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Journal of Computational and Applied Mathematics 81 (1997) 217–227

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Asymptotics of Sobolev orthogonal polynomials for symmetrically coherent pairs of measures with compact support

Francisco Marcellán[a,*], Andrei Martínez-Finkelshtein[b,c,1], Juan J. Moreno-Balcázar[c,1]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain*

[b] *Departamento de Estadística y Matemática Aplicada, Universidad de Almería, Spain*

[c] *Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Spain*

Abstract

We study the strong asymptotics for the sequence of monic polynomials $Q_n(x)$, orthogonal with respect to the inner product

$$(f,g)_S = \int f(x)g(x)\,\mathrm{d}\mu_1(x) + \lambda \int f'(x)g'(x)\,\mathrm{d}\mu_2(x), \quad \lambda > 0,$$

with x outside of the support of the measure μ_2. We assume that μ_1 and μ_2 are symmetric and compactly supported measures on $\mathbb{R}$ satisfying a coherence condition. As a consequence, the asymptotic behaviour of $(Q_n, Q_n)_S$ and of the zeros of Q_n is obtained.

Keywords: Sobolev orthogonal polynomials; Asymptotics; Coherent pairs of measures; Symmetrically coherent pairs of measures

AMS classification: 33C25; 42C05

1. Introduction

Consider the Sobolev inner product

$$(p,q)_S = \int p(x)q(x)\,\mathrm{d}\mu_1(x) + \lambda \int p'(x)q'(x)\,\mathrm{d}\mu_2(x) = \langle p,q\rangle_1 + \lambda\langle p',q'\rangle_2,$$

* Corresponding author. E-mail: pacomarc@ing.uc3m.es. Partially supported by Dirección General de Investigación Científica y Técnica (DGICYT) of Spain under grant PB93-0228-C02-01.

[1] Partially supported by Junta de Andalucía, Grupo de Investigación FQM 0229.

PII S0377-0427(97)00057-5

when both measures μ_i have infinite points of increase and are compactly supported on $\mathbb{R}$. It is easy to see that for $\lambda > 0$, $(\cdot,\cdot)_S$ is actually an inner product, so that there exists a monic orthogonal polynomial sequence, $\{Q_n\}_n$, with respect to $(\cdot,\cdot)_S$ (so-called Sobolev orthogonal polynomials), whose algebraic properties have been studied in depth (see [9] for a wide bibliography on this subject). Nevertheless, general asymptotic results are much more scarce, especially in the case when both measures have a non-trivial absolutely continuous component. In the previous paper [11] we established the strong asymptotics for $\{Q_n\}_n$ for a class of measures, namely under the additional assumption that (μ_1, μ_2) form a coherent pair. Let us recall this concept for a pair of positive measures, introduced by Iserles et al. in [4].

Definition 1. Let μ_i, $i = 1, 2$, be two positive measures and let $\{P_n\}_n$ and $\{T_n\}_n$ be the respective monic orthogonal polynomial sequences (MOPS). Then (μ_1, μ_2) is a coherent pair of measures if there exist nonzero constants $\sigma_1, \sigma_2, \ldots,$ such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad n \geqslant 1. \tag{1}$$

In [11] we obtained the following comparative asymptotics for Q_n, from which the strong asymptotics immediately follows:

Let (μ_1, μ_2) be a coherent pair of measures, $\operatorname{supp}(\mu_1) = [-1, 1]$, $\{T_n\}_n$ the MOPS associated to μ_2 and $\{Q_n\}_n$ the MOPS with respect to $(\cdot,\cdot)_S$. Then,

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{1}{\Phi'(x)} \tag{2}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$, where $\Phi(x) = \frac{1}{2}(x + \sqrt{x^2 - 1})$ with $\sqrt{x^2 - 1} > 0$ when $x > 1$.

In this paper we extend the validity of (2) for another class of measures: those that form a symmetrically coherent pair. This is an analogue of (1) but for measures symmetric with respect to the origin (see [4]).

