Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Functional Analysis 213 (2004) 206–240
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The C^*-algebras of finitely aligned higher-rank graphs ☆

Iain Raeburn,* Aidan Sims, and Trent Yeend

Department of Mathematics, University of Newcastle, NSW 2308, Australia

Received 4 June 2003; accepted 29 October 2003

Communicated by D. Sarason

Abstract

We generalise the theory of Cuntz–Krieger families and graph algebras to the class of *finitely aligned* k-graphs. This class contains in particular all row-finite k-graphs. The Cuntz–Krieger relations for non-row-finite k-graphs look significantly different from the usual ones, and this substantially complicates the analysis of the graph algebra. We prove a gauge-invariant uniqueness theorem and a Cuntz–Krieger uniqueness theorem for the C^*-algebras of finitely aligned k-graphs.

MSC: primary 46L05

Keywords: Graph algebra; Cuntz–Krieger algebra; Uniqueness

1. Introduction

It has been known for many years that the Cuntz–Krieger algebras of (0,1)-matrices [3] can be viewed as the C^*-algebras of directed graphs [4]. More recently, the construction has been extended to cover infinite directed graphs [10,6] and higher-rank analogues, known as k-graphs [9]. The resulting classes of *graph algebras* contain many interesting examples, and have in particular provided a rich

☆This research was supported by the Australian Research Council.
*Corresponding author.
E-mail addresses: iain@maths.newcastle.edu.au (I. Raeburn), aidan@maths.newcastle.edu.au (A. Sims), trent@maths.newcastle.edu.au (T. Yeend).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2003.10.014

supply of models for the classification theory of simple purely infinite nuclear C^*-algebras [15].

Graph algebras have now been associated to all infinite graphs, and an elegant structure theory relates the behaviour of loops in a graph to the properties of its graph algebra. For k-graphs, the current state of affairs is less satisfactory. The object of this paper is to associate graph algebras to a wide class of infinite k-graphs, and to prove versions of the gauge-invariant uniqueness theorem and the Cuntz–Krieger uniqueness theorem for these graph algebras.

Before describing our approach, we recall how the theory of graph algebras developed. A directed graph E consists of a countable vertex set E^0, a countable edge set E^1, and range and source maps $r, s : E^1 \to E^0$. When each vertex receives at most finitely many edges (E is *row-finite*) the graph algebra $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$ and

$$p_v = \sum_{r(e)=v} s_e s_e^* \text{ when } r^{-1}(v) \text{ is nonempty.} \tag{1.1}$$

When $r^{-1}(v)$ is infinite, the sum on the right-hand side of (1.1) cannot converge in a C^*-algebra, and hence the relation must be adjusted. The appropriate adjustment was suggested by the analysis of the Toeplitz algebras of Hilbert bimodules in [7]: impose relation (1.1) only where $r^{-1}(v)$ is finite, and add the requirement that the s_e have orthogonal range projections dominated by $p_{r(e)}$ (which in the row-finite case follows from (1.1)). The resulting family of graph algebras was studied in [6]. That these are the appropriate relations was confirmed when other authors with different points of view arrived at the same conclusion [11,14].

The first work on higher-rank graphs concerned row-finite k-graphs without sources [9]. For directed graphs (that is, when $k = 1$), there is a constructive procedure for extending results to graphs with sources [2, Lemma 1.2]. However when $k > 1$, there are many different kinds of sources, and there is as yet no analogous procedure for dealing with them. In [13], we considered a class of row-finite k-graphs which may have sources provided a local convexity condition is satisfied. In [12], Raeburn and Sims studied infinite k-graphs by viewing them as product systems of graphs, as in [8], and applying the techniques of [5] to the Toeplitz algebras of the associated product system of Hilbert bimodules. The analysis in [12] led to two conclusions. First, it identified an extra Cuntz–Krieger relation which is automatic for row-finite k-graphs, but is not in general. This extra relation is needed to ensure that the algebras generated by Cuntz–Krieger families are spanned by partial isometries of the usual form. Unfortunately, the new relation can involve infinite sums of projections (see [12, Remark 7.2]); the second conclusion of [12] was that we should restrict attention to the *finitely aligned* k-graphs for which the new relation is C^*-algebraic rather than spatial.

In this paper, we introduce Cuntz–Krieger relations which are appropriate for arbitrary finitely aligned k-graphs. We do not assume that our k-graphs are locally convex or row-finite, and we do allow them to have sources. When $k = 1$ or the k-graph is row-finite and locally convex, our new Cuntz–Krieger relations are equivalent to the usual ones. We show that for every finitely aligned k-graph Λ, there is a family of nonzero partial isometries which satisfies the new relations, and we define $C^*(\Lambda)$ to be the universal C^*-algebra generated by such a family. We then prove versions of the gauge-invariant uniqueness theorem and the Cuntz–Krieger uniqueness theorem for $C^*(\Lambda)$. Our analysis is elementary in the sense that we do not use groupoids, partial actions or Hilbert bimodules, though we cheerfully acknowledge that we have gained insight from the models these theories provide.

The results in this paper extend the existing theory of graph algebras in several directions. Since 1-graphs are always finitely aligned, and our new relations are then equivalent to the usual ones (Proposition B.1), our approach provides the first elementary analysis of the C^*-algebra of an arbitrary directed graph. Our results are also new for finitely aligned k-graphs without sources; those interested primarily in this situation may mentally replace all the symbols $\Lambda^{\leqslant n}$ by Λ^n, and thereby avoid several technical complications. Even for row-finite k-graphs we make significant improvements on the existing theory: for non-locally-convex row-finite k-graphs, our Cuntz–Krieger families may have every vertex projection nonzero, unlike those in [13] (see Example A.1).

In Section 2, we describe our new Cuntz–Krieger relations for a finitely aligned k-graph Λ, define $C^*(\Lambda)$ to be the universal C^*-algebra generated by a Cuntz–Krieger family, and investigate some of its basic properties. We discuss a notion of boundary paths which we use to construct a Cuntz–Krieger family in which every vertex projection is nonzero.

The *core* in $C^*(\Lambda)$ is the fixed-point algebra $C^*(\Lambda)^\gamma$ for the gauge action γ of $\mathbb{T}^k$. In Section 3, we show that the core is AF, and deduce that a homomorphism π of $C^*(\Lambda)$ which is nonzero at each vertex projection is injective on the core.

Our proof that $C^*(\Lambda)^\gamma$ is AF is quite different from the argument which we gave for row-finite k-graphs in [13] in that we do not describe $C^*(\Lambda)^\gamma$ as a direct limit over $\mathbb{N}^k$. Instead, we describe $C^*(\Lambda)^\gamma$ as the increasing union of finite-dimensional algebras indexed by finite sets of paths, and produce families of matrix units which span these algebras. In addition to showing that $C^*(\Lambda)$ is AF, this formulation is a key ingredient in our proof of the Cuntz–Krieger uniqueness theorem. The uniqueness theorems themselves are proved in Section 4.

We conclude with three appendices in which we discuss various aspects of our new Cuntz–Krieger relations. In Appendix A, we explain our motivation for introducing these new and apparently substantially different relations; we describe examples illustrating the other possibilities we considered, and their failings. In Appendix B, we show that for ordinary directed graphs (that is, for $k = 1$) and for locally convex row-finite k-graphs, our new Cuntz–Krieger relations are equivalent to the usual ones. Appendix C gives an equivalent formulation of our Cuntz–Krieger relations using only the edges in the 1-skeleton of the k-graph.

