dist}(A, B). \end{aligned}$$

Inductively, we have

$$d(x_n, x_{n+1}) \leqslant k^n d(x_1, x_0) + \big(1-k^n\big)\operatorname{dist}(A, B).$$

Therefore, $d(x_n, x_{n+1}) \to \operatorname{dist}(A, B)$. □

Next, we give a simple existence result for a best proximity point.

Proposition 3.2. *Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists x in A such that $d(x, Tx) = \operatorname{dist}(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converging to some $x \in A$. Now

$$\operatorname{dist}(A, B) \leqslant d(x, x_{2n_k-1}) \leqslant d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Thus $d(x, x_{2n_k-1})$ converges to $\operatorname{dist}(A, B)$. Since

$$\operatorname{dist}(A, B) \leqslant d(x_{2n_k}, Tx) \leqslant d(x_{2n_k-1}, x),$$

$d(x, Tx) = \operatorname{dist}(A, B)$. □

The following proposition leads us to an existence result when one of the sets is boundedly compact.

Proposition 3.3. *Let A and B be nonempty subsets of a metric space X, let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then for $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots,$ the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose $x_0 \in A$ (the proof when x_0 in B is similar), then, since by Proposition 3.1 $d(x_{2n}, x_{2n+1})$ converges to $\operatorname{dist}(A, B)$, it is enough to prove that $\{x_{2n+1}\}$ is bounded.

Suppose $\{x_{2n+1}\}$ is not bounded, then there exists N_0 such that

$$d\big(T^2x_0, T^{2N_0+1}x_0\big) > M \quad \text{and} \quad d\big(T^2x_0, T^{2N_0-1}x_0\big) \leqslant M,$$

where $M > \max(\frac{2d(x_0, Tx_0)}{1/k^2-1} + \operatorname{dist}(A, B), d(T^2x_0, Tx_0))$. By the cyclic contraction property of T,

$$\begin{aligned} \frac{M - \operatorname{dist}(A, B)}{k^2} + \operatorname{dist}(A, B) &< d\big(x_0, T^{2N_0-1}x_0\big) \\ &\leqslant d\big(x_0, T^2x_0\big) + d\big(T^2x_0, T^{2N_0-1}x_0\big) \\ &\leqslant 2d(x_0, Tx_0) + M. \end{aligned}$$

Thus, $M < \frac{2d(x_0, Tx_0)}{1/k^2-1} + \operatorname{dist}(A, B)$, which is a contradiction. □

Theorem 3.4. *Let A and B be nonempty closed subsets of a metric space (X, d) and let T: $A \cup B \to A \cup B$ be a cyclic contraction. If either A or B is boundedly compact, then there exists x in $A \cup B$ with $d(x, Tx) = \operatorname{dist}(A, B)$.*

Proof. It follows directly from Propositions 3.2 and 3.3. $\square$

Corollary 3.5. *Let A and B be nonempty closed subsets of a normed linear space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction. If either the span of A or the span of B is a finite dimensional subspace of X, then there exists x in $A \cup B$ with $d(x, Tx) = \operatorname{dist}(A, B)$.*

Corollary 3.5 need not hold when both A and B span infinite dimensional subspaces.

Example 3.6. Given k in $(0, 1)$, let A and B be subsets of l^p, $1 \leqslant p \leqslant \infty$, defined by $A = \{((1 + k^{2n})e_{2n})\colon n \in \mathbb{N}\}$ and $B = \{((1 + k^{2m-1})e_{2m-1})\colon m \in \mathbb{N}\}$. Suppose

$$T\big((1 + k^{2n})e_{2n}\big) = \big(1 + k^{2n+1}\big)e_{2n+1} \quad \text{and} \quad T\big((1 + k^{2m-1})e_{2m-1}\big) = \big(1 + k^{2m}\big)e_{2m}.$$

Then T is a cyclic contraction on $A \cup B$.