Definition 2. Let μ_i, $i = 1, 2$, be two positive symmetric measures and let $\{P_n\}_n$ and $\{T_n\}_n$ be the respective monic orthogonal polynomial sequences. Then (μ_1, μ_2) is a symmetrically coherent pair of measures if there exist nonzero constants $\sigma_1, \sigma_2, \ldots,$ such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_{n-1} \frac{P'_{n-1}(x)}{n-1}, \quad n \geqslant 2. \tag{3}$$

The algebraic properties of the Sobolev orthogonal polynomials with respect to symmetrically coherent pairs have been studied by several authors (see [6, 7, 16, 19]). In some particular cases, for example, for Gegenbauer–Sobolev and Gegenbauer–Sobolev type orthogonal polynomials, the strong asymptotics has been obtained in [12, 13].

The structure of the paper is as follows. In the next section we introduce some auxiliary results, including a partial answer to the problem of classification of all symmetrically coherent pairs. The main theorem is proved in Section 3. Finally, in Section 4 we study the norm and zero asymptotic behaviour of $Q_n(x)$ and establish that the zero asymptotics, as expected, can be described in terms of the support of the measure μ_2.

We should point out that this result supports the conjecture made in [11] that (2) actually holds under very general assumptions on the measures involved.

2. Auxiliary results

The method used in [11] to prove (2) heavily relies on the complete classification of all possible coherent pairs, established by Meijer in [16]. Although [16] also classifies all the symmetrically coherent pairs of measures, we will not use it here. Instead of this, it is instructive to exploit the connection between coherence and symmetric coherence via the symmetrization procedure, described in [7]: if (μ_1, μ_2) is a symmetrically coherent pair, then $(\tilde{\mu}_1, \tilde{\mu}_2)$ is a coherent pair, where $\tilde{\mu}_1(x) = \mu_1(\sqrt{x})$ and $\tilde{\mu}_2(x) = x\mu_2(\sqrt{x})$. Now, starting from the classification of all coherent pairs of measures (see, e.g., [11, Proposition 1]) we can describe all the "candidates" to be symmetrically coherent pairs on $[-1, 1]$. The point is that, as will be shown, even the existence superfluous pairs of measures in the described class does not affect the asymptotics (2). Moreover, the classification is restated in a form comfortable for the further proof of the main result.

Let w_1, w_2 be two nonnegative weights on $(-1, 1)$ and v_1, v_2 be the corresponding absolutely continuous measures on $[-1, 1]$:

$$\mathrm{d}v_i(x) = w_i(x)\,\mathrm{d}x, \quad i = 1, 2. \tag{4}$$

Proposition 3. *Let μ_1, μ_2 be two measures, and the support* $\operatorname{supp}(\mu_1) = [-1, 1]$. *Then, if (μ_1, μ_2) constitutes a symmetrically coherent pair of measures, one of the following cases takes place*:

Case 1:

$$\mu_1 = v_1, \qquad \mu_2 = v_2 + M\delta_0, \quad M \geqslant 0,$$

where v_i are given by (4) *and either*

$$w_1(x) = \rho^{(\alpha,\beta)}(x)$$

or

$$w_2(x) = \rho^{(\alpha,\beta)}(x)$$

with

$$\rho^{(\alpha,\beta)}(x) = (1 - x^2)^{\alpha}|x|^{\beta} \quad \textit{and} \quad \frac{w_2(x)}{w_1(x)} = \frac{1 - x^2}{\xi^2 - x^2} \quad \textit{with} \quad \xi \in \mathbb{R}\setminus(-1, 1).$$

Case 2:

$$\mu_1 = v_1, \qquad \mu_2 = v_2 + M\delta_0, \quad M \geqslant 0,$$

where either

$$w_1(x) = \rho^{(\alpha,\beta)}(x)$$

or

$$w_2(x) = \rho^{(\alpha,\beta)}(x),$$

with

$$\frac{w_2(x)}{w_1(x)} = \frac{1-x^2}{x^2+\xi^2} \quad \textit{and} \quad \xi \in \mathbb{R}\backslash\{0\}.$$