2. k-Graphs and Cuntz–Krieger families

We regard $\mathbb{N}^k$ as a semigroup with identity 0. For $1 \leqslant i \leqslant k$, we write e_i for the ith generator of $\mathbb{N}^k$, and for $n \in \mathbb{N}^k$ we write n_i for the ith coordinate of n. We use $\leqslant$ for the partial order on $\mathbb{N}^k$ given by $m \leqslant n$ if $m_i \leqslant n_i$ for all i. The expression $m < n$ means $m \leqslant n$ and $m \neq n$, and does not necessarily indicate that $m_i < n_i$ for all i. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum and $m \wedge n$ for their coordinate-wise minimum.

A *k-graph* is a pair (Λ, d) consisting of a countable small category Λ and a *degree functor* $d : \Lambda \to \mathbb{N}^k$ which satisfy the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$ there exist unique $\mu, \sigma \in \Lambda$ such that $d(\mu) = m$, $d(\sigma) = n$ and $\lambda = \mu\sigma$.

Since we are regarding Λ as a type of graph, we refer to the morphisms of Λ as paths and to the objects of Λ as vertices, and write s and r for the domain and codomain maps. For a thorough introduction to the structure of k-graphs, see [13, Section 2].

Notation 2.1. We use lower-case Greek letters to denote paths in k-graphs. However, we reserve δ for the Kronecker delta, and γ for the gauge action (see Section 3).

Given k-graphs (Λ, d_Λ) and (Γ, d_Γ), a *graph morphism* from Λ to Γ is a functor $x : \Lambda \to \Gamma$ such that $d_\Gamma(x(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$. For $n \in \mathbb{N}^k$, Λ^n is the collection of all paths of degree n; that is

$$\Lambda^n \coloneqq \{\lambda \in \Lambda : d(\lambda) = n\}.$$

The factorisation property ensures that associated to each vertex $v \in \mathrm{Obj}(\Lambda)$ there is a unique element of Λ^0 whose range (and hence source) is v; we call this morphism v as well, identifying $\mathrm{Obj}(\Lambda)$ with Λ^0. For $E \subset \Lambda$ and $\lambda \in \Lambda$, we define

$$\lambda E \coloneqq \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$$

and

$$E\lambda \coloneqq \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}.$$

Hence, for $v \in \Lambda^0$ and $E \subset \Lambda$, $vE = \{\mu \in E : r(\mu) = v\}$ and $Ev = \{\mu \in E : s(\mu) = v\}$.

For $n \in \mathbb{N}^k$, we define

$$\Lambda^{\leqslant n} \coloneqq \{\lambda \in \Lambda : d(\lambda) \leqslant n, \text{ and } d(\lambda)_i < n_i \Rightarrow s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

For $\lambda \in \Lambda$ and $m \leqslant n \leqslant d(\lambda)$, the factorisation property gives unique paths $\lambda' \in \Lambda^m$, $\lambda'' \in \Lambda^{n-m}$ and $\lambda''' \in \Lambda^{d(\lambda)-n}$ such that $\lambda = \lambda'\lambda''\lambda'''$. We denote λ'' by $\lambda(m, n)$, so $\lambda' = \lambda(0, m)$ and $\lambda''' = \lambda(n, d(\lambda))$. More generally, for all $m \leqslant n \in \mathbb{N}^k$, $\lambda(m, n) \coloneqq \lambda(m \wedge d(\lambda), n \wedge d(\lambda))$.

Definition 2.2. For $\lambda, \mu \in \Lambda$, we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) : \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the collection of pairs which give *minimal common extensions* of λ and μ. We say that Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

Remark 2.3. For $\lambda, \mu \in \Lambda$, the map, $(\alpha, \beta) \mapsto \lambda\alpha$ is a bijection between $\Lambda^{\min}(\lambda, \mu)$ and the set $\mathrm{MCE}(\lambda, \mu)$ defined in [12, Definition 5.3]. Hence our definition of a finitely aligned k-graph agrees with that of [12].

Definition 2.4. Let (Λ, d) be a k-graph, let $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that E is *exhaustive* if for every $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$.

Definition 2.5. Let (Λ, d) be a finitely aligned k-graph. A *Cuntz–Krieger Λ-family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra satisfying

(i) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(ii) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(iii) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$; and
(iv) $\prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0$ for all $v \in \Lambda^0$ and finite exhaustive $E \subset v\Lambda$.

Remark 2.6. A number of aspects of these Cuntz–Krieger relations are worth commenting on.

- As seen in [12], the restriction to finitely aligned k-graphs is necessary for the sum in relation (iii) to make sense.
- Relation (iii) implies that $t_\lambda^* t_\lambda = t_{s(\lambda)}$, and that $t_\lambda^* t_\mu = 0$ if $\Lambda^{\min}(\lambda, \mu) = \emptyset$.
- Relations (iii) and (iv) have been significantly changed from their usual form (see [2, Section 1, 13, Definition 3.3]), and we feel they require explanation. The short explanation is that they are the right relations for generating tractable Cuntz–Krieger algebras for which a homomorphism is injective on the core if and only if it is nonzero at each vertex projection (Theorem 3.1). A much more detailed explanation is contained in Appendix A.
- In Appendix B, we prove that for 1-graphs and for locally convex row-finite k-graphs, our relations are equivalent to those set forth in [6,13] respectively.
- Previous treatments of k-graph C^*-algebras have shown that the Cuntz–Krieger relations can be formulated in terms of the 1-skeleton of Λ; that is in terms of vertices and paths of degree e_i. We show in Appendix C that the same is true for our relations.

Given a finitely aligned k-graph (Λ, d), there exists a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the following sense:

given a Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, there exists a unique homomorphism π_t of $C^*(\Lambda)$ such that $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

The following lemma sets forth some useful consequences of Definition 2.5(i)–(iii).

Lemma 2.7. *Let (Λ, d) be a finitely aligned k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a family of partial isometries satisfying Definition* 2.5(i)–(iii). *Then*

(i) $t_\lambda t_\lambda^* t_\mu t_\mu^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_{\lambda\alpha} t_{\lambda\alpha}^*$ *for all* $\lambda, \mu \in \Lambda$. *In particular,* $\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ *is a family of commuting projections.*
(ii) *For* $\lambda, \mu \in \Lambda^{\leqslant n}$, *we have* $t_\lambda^* t_\mu = \delta_{\lambda,\mu} t_{s(\lambda)}$.
(iii) *If* $E \subset v\Lambda^{\leqslant n}$ *is finite, then* $t_v \geqslant \sum_{\lambda\in E} t_\lambda t_\lambda^*$.
(iv) $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\} = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$.

Proof. Part (i) is obtained by multiplying both sides of the equation in Definition 2.5(iii) on the left by t_λ and on the right by t_μ^*.

For (ii), suppose that $t_\lambda^* t_\mu \neq 0$. Then Definition 2.5(iii) ensures that there exists $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, so $\lambda\alpha = \mu\beta$ and $d(\lambda\alpha) \leqslant n$. Since $\lambda, \mu \in \Lambda^{\leqslant n}$, it follows that $\alpha = \beta = s(\lambda)$, so $\lambda = \mu$.