Proof. The case when $p = \infty$ is easy to check, so we consider $1 \leqslant p < \infty$. Here $\operatorname{dist}(A, B) = 2^{1/p}$. Now, by the triangle inequality for the l_p norm on $\mathbb{R}^2$,

$$\begin{aligned}
&\big(\big(1 + k^{2n+1}\big)^p + \big(1 + k^{2m}\big)^p\big)^{1/p} \\
&\quad = \big(\big(k + k^{2n+1} + (1 - k)\big)^p + \big(k + k^{2m} + (1 - k)\big)^p\big)^{1/p} \\
&\quad \leqslant \big(\big(k + k^{2n+1}\big)^p + \big(k + k^{2m}\big)^p\big)^{1/p} + 2^{1/p}(1 - k) \\
&\quad \leqslant k\big(\big(1 + k^{2n}\big)^p + \big(1 + k^{2m-1}\big)^p\big)^{1/p} + 2^{1/p}(1 - k). \qquad \square
\end{aligned}$$

Note that A and B defined above are closed sets but $A_0 = B_0 = \emptyset$, so there does not exist a best proximity point.

Next we proceed to our main result of this paper which gives existence, uniqueness and convergence for best proximity points. The following convergence lemma forms the basis for our result.

Lemma 3.7. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying*:

(i) $\|z_n - y_n\| \to \operatorname{dist}(A, B)$.
(ii) *For every $\epsilon > 0$ there exists N_0 such that for all $m > n \geqslant N_0$, $\|x_m - y_n\| \leqslant \operatorname{dist}(A, B) + \epsilon$.*

Then, for every $\epsilon > 0$ there exists N_1 such that for all $m > n \geqslant N_1$, $\|x_m - z_n\| \leqslant \epsilon$.

Proof. Assume the contrary, then there exists $\epsilon_0 > 0$ such that for every $k \in \mathbb{N}$, there exists $m_k > n_k \geqslant k$, for which $\|x_{m_k} - z_{n_k}\| \geqslant \epsilon_0$.

Choose $0 < \gamma < 1$ such that $\epsilon_0/\gamma > \operatorname{dist}(A, B)$ and choose ϵ such that $0 < \epsilon < \min(\frac{\epsilon_0}{\gamma} - \operatorname{dist}(A, B), \frac{\operatorname{dist}(A,B)\delta(\gamma)}{1-\delta(\gamma)})$.

For this $\epsilon > 0$ there exists N_0 such that for all $m_k > n_k \geqslant N_0$, $\|x_{m_k} - y_{n_k}\| \leqslant \operatorname{dist}(A, B) + \epsilon$. Also, there exists N_2 such that $\|z_{n_k} - y_{n_k}\| \leqslant \operatorname{dist}(A, B) + \epsilon$ for all $n_k \geqslant N_2$. Choose $N_1 = \max(N_0, N_2)$.

By uniform convexity, for all $m_k > n_k \geqslant N_1$,

$$\left\| \frac{x_{m_k} + z_{n_k}}{2} - y \right\| \leqslant \left(1 - \delta\left(\frac{\epsilon_0}{\operatorname{dist}(A, B) + \epsilon}\right)\right)\left(\operatorname{dist}(A, B) + \epsilon\right).$$

Using the fact that δ is strictly increasing and by the choice of ϵ, we have $\|\frac{z_{n_k} + x_{m_k}}{2} - y\| < \operatorname{dist}(A, B)$, for all $m_k > n_k \geqslant N_1$, which is a contradiction, hence the lemma. □

In a similar way we can prove the following lemma.

Lemma 3.8. *Let A be a nonempty closed and convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying*:

(i) $\|x_n - y_n\| \to \operatorname{dist}(A, B)$.
(ii) $\|z_n - y_n\| \to \operatorname{dist}(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

Corollary 3.9. *Let A be a nonempty closed and convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in A and $y_0 \in B$ such that $\|x_n - y_0\| \to \operatorname{dist}(A, B)$. Then x_n converges to $P_A(y_0)$.*

Proof. Since $\operatorname{dist}(A, B) \leqslant \|y_0 - P_A(y_0)\| \leqslant \|y_0 - x_n\|$, we have $\|y_0 - P_A(y_0)\| = \operatorname{dist}(A, B)$. Now put $y_n = y_0$ and $z_n = P_A(y_0)$ in Lemma 3.8. □

Theorem 3.10. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map, then there exists a unique best proximity point x in A (that is with $\|x - Tx\| = \operatorname{dist}(A, B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $\operatorname{dist}(A, B) = 0$, then $A \cap B \neq \emptyset$ and the theorem follows from Banach contraction theorem, as T is a contraction map on $A \cap B$. Therefore assume $\operatorname{dist}(A, B) \neq 0$.