Case 3:

$$\mu_1 = \nu_1 + N(\delta_{-1}+\delta_1), \qquad \mu_2 = \nu_2 + M\delta_0, \quad M,N \geqslant 0,$$

where

$$w_1(x) = w_2(x) = |x|^{\beta} = \rho^{(0,\beta)}(x).$$

Case 4:

$$\mu_1 = \nu_1, \qquad \mu_2 = \nu_2 + N(\delta_{\xi}+\delta_{-\xi}) + M\delta_0, \quad N>0,\ M\geqslant 0,\ \xi \in \mathbb{R}\backslash(-1,1),$$

where

$$w_1(x) = \rho^{(\alpha,\beta)}(x) \quad \textit{and} \quad \frac{w_2(x)}{w_1(x)} = \frac{1-x^2}{\xi^2-x^2}.$$

In all the cases $\alpha,\beta \in \mathbb{R}$ *can take any admissible value (i.e., such that* $w_1, w_2 \in L_1[-1,1]$*).*

Remark 4. Due to a freedom in the selection of parameters, the class of measures described in Proposition 3 along with all symmetrically coherent pairs (according to [16], $\beta = M = 0$) includes some "additional" cases. Nevertheless, for the proof of the asymptotic result this will be irrelevant.

Remark 5. The choice $\xi = 1$, $\beta = 0$ and $M = 0$ in case 1 gives us the sequence of Gegenbauer–Sobolev orthogonal polynomials studied in [12]. Analogously, setting $\xi=1$, $\beta=0$ and $N>0$, $M=0$, in case 4 leads to the so-called Gegenbauer–Sobolev-type orthogonal polynomials (see [13]).

The comparative asymptotics, established in the next section, is a direct consequence of the following relation, obtained in [4] (see also [6, 19]). Its proof is totally analogous to that of Proposition 2 in [11].

Proposition 6. *Let* (μ_1,μ_2) *be a symmetrically coherent pair of measures. Then, with the notation introduced in Section* 1, *the following relation is verified:*

$$P_{n+2}(x) - \sigma_n \frac{n+2}{n} P_n(x) = Q_{n+2}(x) - \alpha_n(\lambda) Q_n(x), \quad n \geqslant 1, \tag{5}$$

where

$$\alpha_n(\lambda) = \sigma_n \frac{n+2}{n} \frac{k_n}{\tilde{k}_n} \neq 0, \quad n \geqslant 1, \tag{6}$$

with $k_n = \langle P_n(x), P_n(x)\rangle_1$ *and* $\tilde{k}_n = (Q_n, Q_n)_S$.

3. Relative and strong asymptotics

Now, we are ready to prove the main result of this paper:

Theorem 7. *Let* (μ_1, μ_2) *be a symmetrically coherent pair of measures,* $\operatorname{supp}(\mu_1) = [-1,1]$, $\{T_n\}_n$ *the MOPS associated to* μ_2 *and* $\{Q_n\}_n$ *the MOPS with respect to* $(\cdot,\cdot)_S$. *Then,*

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{1}{\Phi'(x)} \tag{7}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$, *where* $\Phi(x) = \frac{1}{2}(x + \sqrt{x^2-1})$ *with* $\sqrt{x^2-1} > 0$ *when* $x > 1$.

In order to obtain from (5) the asymptotics of $Q_n(x)$ it is essential to study the limit behaviour of the parameters σ_n and $\alpha_n(\lambda)$.