For (iii), note that if $\lambda, \mu \in E$ and $\lambda \neq \mu$, then $t_\lambda t_\lambda^* t_\mu t_\mu^* = 0$ by (ii), and $t_v t_\lambda t_\lambda^* = t_\lambda t_\lambda^*$ for all $\lambda \in E$ by Definition 2.5(ii).

For part (iv), note that $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\}$ is clearly closed under adjoints and contains $\{t_\lambda : \lambda \in \Lambda\}$. Furthermore, $\overline{\operatorname{span}}\{t_\lambda : \lambda \in \Lambda\}$ is closed under multiplication by Definition 2.5(iii). To see that $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\} = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$, note that if $s(\lambda) \neq s(\mu)$ then $t_\lambda t_\mu^* = t_\lambda t_{s(\lambda)} t_{s(\mu)}^* t_\mu^* = 0$ by Definition 2.5(i). □

We define our prototypical Cuntz–Krieger Λ-family using a boundary-path space associated to Λ. For $m \in (\mathbb{N} \cup \{\infty\})^k$, recall from [13, Examples 2.2(ii)] the definition of the k-graph $\Omega_{k,m}$

$$\operatorname{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p \leqslant m\},$$

$$\operatorname{Hom}(\Omega_{k,m}) = \{(p,q) \in \operatorname{Obj}(\Omega_{k,m}) \times \operatorname{Obj}(\Omega_{k,m}) : p \leqslant q\},$$

$$r(p,q) = p, \quad s(p,q) = q, \quad d(p,q) = q - p.$$

If $x : \Omega_{k,m} \to \Lambda$ is a graph morphism and $\lambda \in \Lambda$ with $s(\lambda) = x(0)$, then there is a unique graph morphism $\lambda x : \Omega_{k,m+d(\lambda)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$, and $(\lambda x)(d(\lambda), n) = x(0, n - d(\lambda))$ for all $n \geqslant d(\lambda)$. If $x : \Omega_{k,m} \to \Lambda$ is a graph morphism and $n \in \mathbb{N}^k$ with $n \leqslant m$, then there is a unique graph morphism $x(n,m)$: $\Omega_{k,m-n} \to \Lambda$ such that $(x(n,m))(0,l) = x(n, n+l)$ for all $l \in \mathbb{N}^k$. Notice that these two constructions are inverse in the sense that $(\lambda x)(d(\lambda), d(\lambda x))$ and $x(0,n)x(n,m)$ are both equal to x.

Definition 2.8. Let (Λ, d) be a k-graph, let $m \in (\mathbb{N} \cup \{\infty\})^k$, and let $x : \Omega_{k,m} \to \Lambda$ be a graph morphism. We call x a *boundary path* if there exists $n_x \in \mathbb{N}^k$ such that $n_x \leqslant m$ and

$$n \in \mathbb{N}^k, n_x \leqslant n \leqslant m \text{ and } n_i = m_i \text{ imply that } x(n)\Lambda^{e_i} = \emptyset. \tag{2.1}$$

We extend the range and degree maps to boundary paths $x : \Omega_{k,m} \to \Lambda$ by setting $r(x) := x(0)$ and $d(x) := m$. We write $\Lambda^{\leqslant\infty}$ for the collection of all boundary paths of Λ, and $v\Lambda^{\leqslant\infty}$ for $\{x \in \Lambda^{\leqslant\infty} : r(x) = v\}$.

Remark 2.9. If Λ has no sources, then the boundary path space $\Lambda^{\leqslant\infty}$ is the usual infinite path space Λ^{∞} of [9, Definitions 2.1] consisting of all graph morphisms $x : \Omega_{k,(\infty,\dots,\infty)} \to \Lambda$.

Lemma 2.10. *Let (Λ, d) be a k-graph, and let $x \in \Lambda^{\leqslant\infty}$.*

(i) *If $\lambda \in \Lambda$ with $s(\lambda) = r(x)$, then $\lambda x \in \Lambda^{\leqslant\infty}$.*
(ii) *If $n \in \mathbb{N}^k$ with $n \leqslant d(x)$, then $x(n, d(x)) \in \Lambda^{\leqslant\infty}$.*

Proof. We need only show that there exist $n_{\lambda x}$ and $n_{x(n,d(x))}$ satisfying (2.1). This works with $n_{\lambda x} := n_x + d(\lambda)$ and $n_{x(n,d(x))} := (n_x - n) \vee 0$. $\square$

Lemma 2.11. *Let (Λ, d) be a k-graph. Then $v\Lambda^{\leqslant\infty}$ is nonempty for all $v \in \Lambda^0$.*

Proof. For $i \in \mathbb{N}$ write $[i]$ for the element of $\{1, \dots, k\}$ which is congruent to $i \pmod k$. Fix $v \in \Lambda^0$. Construct a sequence of paths with range v as follows: $\lambda_0 := v$, and given λ_{i-1},

$$\lambda_i := \lambda_{i-1}\nu \quad \text{for some } \nu \in s(\lambda_{i-1})\Lambda^{\leqslant e_{[i]}},$$

so at the ith step, we append a segment of degree $e_{[i]}$ if possible, and append nothing otherwise.

Define $m := \lim_{i\to\infty} d(\lambda_i) \in (\mathbb{N} \cup \{\infty\})^k$. Then there is a unique graph morphism $x : \Omega_{k,m} \to \Lambda$ such that $x(0, d(\lambda_i)) = \lambda_i$ for all $i \in \mathbb{N}$. To show that x is a boundary path, we need only produce $n_x \in \mathbb{N}^k$ with $n_x \leqslant m$ which satisfies (2.1).

For each $j \in \{1, \dots, k\}$ such that $s(\lambda_{i-1})\Lambda^{e_j} = \emptyset$ for some i, let

$$i(j) := \min\{i \in \mathbb{N} : [i] = j \text{ and } s(\lambda_{i-1})\Lambda^{e_j} = \emptyset\}.$$

Let $I := \max\{i(j) : m_j < \infty\}$, and let $n_x := d(\lambda_I)$.

Suppose that $n \in \mathbb{N}^k$ with $n_x \leqslant n \leqslant m$, and that $n_j = m_j$. Then $m_j < \infty$ so $i(j)$ is defined and $I \geqslant i(j)$ by definition. Since $n \geqslant n_x = d(\lambda_I)$, it follows that $n \geqslant d(\lambda_{i(j)-1})$. But $s(\lambda_{i(j)-1})\Lambda^{e_j} = \emptyset$, which implies $x(n)\Lambda^{e_j} = \emptyset$ by the factorisation property. $\square$

Proposition 2.12. *Let (Λ, d) be a finitely aligned k-graph. For $\lambda \in \Lambda$, define*

$$S_\lambda e_x := \begin{cases} e_{\lambda x} & \text{if } s(\lambda) = r(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family on $\ell^2(\Lambda^{\leqslant\infty})$ called the boundary-path representation. *Furthermore, every S_v is nonzero.*

Proof. It follows from Lemma 2.11 that each S_v is nonzero.

A simple calculation using inner products in $\ell^2(\Lambda^{\leqslant\infty})$ shows that

$$S_\lambda^* e_x = \begin{cases} e_{x(d(\lambda), d(x))} & \text{if } x(0, d(\lambda)) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We need to check (i)–(iv) of Definition 2.5.

Relation (i) holds since S_v is the projection onto $\overline{\text{span}}\{e_x : x \in v\Lambda^{\leqslant\infty}\}$.