Since

$$\|x_{2n} - Tx_{2n}\| \to \operatorname{dist}(A, B) \quad \text{and} \quad \left\|T^2 x_{2n} - Tx_{2n}\right\| \to \operatorname{dist}(A, B).$$

By Lemma 3.8, $\|x_{2n} - x_{2(n+1)}\| \to 0$. Similarly we can show that $\|Tx_{2n} - Tx_{2(n+1)}\| \to 0$. We now show that for every $\epsilon > 0$ there exists N_0 such that for all $m > n \geqslant N_0$, $\|x_{2m} - Tx_{2n}\| \leqslant \operatorname{dist}(A, B) + \epsilon$.

Suppose not, then there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geqslant k$ for which

$$\|x_{2m_k} - Tx_{2n_k}\| \geqslant \operatorname{dist}(A, B) + \epsilon$$

this m_k can be chosen such that it is the least integer greater than n_k to satisfy the above inequality. Now

$$\begin{aligned}\operatorname{dist}(A,B)+\epsilon &\leqslant \|x_{2m_k}-Tx_{2n_k}\| \\ &\leqslant \|x_{2m_k}-x_{2(m_k-1)}\|+\|x_{2(m_k-1)}-Tx_{2n_k}\|.\end{aligned}$$

Hence $\lim_{k\to\infty}\|x_{2m_k}-Tx_{2n_k}\|=\operatorname{dist}(A,B)+\epsilon$. Consequently,

$$\begin{aligned}\|x_{2m_k}-Tx_{2n_k}\| &\leqslant \|x_{2m_k}-x_{2(m_k+1)}\|+\|x_{2(m_k+1)}-Tx_{2(n_k+1)}\| \\ &\quad+\|Tx_{2(n_k+1)}-Tx_{2n_k}\| \\ &\leqslant \|x_{2m_k}-x_{2(m_k+1)}\|+k^2\|x_{2m_k}-Tx_{2n_k}\| \\ &\quad+\left(1-k^2\right)\operatorname{dist}(A,B)+\|Tx_{2(n_k+1)}-Tx_{2n_k}\|.\end{aligned}$$

Hence

$$\operatorname{dist}(A,B)+\epsilon\leqslant k^2\big(\operatorname{dist}(A,B)+\epsilon\big)+\left(1-k^2\right)\operatorname{dist}(A,B)=\operatorname{dist}(A,B)+k^2\epsilon,$$

which is a contradiction. Therefore $\{x_{2n}\}$ is a Cauchy sequence by Lemma 3.7 and hence converges to some $x\in A$. From Proposition 3.2, it follows that $\|x-Tx\|=\operatorname{dist}(A,B)$.

Suppose $x,y\in A$ and $x\neq y$ such that $\|x-Tx\|=\operatorname{dist}(A,B)$ and $\|y-Ty\|=\operatorname{dist}(A,B)$ where necessarily, $T^2x=x$ and $T^2y=y$. Therefore

$$\begin{aligned}\|Tx-y\|&=\left\|Tx-T^2y\right\|\leqslant\|x-Ty\|,\\ \|Ty-x\|&=\left\|Ty-T^2x\right\|\leqslant\|y-Tx\|,\end{aligned}$$

which implies $\|Ty-x\|=\|y-Tx\|$. But, since $\|y-Tx\|>\operatorname{dist}(A,B)$, it follows that $\|Ty-x\|<\|y-Tx\|$, a contradiction. Therefore $x=y$. Hence the theorem. □

Remark 3.11. If the convexity assumption is dropped from Theorem 3.10, then the convergence and uniqueness is not guaranteed even in finite dimensional spaces. Consider $X=R^4$, $A=\{e_1,e_3\}$ and $B=\{e_2,e_4\}$. Define $T(e_i)=e_{i+1}$, where $e_{4+i}=e_i$.

It is also interesting to ask whether a best proximity point exists when A and B are nonempty closed and convex subsets of a reflexive Banach space.

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