Proposition 8. *The parameters* σ_n *of the symmetric coherence relation* (3) *verify*:
1. *In cases* 1 *and* 3 *of Proposition* 3,

$$\lim_{n\to\infty} \sigma_n = \frac{1}{16\Phi^2(\xi)}. \tag{8}$$

Here and below we assume $\Phi(\pm 1) = \pm\frac{1}{2}$.
2. *In case* 2,

$$\lim_{n\to\infty} \sigma_n = \frac{-1}{4\eta^2(\xi)} = -\frac{1}{16|\Phi(\mathrm{i}\xi)|^2}, \tag{9}$$

where we denote $\eta(\xi) = \xi + \sqrt{\xi^2+1} > 1$ *for* $\xi > 0$.
3. *In case* 4,

$$\lim_{n\to\infty} \sigma_n = \Phi^2(\xi). \tag{10}$$

Proof. The symmetric coherence condition (3) can be rewritten as follows:

$$\sigma_{n-1} = \frac{\dfrac{1}{n+1}\dfrac{P'_{n+1}(x)}{P_{n+1}(x)}\dfrac{P_{n+1}(x)}{P_{n-1}(x)} - \dfrac{T_n(x)}{P_n(x)}\dfrac{P_n(x)}{P_{n-1}(x)}}{\dfrac{1}{n-1}\dfrac{P'_{n-1}(x)}{P_{n-1}(x)}}. \tag{11}$$

Notice that when (μ_1, μ_2) belong to cases 1, 2 or 3 of Proposition 3, both μ_1, μ_2 satisfy the Szegő's condition in $[-1,1]$ (fact that we denote by $\mu_1, \mu_2 \in S$). Then, using the Szegő's theory as in [11], we have that the relative asymptotics $T_n(x)/P_n(x)$ when μ_1, $\mu_2 \in S$ is

$$\frac{T_n(x)}{P_n(x)} \to \frac{D(0,\mu_2)}{D(0,\mu_1)} \frac{D(\frac{1}{2\Phi(x)}, \mu_1)}{D(\frac{1}{2\Phi(x)}, \mu_2)}, \tag{12}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$ where $D(z,\mu)$ is the Szegő function in $\overline{\mathbb{C}} \setminus [-1,1]$ corresponding to the measure μ. In consequence, it only depends on the ratio of the weights $w_2(x)/w_1(x)$.

In cases 1 and 3 of Proposition 3 this ratio is given by $(1-x^2)/(\xi^2-x^2)$. Using Jensen–Poisson formula and straightforward computations, we can verify that in these cases

$$\frac{D(z,\mu_2)}{D(z,\mu_1)}=\begin{cases}\dfrac{2(1-z^2)\Phi(\xi)}{(z-2\Phi(\xi))(z+2\Phi(\xi))}, & \xi\leqslant -1,\\[2ex] -\dfrac{2(1-z^2)\Phi(\xi)}{(z-2\Phi(\xi))(z+2\Phi(\xi))}, & \xi\geqslant 1.\end{cases} \tag{13}$$

Hence,

$$\frac{T_n(x)}{P_n(x)}\to\Phi'(x)\left(1-\frac{1}{16\Phi^2(\xi)\Phi^2(x)}\right). \tag{14}$$

Taking limits in (11), using (14) and the explicit strong asymptotics for $\{P_n\}_n$ given by Szegő's theory we obtain that

$$\lim_{n\to\infty}\sigma_n=\frac{1}{16\Phi^2(\xi)}.$$

In case 2, $w_2(x)/w_1(x)=(1-x^2)/(x^2+\xi^2)$, and as above, we prove that

$$\frac{D(z,\mu_2)}{D(z,\mu_1)}=\begin{cases}-\dfrac{(1-z^2)\eta(\xi)}{z^2+\eta^2(\xi)}, & \eta(\xi)<0,\\[2ex] \dfrac{(1-z^2)\eta(\xi)}{z^2+\eta^2(\xi)}, & \eta(\xi)>0,\end{cases} \tag{15}$$

where $\eta(\xi)=\xi+\sqrt{\xi^2+1}$. Hence,

$$\frac{T_n(x)}{P_n(x)}\to\frac{1}{\eta^2(\xi)}\frac{1+4\Phi^2(x)\eta^2(\xi)}{(2\Phi(x)-1)(2\Phi(x)+1)}=\Phi'(x)\left(1+\frac{1}{4\eta^2(\xi)\Phi^2(x)}\right). \tag{16}$$