Checking (ii) amounts to showing that the boundary path $\lambda(\mu x)$ is equal to the boundary path $(\lambda\mu)x$. This follows from associativity of composition in the category Λ.

Relation (iii) follows from a simple calculation involving inner products (see [12, Example 7.4]).

To check that (iv) holds, let $E \subset v\Lambda$ be finite and exhaustive and let $x \in v\Lambda^{\leqslant\infty}$. It suffices to show that $\prod_{\lambda \in E}(S_v - S_\lambda S_\lambda^*)e_x = 0$. Let

$$N := \left(\bigvee_{\lambda \in E} d(\lambda)\right) \vee n_x,$$

in particular, $N \geqslant n_x$ so (2.1) implies that $x(N)\Lambda^{e_j} = \emptyset$ whenever $m_j < \infty$. Since E is exhaustive, there exists $\lambda_x \in E$ such that $\Lambda^{\min}(x(0,N), \lambda_x) \neq \emptyset$; let $(\alpha, \beta) \in \Lambda^{\min}(x(0,N), \lambda_x)$. We claim that $\alpha = x(N)$. Suppose for contradiction $d(\alpha)_i > 0$ for some i. Then $d(x(0,N))_i < d(\lambda_x)_i$. But $N_i \geqslant d(\lambda_x)_i$ by definition, and hence we must have $d(x)_i < N_i$, so $m_i < \infty$. Hence $x(N)\Lambda^{e_i} = \emptyset$ contradicting $d(\alpha)_i > 0$. This establishes the claim, giving $x(0,N) = \lambda_x\beta$, and hence $x(0, d(\lambda_x)) = \lambda_x$. But then

$$\left(\prod_{\lambda \in E}(S_v - S_\lambda S_\lambda^*)\right)e_x = \left(\prod_{\lambda \in E \setminus \{\lambda_x\}}(S_v - S_\lambda S_\lambda^*)\right)(S_v - S_{\lambda_x}S_{\lambda_x}^*)e_x = 0$$

because $S_v e_x = e_x = S_{\lambda_x}S_{\lambda_x}^* e_x$. $\square$

3. Analysis of the core

Given a finitely aligned k-graph (Λ, d), there is a strongly continuous *gauge action* $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ determined by $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ where $z^m = z_1^{m_1} \cdots z_k^{m_k} \in \mathbb{T}$. The fixed-point algebra $C^*(\Lambda)^\gamma$ is equal to $\overline{\mathrm{span}}\{s_\lambda s_\mu^* : d(\lambda) = d(\mu)\}$ and is called the *core* of $C^*(\Lambda)$.

Theorem 3.1. *Let* $(\Lambda,\ d)$ *be a finitely aligned k-graph. Then* $C^*(\Lambda)^\gamma$ *is AF. If* $\{t_\lambda : \lambda \in \Lambda\}$ *is a Cuntz–Krieger* Λ*-family with* $t_v \neq 0$ *for all* $v \in \Lambda^0$, *then the homomorphism* π_t *of* $C^*(\Lambda)$ *such that* $\pi_t(s_\lambda) = t_\lambda$ *is injective on* $C^*(\Lambda)^\gamma$.

The remainder of this section is devoted to proving Theorem 3.1. We therefore fix a finitely aligned k-graph (Λ, d) and a Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$. We also fix a finite set $E \subset \Lambda$. We want to identify a finite set ΠE containing E such that

$$\mathrm{span}\ \{s_\lambda s_\mu^* : \lambda, \mu \in \Pi E, d(\lambda) = d(\mu)\}$$

is closed under multiplication, and hence is a finite-dimensional subalgebra of $C^*(\Lambda)^\gamma$. The next lemma implies that such sets exist.

Lemma 3.2. *There exists a finite set* $F \subset \Lambda$ *which contains E and satisfies*

$$\lambda, \mu, \sigma, \tau \in F, \quad d(\lambda) = d(\mu), \quad d(\sigma) = d(\tau), \quad s(\lambda) = s(\mu)$$

and

$$s(\sigma) = s(\tau) \quad \textit{imply} \quad \{\lambda\alpha, \tau\beta : (\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)\} \subset F. \tag{3.1}$$

Moreover, for any finite F which contains E and satisfies (3.1),

$$M_F^t := \mathrm{span}\{t_\lambda t_\mu^* : \lambda, \mu \in F, d(\lambda) = d(\mu)\}$$

is a finite-dimensional C^**-subalgebra of* $C^*(\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\})$.

Before proving Lemma 3.2, we recall from [12, Definition 8.3] that for $F \subset \Lambda$,

$$\mathrm{MCE}(F) := \left\{ \lambda \in \Lambda : d(\lambda) = \bigvee_{\alpha \in F} d(\alpha) \text{ and } \lambda(0, d(\alpha)) = \alpha \text{ for all } \alpha \in F \right\}$$

and that $\vee F := \bigcup_{G \subset F} \mathrm{MCE}(G)$. Lemma 8.4 of [12] shows that $\vee F$ contains F, is finite whenever F is, and is closed under taking minimal common extensions.

Proof of Lemma 3.2. To begin with, notice that (3.1) is equivalent to

$$\lambda, \mu, \sigma \in F, \quad d(\lambda) = d(\mu), \quad s(\lambda) = s(\mu) \text{ and } (\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma) \text{ imply } \lambda\alpha \in F.$$

Let $N := \bigvee_{\lambda \in E} d(\lambda)$. Let $E_0 := E$, and let

$$E_1 := \{\lambda_1(0, d(\lambda_1))\lambda_2(d(\lambda_1), d(\lambda_2)) \cdots \lambda_j(d(\lambda_{j-1}), d(\lambda_j)) : \lambda_l \in \vee E_0,\\ d(\lambda_l) \leqslant d(\lambda_{l+1}), s(\lambda_l) = \tau(\lambda_{l+1}(d(\lambda_l), d(\lambda_{l+1}))) \text{ for } 1 \leqslant l \leqslant j\}.$$

The set E_1 is finite because $\vee E_0$ is finite. Furthermore E_1 contains $E = E_0$ by definition. Suppose that $\lambda \in E_1$. Then $d(\lambda) = d(\lambda_j)$ for some $\lambda_j \in \vee E_0$, so $d(\lambda) \leqslant N$. If $\lambda, \mu, \sigma \in E_0$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, and if $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, then $\lambda, \mu\alpha \in \vee E_0$ and hence $\lambda\alpha \in E_1$.

Iteratively construct sets $E_i \subset \Lambda, i \geqslant 2$ by

$$E_i := \{\lambda_1(0, d(\lambda_1)) \cdots \lambda_j(d(\lambda_{j-1}), d(\lambda_j)) : \lambda_l \in \vee E_{i-1},\\ d(\lambda_l) \leqslant d(\lambda_{l+1}), s(\lambda_l) = r(\lambda_{l+1}(d(\lambda_l), d(\lambda_{l+1}))) \text{ for } 1 \leqslant l \leqslant j\}.$$

We claim that for all $i \geqslant 2$,

(a) E_i is finite,
(b) $E_{i-1} \subset E_i$,
(c) $d(\lambda) \leqslant N$ for all $\lambda \in E_i$,
(d) if $\lambda, \mu, \sigma \in E_{i-1}$ satisfy $d(\lambda) = d(\mu)$, $s(\lambda) = s(\mu)$, and if $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, then $\lambda\alpha \in E_i$, and
(e) If $E_{i-1} \neq E_i$, then $\min_{\lambda \in E_i \setminus E_{i-1}} |d(\lambda)| > \min_{\mu \in E_{i-1} \setminus E_{i-2}} |d(\mu)|$.

Once we have established (a)–(e), conditions (b), (c) and (e) combine to ensure that $E_{|N|+1} = E_{|N|}$. With $F := E_{|N|}$, it then follows that $E \subset F$ by (b), F is finite by (a), and F satisfies (3.1) by (d).

Let $h \geqslant 1$ and suppose that (a)–(d) hold for $i = h$. We will show that (a)–(d) hold for $i = h + 1$. Since we have already established (a)–(d) for $i = 1$, (a)–(d) will then follow for all $i \geqslant 1$ by induction. We have E_{h+1} finite because Λ is finitely aligned and E_h is finite, giving (a). The inclusion $E_h \subset \vee E_h \subset E_{h+1}$ gives (b). If $\lambda \in E_{h+1}$, then $d(\lambda) = d(\lambda_j)$ for some $\lambda_j \in \vee E_h$, so $d(\lambda) \leqslant N$ by definition of $\vee E_h$, and by (c) for $i = h$. Now suppose that λ, μ, σ and (α, β) are as in (d) for $i = h + 1$. Then $\mu\alpha \in \vee E_h$, and $\lambda\alpha = \lambda(0, d(\lambda))(\mu\alpha)(d(\mu), d(\mu\alpha)) \in E_{h+1}$, giving (d) for $i = h + 1$.

To establish (e), suppose that $i \geqslant 2$ and $\lambda \in E_i \setminus E_{i-1}$. Then

$$\lambda = \lambda_1(0, d(\lambda_1)) \cdots \lambda_j(d(\lambda_{j-1}), d(\lambda_j)),$$

where each $\lambda_l \in \vee E_{i-1}$. If every $\lambda_l \in E_{i-1}$, then each λ_l may be written as

$$\lambda_l = \lambda_{l,1}(0, d(\lambda_{l,1})) \cdots \lambda_{l,h_l}(d(\lambda_{l,h_l-1}),\ d(\lambda_{l,h_l})),$$

where each $\lambda_{l,m} \in \vee E_{i-2}$, and then

$$\lambda = \lambda_{1,1}(0, d(\lambda_{1,1}))\lambda_{1,2}(d(\lambda_{1,1}), d(\lambda_{1,2})) \cdots \lambda_{j,h_j}(d(\lambda_{j,h_j-1}), d(\lambda_{j,h_j}))$$

belongs to E_{i-1} contradicting $\lambda \in E_i \setminus E_{i-1}$. Hence there must be some l such that $\lambda_l \in (\vee E_{i-1}) \setminus E_{i-1}$. By definition of $\vee E_{i-1}$, there exists $G \subset E_{i-1}$ such that

$\lambda_l \in \mathrm{MCE}(G)$. Furthermore, $d(\lambda_l) > d(\sigma)$ for all $\sigma \in G$, for if not we have $\lambda_l \in G \subset E_{i-1}$. If $G \subset E_{i-2}$, then $\lambda_l \in E_{i-1}$, so there exists $\sigma \in (G \backslash E_{i-2}) \subset (E_{i-1} \backslash E_{i-2})$. Hence $|d(\lambda)| \geqslant |d(\lambda_l)| > |d(\sigma)| \geqslant \min_{\mu \in E_{i-1} \backslash E_{i-2}} |d(\mu)|$, proving the claim.

Now suppose that F is any finite set containing E and satisfying (3.1). Then M_F^t is a finite-dimensional subspace of $C^*(\Lambda)^\gamma$ which is closed under taking adjoints.

Hence we need only check that M_F^t is closed under multiplication. But if $t_\lambda t_\mu^*$ and $t_\sigma t_\tau^*$ are generators of M_F^t, then $\lambda, \mu, \sigma, \tau$ are as in (3.1). Since

$$t_\lambda t_\mu^* t_\sigma t_\tau^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} t_{\lambda\alpha} t_{\tau\beta}^*$$

and since each $\lambda\alpha$ and each $\tau\beta$ belong to F by (3.1), it follows that $t_\lambda t_\mu^* t_\sigma t_\tau^* \in M_F^t$. $\square$

The intersection of a family of sets satisfying (3.1) also satisfies (3.1), so we can make the following definition.

Definition 3.3. For any Λ and E, we define ΠE to be the smallest set containing E which satisfies (3.1); that is

$$\Pi E := \bigcap \{F \subset \Lambda : E \subset F \text{ and } F \text{ satisfies (3.1)}\}.$$

Remark 3.4. The following consequences of Lemma 3.2 will prove useful:

(i) ΠE is finite.

(ii) For $\rho, \xi \in \Pi E$ with $d(\rho) = d(\xi)$ and $s(\rho) = s(\xi)$, and for all $v \in s(\rho)\Lambda$,

$$\rho v \in \Pi E \quad \text{if and only if} \quad \xi v \in \Pi E :$$

the "if" direction follows from (3.1) with $\lambda = \rho$, $\mu = \xi$, and $\sigma = \tau = \xi v$, and the "only if" direction follows from (3.1) $\lambda = \mu = \rho v$, $\sigma = \rho$, and $\tau = \xi$.

(iii) If $\rho, \xi \in \Pi E$ and $(\alpha, \beta) \in \Lambda^{\min}(\rho, \xi)$, then (3.1) with $\lambda = \mu = \rho$ and $\sigma = \tau = \xi$ gives $\rho\alpha = \xi\beta \in \Pi E$; that is to say, ΠE is closed under taking minimal common extensions, so $\Pi E = \vee(\Pi E)$.

The next step is to find a family of matrix units for $M_{\Pi E}^t$. The trick is first to express each t_v as a sum of orthogonalised range projections associated to paths in ΠE.

Proposition 3.5. *For each* $\lambda \in \Pi E$, *define*

$$Q(t)_\lambda^{\Pi E} := t_\lambda t_\lambda^* \prod_{\substack{\lambda v \in \Pi E \\ d(v) > 0}} (t_\lambda t_\lambda^* - t_{\lambda v} t_{\lambda v}^*).$$

Then $\{Q(t)_\lambda^{\Pi E} : \lambda \in \Pi E\}$ *is a family of mutually orthogonal projections such that*

$$\prod_{\lambda \in v\Pi E} (t_v - t_\lambda t_\lambda^*) + \sum_{\mu \in v\Pi E} Q(t)_\mu^{\Pi E} = t_v \tag{3.2}$$

for all $v \in r(\Pi E)$.

Proof. Fix $v \in r(\Pi E)$. Any $G \subset \Lambda$ satisfies (3.1) if and only if $G \cup \{v\}$ satisfies (3.1). Hence, by Definition 3.3, $(\Pi E) \cup \{v\} = \Pi(E \cup \{v\})$.

If $v \in \Pi E$, then $\prod_{\lambda \in v\Pi E}(t_v - t_\lambda t_\lambda^*) = 0$, so setting $F := v\Pi E$, the left-hand side of (3.2) is equal to $\sum_{\lambda \in F} Q(t)_\lambda^F$.