Again, from (11), using (16) and the asymptotics of $\{P_n\}_n$ we have that

$$\lim_{n\to\infty}\sigma_n=-\frac{1}{4\eta^2(\xi)}.$$

It remains to consider case 4. Denote by $\{T_n^*\}_n$ the sequence of monic orthogonal polynomials with respect to the absolutely continuous measure $\nu_2\in S$. The symmetric coherence condition (3) now can be rewritten as

$$\sigma_{n-1}=\frac{\dfrac{1}{n+1}\dfrac{P'_{n+1}(x)}{P_{n+1}(x)}\dfrac{P_{n+1}(x)}{P_{n-1}(x)}-\dfrac{T_n(x)}{T_n^*(x)}\dfrac{T_n^*(x)}{P_n(x)}\dfrac{P_n(x)}{P_{n-1}(x)}}{\dfrac{1}{n-1}\dfrac{P'_{n-1}(x)}{P_{n-1}(x)}}. \tag{17}$$

Since the asymptotics of $T_n^*(x)/P_n(x)$ is given by (14), we only need to know the behavior of the ratio $T_n(x)/T_n^*(x)$.

Provided the orthogonality measures of T_n and T_n^* differ only in two mass points at ξ and $-\xi$, we have (see [17, p. 132])

$$\lim_{n\to\infty} \frac{T_n(x)}{T_n^*(x)} = \frac{(\Phi^2(x) - \Phi^2(\xi))^2}{(x^2 - \xi^2)\Phi^2(x)}, \tag{18}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus([-1,1]\cup\{-\xi,\xi\})$.

Then, taking limits in (17) and using (16) and (18), we prove that

$$\lim_{n\to\infty} \sigma_n = \Phi^2(x) - \frac{(\Phi^2(x) - \Phi^2(\xi))^2}{x^2 - \xi^2}\left(1 - \frac{1}{16\Phi^2(x)\Phi^2(\xi)}\right) = \Phi^2(\xi). \qquad \square \tag{19}$$

Proposition 9. *The sequence* $\alpha_n(\lambda)$ *defined in* (6) *verifies*

$$\lim_{n\to\infty} \alpha_n(\lambda) = 0. \tag{20}$$

Proof. This is totally analogous to the proof of Proposition 4 in [11]. $\square$

Now, we are ready to prove the main result in one step.

Proof of Theorem 7. With the notation

$$Y_n(x) := \frac{Q_n(x)}{P_n(x)}, \qquad \delta_n(x) := \alpha_{n-2}(\lambda)\frac{P_{n-2}(x)}{P_n(x)},$$

$$\beta_n := 1 - \sigma_{n-2}\frac{n}{n-2}\frac{P_{n-2}(x)}{P_n(x)},$$

Eq. (5) can be rewritten as

$$Y_n(x) - \delta_n(x)Y_{n-2}(x) = \beta_n(x), \tag{21}$$

which uniquely defines the sequence $\{Y_n\}$ of analytic functions in $\overline{\mathbb{C}}\setminus[-1,1]$, with the initial values $Y_0 = Y_1 = Y_2 = 1$. From (21),

$$|Y_n(x)| \leqslant |\delta_n(x)||Y_{n-2}(x)| + |\beta_n(x)|. \tag{22}$$

Using (20) we obtain that there exists $n_0 \in \mathbb{N}$ such that

$$|\delta_n(x)| < \tfrac{1}{2}, \quad n \geqslant n_0, \tag{23}$$

locally uniformly in $\overline{\mathbb{C}}\setminus[-1,1]$.

On the other hand,

$$|\beta_n(x)| = \left|1 - \sigma_{n-2}\frac{n}{n-2}\frac{P_{n-2}(x)}{P_n(x)}\right| \leqslant 1 + \frac{n}{n-2}|\sigma_{n-2}|\left|\frac{P_{n-2}(x)}{P_n(x)}\right|.$$

From (8)–(10) and $|\Phi(x)| > \frac{1}{2}$ for $x \notin [-1,1]$ we deduce the existence of $B > 0$ and $n_1 \in \mathbb{N}$ such that

$$|\beta_n(x)| < B, \quad n \geqslant n_1. \tag{24}$$

Then, by (23) and (24) in (22), we have for $n \geqslant n_2 = \max\{n_0, n_1\}$

$$|Y_n(x)| < \tfrac{1}{2}|Y_{n-2}(x)| + B. \tag{25}$$

Consider the sequence

$$Z_n(x) = \begin{cases} |Y_n(x)|, & n \leqslant n_2, \\ \tfrac{1}{2}Z_{n-2}(x) + B, & n > n_2. \end{cases}$$

For $m > n_2$,

$$Z_{m+2r} = (\tfrac{1}{2})^r Z_m + 2B(1 - \tfrac{1}{2^r}), \quad r = 1, 2, \ldots \tag{26}$$

Taking limits when $r \longrightarrow \infty$ in (26), we obtain that $\{Z_n\}$ is uniformly bounded for all n sufficiently large. Moreover, $0 < |Y_n(x)| \leqslant Z_n(x)$, for all $n \in \mathbb{N}$. Hence, $\{Y_n\}$ is uniformly bounded. Taking limits in (21) and using (8), we have in cases 1 and 3,

$$Y_n(x) \longrightarrow 1 - \frac{1}{16\Phi^2(\xi)\Phi^2(x)},$$

in case 2, by (9),

$$Y_n(x) \longrightarrow 1 + \frac{1}{4\eta^2(\xi)\Phi^2(x)},$$

and in case 4, by (10),

$$Y_n(x) \longrightarrow 1 - \frac{\Phi^2(\xi)}{\Phi^2(x)},$$

in all the cases uniformly on compact subsets of $\overline{\mathbb{C}} \backslash [-1, 1]$.

In this way, we have established the following assertion that gives the asymptotics of $\{Q_n\}$ relative to $\{P_n\}$:

Proposition 10. *Uniformly on compact subsets of* $\overline{\mathbb{C}} \backslash [-1, 1]$,

1. *in cases* 1 *and* 3 *of Proposition* 3,

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = 1 - \frac{1}{16\Phi^2(\xi)\Phi^2(x)}, \tag{27}$$

2. *in case* 2,

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = 1 + \frac{1}{4\eta^2(\xi)\Phi^2(x)}, \tag{28}$$

3. *in case* 4,

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = 1 - \frac{\Phi^2(\xi)}{\Phi^2(x)}. \tag{29}$$

Now we can derive (7).

Cases 1 *and* 3: Combining (14) and (27) we obtain

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}\lim_{n\to\infty}\frac{P_n(x)}{T_n(x)}$$

$$=\left(1-\frac{1}{16\Phi^2(\xi)\Phi^2(x)}\right)\frac{16\Phi^2(\xi)\Phi^2(x)}{\Phi'(x)(16\Phi^2(\xi)\Phi^2(x)-1)}=\frac{1}{\Phi'(x)},$$

uniformly on compact subsets of $\overline{\mathbf{C}}\setminus[-1,1]$.

Case 2: Combining (16) and (28) we obtain

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}\lim_{n\to\infty}\frac{P_n(x)}{T_n(x)}$$

$$=\left(1+\frac{1}{4\eta^2(\xi)\Phi^2(x)}\right)\frac{4\eta^2(\xi)\Phi^2(x)}{\Phi'(x)(4\eta^2(\xi)\Phi^2(x)+1)}=\frac{1}{\Phi'(x)},$$

uniformly on compact subsets of $\overline{\mathbf{C}}\setminus[-1,1]$.