On the other hand, if $v \notin \Pi E$, then with $F := v((\Pi E) \cup \{v\})$, we have

$$Q(t)_\lambda^F = Q(t)_\lambda^{(\Pi E) \cup \{v\}} = Q(t)_\lambda^{\Pi E}$$

for all $\lambda \in v(\Pi E)$. Furthermore,

$$Q(t)_v^F = \prod_{\lambda \in v\Pi E} (t_v - t_\lambda t_\lambda^*).$$

So the left-hand side of (3.2) is once again equal to $\sum_{\lambda \in F} Q(t)_\lambda^F$.

In either case, $F = \vee F$ and $\lambda \in F \Rightarrow r(\lambda) \in F$. Under the identification of finitely aligned product systems of graphs over $\mathbb{N}^k$ with finitely aligned k-graphs (see [12, Example 3.5]), the proof of [12, Proposition 8.6] with its first sentence removed now proves our result. □

Remark 3.6. For $\lambda \in \Pi E$, we have

$$\begin{aligned} Q(t)_\lambda^{\Pi E} &= t_\lambda t_\lambda^* \prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_\lambda (t_{s(\lambda)} - t_\nu t_\nu^*) t_\lambda^*) \\ &= t_\lambda \left(\prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\lambda^* \end{aligned} \tag{3.3}$$

because $t_\lambda^* t_\lambda = t_{s(\lambda)}$.

Corollary 3.7. *Let* $\mu \in \Pi E$. *Then* $t_\mu t_\mu^* = \sum_{\mu\nu \in \Pi E} Q(t)_{\mu\nu}^{\Pi E}$.

Proof. First notice that

$$t_\mu t_\mu^* = t_\mu t_\mu^* t_{r(\mu)} = t_\mu t_\mu^* \left(\prod_{\lambda \in r(\mu)\Pi E} (t_{r(\mu)} - t_\lambda t_\lambda^*) + \sum_{\sigma \in r(\mu)\Pi E} Q(t)_\sigma^{\Pi E} \right)$$

by Proposition 3.5. By definition of $Q(t)^{\Pi E}_{\mu\nu}$, we have $t_\mu t_\mu^* \geqslant Q(t)^{\Pi E}_{\mu\nu}$ for all ν, so it suffices to show that

(i) $t_\mu t_\mu^* \prod_{\lambda \in r(\mu)\Pi E}(t_{r(\mu)} - t_\lambda t_\lambda^*) = 0$; and
(ii) for $\sigma \in \Pi E$ with $\sigma(0, d(\mu)) \neq \mu$, we have $t_\mu t_\mu^* Q(t)^{\Pi E}_\sigma = 0$.

Claim (i) is straightforward because $\mu \in r(\mu)\Pi E$, and hence

$$t_\mu t_\mu^* \prod_{\lambda \in r(\mu)\Pi E} (t_{r(\mu)} - t_\lambda t_\lambda^*) \leqslant t_\mu t_\mu^* (t_{r(\mu)} - t_\mu t_\mu^*) = 0.$$

It remains to prove Claim (ii). But for σ as in Claim (ii), $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$ implies $d(\beta) > 0$, and the definition of ΠE ensures that $\sigma\beta \in \Pi E$. Hence

$$\begin{aligned}
t_\mu t_\mu^* Q(t)^{\Pi E}_\sigma
&= t_\mu t_\mu^* t_\sigma t_\sigma^* \prod_{\substack{\sigma\nu \in \Pi E \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) \\
&= \left(\sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} t_{\sigma\beta} t_{\sigma\beta}^* \right) \left(\prod_{\substack{\sigma\nu \in \Pi E \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) \right) \\
&= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} \left(t_{\sigma\beta} t_{\sigma\beta}^* (t_\sigma t_\sigma^* - t_{\sigma\beta} t_{\sigma\beta}^*) \prod_{\substack{\sigma\nu \in \Pi E \setminus \{\sigma\beta\} \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) \right) \\
&= 0
\end{aligned}$$

establishing Claim (ii). □

Definition 3.8. For $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, define $\Theta(t)^{\Pi E}_{\lambda,\mu} := Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^*$.

Proposition 3.9. *The set*

$$\{\Theta(t)^{\Pi E}_{\lambda,\mu} : \lambda, \mu \in \Pi E, d(\lambda) = d(\mu), s(\lambda) = s(\mu)\}$$

is a collection of partial isometries which span $M^t_{\Pi E}$ and satisfy

(i) $(\Theta(t)^{\Pi E}_{\lambda,\mu})^* = \Theta(t)^{\Pi E}_{\mu,\lambda}$; *and*
(ii) $\Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} = \delta_{\mu,\sigma} \Theta(t)^{\Pi E}_{\lambda,\tau}$.

To prove Proposition 3.9 we need to establish two lemmas.

Lemma 3.10. *Let λ, $\mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Then*

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = t_\lambda \left(\prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\mu^* = t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu.$$

Proof. We begin by calculating

$$\begin{aligned}
\Theta(t)^{\Pi E}_{\lambda,\mu} &= Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^* \\
&= t_\lambda \left(\prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\lambda^* t_\lambda t_\mu^* \quad \text{by (3.3)} \\
&= t_\lambda \left(\prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\mu^*, \qquad (3.4)
\end{aligned}$$

which establishes the first equality. For the second equality, we continue the calculation as follows:

$$\begin{aligned}
\Theta(t)^{\Pi E}_{\lambda,\mu} &= t_\lambda \left(\prod_{\substack{\lambda\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\mu^* \quad \text{by (3.4)} \\
&= t_\lambda \left(\prod_{\substack{\mu\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) \right) t_\mu^* \quad \text{by Remark 3.4(ii)} \\
&= t_\lambda t_\mu^* \left(t_\mu \prod_{\substack{\mu\nu \in \Pi E \\ d(\nu) > 0}} (t_{s(\lambda)} - t_\nu t_\nu^*) t_\mu^* \right) \\
&= t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu \quad \text{by (3.3).} \qquad \square
\end{aligned}$$

Lemma 3.11. *Let λ, $\mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Then*

$$t_\lambda t_\mu^* = \sum_{\lambda\nu \in \Pi E} \Theta(t)^{\Pi E}_{\lambda\nu,\mu\nu}.$$

Proof. Just calculate

$$\begin{aligned}
t_\lambda t_\mu^* &= t_\lambda t_\mu^* t_\mu t_\mu^* \\
&= t_\lambda t_\mu^* \left(\sum_{\mu\nu \in \Pi E} Q(t)^{\Pi E}_{\mu\nu} \right) \quad \text{by Corollary 3.7} \\
&= \sum_{\mu\nu \in \Pi E} \left(t_\lambda t_\mu^* t_{\mu\nu} \left(\prod_{\substack{\mu\nu\nu' \in \Pi E \\ d(\nu')>0}} (t_{s(\nu)} - t_{\nu'} t_{\nu'}^*) t_{\mu\nu}^* \right) \right) \quad \text{by (3.3)} \\
&= \sum_{\lambda\nu \in \Pi E} \left(t_{\lambda\nu} \left(\prod_{\substack{\lambda\nu\nu' \in \Pi E \\ d(\nu')>0}} (t_{s(\nu)} - t_{\nu'} t_{\nu'}^*) \right) t_{\mu\nu}^* \right) \\
&\qquad\qquad \text{by two applications of Remark 3.4(ii)} \\
&= \sum_{\lambda\nu \in \Pi E} \Theta(t)^{\Pi E}_{\lambda\nu,\mu\nu} \quad \text{by Lemma 3.10.} \qquad \square
\end{aligned}$$

Proof of Proposition 3.9. The $\Theta(t)^{\Pi E}_{\lambda,\mu}$ are clearly partial isometries. It follows from Lemma 3.11 that they span $M^t_{\Pi E}$. It remains to show that the $\Theta(t)^{\Pi E}_{\lambda,\mu}$ satisfy (i) and (ii).