Case 4: Using (14), (18) and (29) we have

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\lim_{n\to\infty}\frac{Q_n(x)}{P_n(x)}\lim_{n\to\infty}\frac{P_n(x)}{T_n^*(x)}\lim_{n\to\infty}\frac{T_n^*(x)}{T_n(x)}$$

$$=\left(1-\frac{\Phi^2(\xi)}{\Phi^2(x)}\right)\frac{\Phi^2(x)}{\Phi'(x)(\Phi^2(x)-\Phi^2(\xi))}=\frac{1}{\Phi'(x)}, \tag{30}$$

with $x\in\overline{\mathbf{C}}\setminus([-1,1]\cup\{-\xi,\xi\})$; clearly, (30) holds also in a neighborhood of $-\xi$ or ξ.

Thus, the theorem is proved. □

As we pointed out above, Theorem 7 allows to establish the strong outer asymptotics of the sequence $\{Q_n\}_n$:

Corollary 11. *With the hypothesis of Theorem* 1 *and notation introduced above,*

1. *If* $\operatorname{supp}(\mu_2)=[-1,1]$,

$$Q_n(x)=\frac{1}{\Phi'(x)}\frac{D(0,\mu_2)\Phi^n(x)}{D(\frac{1}{2\Phi(x)},\mu_2)}(1+o(1)), \tag{31}$$

uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$.

2. *If the pair* (μ_1,μ_2) *belongs to case* 4 *of Proposition* 3 *with* $|\xi|>1$,

$$Q_n(x)=\frac{1}{\Phi'(x)}\frac{(\Phi^2(x)-\Phi^2(\xi))^2}{x^2-\xi^2}\frac{D(0,\nu_2)}{D(\frac{1}{2\Phi(x)},\nu_2)}\Phi^{n-2}(x)(1+o(1)), \tag{32}$$

uniformly on compact subsets of $\mathbf{C}\setminus([-1,1]\cup\{-\xi,\xi\})$.

4. Norm and zero asymptotics

Now we study the (Sobolev) norm behaviour of $Q_n(x)$. With the notation $k_n = \langle P_n(x), P_n(x)\rangle_1$, $k_n' = \langle T_n(x), T_n(x)\rangle_2$ and $\tilde{k}_n = (Q_n, Q_n)_S$, introduced above, the following theorem holds.

Theorem 12.

$$k_n + \lambda n^2 k_{n-1}' \leqslant \tilde{k}_n \leqslant k_n + \sigma_{n-2}^2 \left(\frac{n}{n-2}\right)^2 k_{n-2} + \lambda n^2 k_{n-1}', \quad n \geqslant 3.$$

In particular,

$$\lim_{n\to\infty} \frac{\tilde{k}_n}{n^2 k_{n-1}'} = \lambda.$$

Proof. This is totally analogous to the proof of Theorem 2 in [11]. □

Finally, we make some remarks on the behaviour of the zeros of $Q_n(x)$.

First, strong asymptotics (31) implies weak asymptotics. That is, if we associate with each $Q_n(x)$ the discrete unit measure with equal positive masses at its zeros (with account of multiplicity)

$$\omega_n = \frac{1}{n} \sum_{Q_n(\xi)=0} \delta_\xi,$$

then if $\text{supp}(\mu_2) = [-1, 1]$,

$$\mathrm{d}\omega_n(x) \longrightarrow \frac{1}{\pi} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

in the weak-* topology. This a particular case of a nice result recently established in [3].

Furthermore, Corollary 11 implies the following assertion:

Corollary 13. *The zeros of Sobolev monic orthogonal polynomials are, in all the cases, dense in* $\text{supp}(\mu_2)$, *i.e.*,

$$\bigcap_{n\geqslant 1} \bigcup_{k=n}^{\infty} \{x : Q_k(x) = 0\} = \text{supp}(\mu_2).$$

Moreover, if μ_2 has mass points $-\xi, \xi \in \mathbb{R}\backslash[-1,1]$, exactly one zero of $Q_n(x)$ is attracted by $-\xi$, other zero is attracted by ξ, and the rest accumulate at $[-1,1]$.

Proof. It is an immediate consequence of Theorem 7 and the (known) behaviour of $\{T_n\}_n$. □

References

[1] M. Alfaro, F. Marcellán, M.L. Rezola, Estimates for Jacobi–Sobolev type orthogonal polynomials, submitted.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] W. Gautschi, A. Kuijlaars, Zeros and critical points of Sobolev orthogonal polynomials, preprint.
[4] A. Iserles, P.E. Koch, S. Nørsett, J.M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products, J. Approx. Theory 65 (1991) 151–175.
[5] G. López, F. Marcellán, W. Van Assche, Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product, Constr. Approx. 11 (1995) 107–137.
[6] F. Marcellán, J. Petronilho, Orthogonal polynomials and coherent pairs: the classical case, Indag. Math. N.S. 6 (1995) 287–307.
[7] F. Marcellán, J. Petronilho, T.E. Pérez, M.A. Piñar, What is beyond coherent pairs of orthogonal polynomials?, J. Comput. Appl. Math. 65 (1995) 267–277.
[8] F. Marcellán, T.E. Pérez, M.A. Piñar, Gegenbauer–Sobolev orthogonal polynomials, in: A. Cuyt (Ed.), Nonlinear Numerical Methods and Rational Approximation II, Kluwer, Dordrecht, 1994, pp. 71–82.
[9] F. Marcellán, A. Ronveaux, Orthogonal polynomials and Sobolev inner products: a bibliography, Facultés Universitaires N.D. de la Paix, Namur, 1995, preprint.
[10] F. Marcellán, W. Van Assche, Relative asymptotics for orthogonal polynomials, J. Approx. Theory 72 (1993) 193–209.
[11] A. Martínez-Finkelshtein, J.J. Moreno-Balcázar, T.E. Pérez, M.A. Piñar, Asymptotics of Sobolev orthogonal polynomials for coherent pairs of measures, J. Approx. Theory, to appear.
[12] A. Martínez-Finkelshtein, J.J. Moreno-Balcázar, H. Pijeira-Cabrera, Strong asymptotics for Gegenbauer–Sobolev orthogonal polynomials, J. Comput. Appl. Math., 81 (1997) 211–216.
[13] A. Martínez-Finkelshtein, J.J. Moreno-Balcázar, H. Pijeira-Cabrera, Asymptotics for Gegenbauer–Sobolev type polynomials, in: A. Martínez-Finkelshtein, F. Marcellán, J.J. Moreno-Balcázar (Eds.), Complex Methods in Approximation Theory, Universidad de Almería, 1997, pp. 85–92.
[14] H.G. Meijer, Coherent pairs and zeros of Sobolev-type orthogonal polynomials, Indag. Math. N.S. 4 (2) (1993) 163–176.
[15] H.G. Meijer, Zero distribution of orthogonal polynomials in a certain discrete Sobolev space, J. Math. Anal. Appl. 172 (2) (1993) 520–532.
[16] H.G. Meijer, Determination of all coherent pairs, J. Approx. Theory, to appear.
[17] P.G. Nevai, Orthogonal Polynomials, Memoirs of Amer. Math. Soc., vol. 18, AMS, Providence, RI, 1979.
[18] E.M. Nikishin, V.N. Sorokin, Rational Approximations and Orthogonality, Trans. of Mathematical Monographs, vol. 92, Amer. Math. Soc., Providence, RI, 1991.
[19] T.E. Pérez, Polinomios Ortogonales respecto a productos de Sobolev: el caso continuo, Ph.D. Thesis, Universidad de Granada, 1994.
[20] G. Szegő, Orthogonal Polynomials, 4th ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.