Let λ, $\mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Since the $Q(t)^{\Pi E}_\lambda$ are projections by Proposition 3.5, we can and use Lemma 3.10 to calculate

$$(\Theta(t)^{\Pi E}_{\lambda,\mu})^* = (Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^*)^* = t_\mu t_\lambda^* Q(t)^{\Pi E}_\lambda = \Theta(t)^{\Pi E}_{\mu,\lambda}.$$

Furthermore, if σ, τ also belong to ΠE with $d(\sigma) = d(\tau)$ and $s(\sigma) = s(\tau)$, then

$$\begin{aligned}
\Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} &= t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu Q(t)^{\Pi E}_\sigma t_\sigma t_\tau^* \quad \text{by Lemma 3.10} \\
&= \delta_{\mu,\sigma} t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu t_\mu t_\tau^* \quad \text{by Proposition 3.5} \\
&= \delta_{\mu,\sigma} Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^* t_\mu t_\tau^* \quad \text{by Lemma 3.10} \\
&= \delta_{\mu,\sigma} Q(t)^{\Pi E}_\lambda t_\lambda t_\tau^* \quad \text{since } s(\lambda) = s(\mu) \\
&= \delta_{\mu,\sigma} \Theta(t)^{\Pi E}_{\lambda,\tau}. \qquad \square
\end{aligned}$$

We now need to say which pairs λ, μ satisfy $\Theta(t)^{\Pi E}_{\lambda,\mu} \neq 0$.

Notation 3.12. For $\lambda, \mu \in \Pi E$ with $s(\lambda) = s(\mu) = v$ and $d(\lambda) = d(\mu) = n$, Remark 3.4(ii) ensures that

$$\{v \in v\Lambda : d(v) > 0, \lambda v \in \Pi E\} = \{v \in v\Lambda : d(v) > 0, \mu v \in \Pi E\}.$$

We denote this set by $T^{\Pi E}(n, v)$. For convenience, for $\lambda \in \Pi E$, we write $T(\lambda)$ for $T^{\Pi E}(d(\lambda), s(\lambda))$.

Proposition 3.13. *Suppose that $t_v \neq 0$ for all $v \in \Lambda^0$. Then*

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = 0 \quad \textit{if and only if} \quad T(\lambda) \textit{ is exhaustive.}$$

To prove Proposition 3.13, we need a definition and two lemmas.

Definition 3.14. For each $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ with $T^{\Pi E}(n, v)$ nonexhaustive, fix $\xi^{\Pi E}(n, v) \in v\Lambda$ such that $\Lambda^{\min}(\xi^{\Pi E}(n, v), v) = \emptyset$ for all $v \in T^{\Pi E}(n, v)$. Again for convenience, we will write ξ_λ in place of $\xi^{\Pi E}(d(\lambda), s(\lambda))$ for $\lambda \in \Pi E$.

Lemma 3.15. *For each $\lambda \in \Pi E$ such that $T(\lambda)$ is not exhaustive, $t_{\lambda\xi_\lambda} t^*_{\lambda\xi_\lambda} \leqslant Q(t)^{\Pi E}_\lambda$.*

Proof. Set $\xi = \xi_\lambda$, and calculate

$$\begin{aligned}
t_{\lambda\xi} t^*_{\lambda\xi} Q(t)^{\Pi E}_\lambda &= t_{\lambda\xi} t^*_{\lambda\xi} t_\lambda t^*_\lambda \prod_{\substack{\lambda v \in \Pi E \\ d(v) > 0}} (t_\lambda t^*_\lambda - t_{\lambda v} t^*_{\lambda v}) \\
&= \prod_{\substack{\lambda v \in \Pi E \\ d(v) > 0}} (t_{\lambda\xi} t^*_{\lambda\xi} (t_\lambda t^*_\lambda - t_{\lambda v} t^*_{\lambda v})) \\
&= \prod_{\substack{\lambda v \in \Pi E \\ d(v) > 0}} \left(t_{\lambda\xi} t^*_{\lambda\xi} - \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda\xi, \lambda v)} t_{\lambda v\beta} t^*_{\lambda v\beta} \right) \\
&= \prod_{\substack{\lambda v \in \Pi E \\ d(v) > 0}} t_{\lambda\xi} t^*_{\lambda\xi} \\
&\qquad \text{since each } \Lambda^{\min}(\lambda\xi, \lambda v) = \Lambda^{\min}(\xi, v) = \emptyset \\
&\qquad \text{by choice of } \xi = \xi_\lambda \\
&= t_{\lambda\xi} t^*_{\lambda\xi}. \qquad \square
\end{aligned}$$

Lemma 3.16. *Let $\lambda \in \Pi E$ and suppose that $T(\lambda)$ is not exhaustive. Let $\sigma, \tau \in \Pi E$ with $d(\sigma) = d(\tau)$ and $s(\sigma) = s(\tau)$. Then*

$$t_{\lambda\xi_\lambda} t^*_{\lambda\xi_\lambda} \Theta(t)^{\Pi E}_{\sigma,\tau} = \delta_{\lambda,\sigma} t_{\lambda\xi_\lambda} t^*_{\tau\xi_\lambda}.$$

Proof. Set $\xi = \xi_\lambda$ and calculate

$$\begin{aligned} t_{\lambda\xi}t^*_{\lambda\xi}\Theta(t)^{\Pi E}_{\sigma,\tau} &= t_{\lambda\xi}t^*_{\lambda\xi}Q(t)^{\Pi E}_{\sigma}t_\sigma t^*_\tau \\ &= t_{\lambda\xi}t^*_{\lambda\xi}Q(t)^{\Pi E}_{\lambda}Q(t)^{\Pi E}_{\sigma}t_\sigma t^*_\tau \quad \text{by Lemma 3.15} \\ &= \delta_{\lambda,\sigma}t_{\lambda\xi}t^*_{\lambda\xi}Q(t)^{\Pi E}_{\lambda}t_\lambda t^*_\tau \quad \text{by Proposition 3.5} \\ &= \delta_{\lambda,\sigma}t_{\lambda\xi}t^*_{\tau\xi} \quad \text{by Lemma 3.15.} \qquad \square \end{aligned}$$

Proof of Proposition 3.13. For the "if" direction, note that $T(\lambda)$ is certainly finite and if it is also exhaustive then

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = t_\lambda\left(\prod_{v\in T(\lambda)}(t_{s(\lambda)} - t_v t^*_v)\right)t_\mu = 0$$

by Definition 2.5(iv). For the "only if" direction, suppose that $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, and suppose that $T(\lambda)$ is not exhaustive. Then Lemma 3.16 ensures that

$$t_{\lambda\xi_\lambda}t^*_{\lambda\xi_\lambda}\Theta(t)^{\Pi E}_{\lambda,\mu} = t_{\lambda\xi_\lambda}t^*_{m\xi_\lambda},$$

which is nonzero because each $t_v \neq 0$. Hence $\Theta(t)^{\Pi E}_{\lambda,\mu} \neq 0$. $\square$

Corollary 3.17. *Suppose that $t_v \neq 0$ for all $v \in \Lambda^0$. Suppose $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Then $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$ if and only if $\Theta(s)^{\Pi E}_{\lambda,\mu} = 0$.*

Proof. We know from the boundary path representation that each s_v is nonzero. The result then follows from Proposition 3.13 applied to both $\{s_\lambda\}$ and $\{t_\lambda\}$. $\square$

Proof of Theorem 3.1. Since

$$C^*(\Lambda)^\gamma = \overline{\text{span}}\{s_\lambda s^*_\mu : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\},$$

we have

$$C^*(\Lambda)^\gamma = \overline{\bigcup_{E\subset\Lambda \text{ finite}} M^s_{\Pi E}}.$$

Since each $M^s_{\Pi E}$ is finite-dimensional, it follows that $C^*(\Lambda)^\gamma$ is AF. Furthermore, since $\pi_t(\Theta(s)^{\Pi E}_{\lambda,\mu}) = \Theta(t)^{\Pi E}_{\lambda,\mu}$ for all finite $E \subset \Lambda$ and $\Theta(t)^{\Pi E}_{\lambda,\mu} \in M^t_{\Pi E}$, Corollary 3.17 ensures that π_t maps nonzero matrix units $\Theta(s)^{\Pi E}_{\lambda,\mu}$ to nonzero matrix units $\Theta(t)^{\Pi E}_{\lambda,\mu}$, and hence is faithful on each $M^s_{\Pi E}$. The result now follows from [1, Lemma 1.3]. $\square$

4. The uniqueness theorems

Write Φ for the linear map from $C^*(\Lambda)$ to $C^*(\Lambda)^\gamma$ obtained by averaging over the gauge action; that is, $\Phi(a) := \int_{\mathbb{T}^k} \gamma_z(a)\, dz$. The map Φ is faithful on positive elements and satisfies $\Phi(s_\lambda s_\mu^*) = \delta_{d(\lambda),d(\mu)} s_\lambda s_\mu^*$.

Proposition 4.1. *Let (Λ, d) be a finitely aligned k-graph. Suppose that π is a homomorphism of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$ and*

$$||\pi(\Phi(a))|| \leqslant ||\pi(a)|| \quad \text{for all } a \in C^*(\Lambda). \tag{4.1}$$

Then π is injective.

Proof. Eq. (4.1), Theorem 3.1, and the properties of Φ show that $\pi(a^*a) = 0 \Rightarrow a^*a = 0$. $\square$

4.1. The gauge-invariant uniqueness theorem

Theorem 4.2. *Let (Λ, d) be a finitely aligned k-graph, and let π be a homomorphism of $C^*(\Lambda)$. Suppose that there is a strongly continuous action $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\{\pi(s_\lambda) : \lambda \in \Lambda\}))$ such that $\theta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}^k$. If $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$, then π is injective.*

Proof. Averaging over θ is norm-decreasing and implements $\pi(a) \mapsto \pi(\Phi(a))$. Hence Eq. (4.1) holds, and the result follows from Proposition 4.1. $\square$

Corollary 4.3. (The gauge-invariant uniqueness theorem). *Let (Λ, d) be a finitely aligned k-graph. There exists a Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_v \neq 0$ for every $v \in \Lambda^0$, and such that there exists a strongly continuous action $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\{t_\lambda : \lambda \in \Lambda\}))$ satisfying $\theta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all $\lambda \in \Lambda$. Furthermore, any two such families generate canonically isomorphic C^*-algebras.*

Proof. Proposition 2.12 shows that there is a Cuntz–Krieger Λ-family consisting of nonzero partial isometries. It follows that each $s_v \in C^*(\Lambda)$ is nonzero, so $t_\lambda := s_\lambda$ and $\theta := \gamma$ gives existence. The last statement follows from Theorem 4.2. $\square$

Recall from [9] that if (Λ_1, d_1) is a k_1-graph and (Λ_2, d_2) is a k_2-graph, then the pair $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a $(k_1 + k_2)$-graph. It is easy to check that if Λ_1 and Λ_2 are finitely aligned, then so is $\Lambda_1 \times \Lambda_2$.

Corollary 4.4. *Let Λ_1 be finitely aligned k_1-graph and let Λ_2 be a finitely aligned k_2-graph. Then $C^*(\Lambda_1 \times \Lambda_2)$ is canonically isomorphic to $C^*(\Lambda_1) \otimes C^*(\Lambda_2)$.*

Proof. Implicit in the statement of the corollary is that all tensor products of $C^*(\Lambda_1)$ and $C^*(\Lambda_2)$ coincide. The bilinearity of tensor products ensures that $\{s_{\lambda_1} \otimes s_{\lambda_2} : (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\}$ is a Cuntz–Krieger $(\Lambda_1 \times \Lambda_2)$-family regardless of the tensor product in question. Separate arguments for the spatial tensor product and the universal tensor product show that for either one, the formula

$$\theta_z(s_{\lambda_1} \otimes s_{\lambda_2}) \coloneqq (z_1^{d(\lambda_1)_1} \cdots z_{k_1}^{d(\lambda_1)_{k_1}} z_{k_1+1}^{d(\lambda_2)_1} \cdots z_{k_1+k_2}^{d(\lambda_2)_{k_2}}) s_{\lambda_1} \otimes s_{\lambda_2}$$

extends to a strongly continuous action θ of $\mathbb{T}^{k_1+k_2}$ on $C^*(\{s_{\lambda_1} \otimes s_{\lambda_2} : (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\})$ which is equivariant with the gauge action on $C^*(\Lambda_1 \times \Lambda_2)$. The vertex projections $s_{v_1} \otimes s_{v_2}$, are all nonzero because each s_{v_1} is nonzero and each s_{v_2} is nonzero. Corollary 4.3 shows that the two tensor products coincide, and Theorem 4.2 shows they are canonically isomorphic to $C^*(\Lambda_1 \times \Lambda_2)$. □

4.2. The Cuntz–Krieger uniqueness theorem

Theorem 4.5. *Let (Λ, d) be a finitely aligned k-graph, and suppose that*

$$\begin{aligned} &\text{for each } v \in \Lambda^0 \text{ there exists } x \in v\Lambda^{\leqslant \infty} \text{ such that} \\ &\lambda, \mu \in \Lambda v \text{ and } \lambda \neq \mu \text{ imply } \lambda x \neq \mu x. \end{aligned} \tag{B}$$

Suppose that π is a homomorphism of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$. Then π is injective.

Corollary 4.6. (The Cuntz–Krieger uniqueness theorem). *Let (Λ, d) be a finitely aligned k-graph which satisfies* (B). *There exists a Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_v \neq 0$ for all $v \in \Lambda^0$. Furthermore, any two such families generate canonically isomorphic C^*-algebras.*

Proof. The existence of a nonzero Cuntz–Krieger Λ-family follows from Proposition 2.12. The last statement of the corollary follows from Theorem 4.5. □

The rest of this section is devoted to proving Theorem 4.5. For the remainder of this section, let (Λ, d) and π be as in Theorem 4.5 and fix a finite set $E \subset \Lambda$ and a linear combination $a = \sum_{\lambda,\mu \in E} a_{\lambda,\mu} s_\lambda s_\mu^* \in C^*(\Lambda)$. Notice that $\Phi(a) = \sum_{\lambda,\mu \in E, d(\lambda)=d(\mu)} a_{\lambda,\mu} s_\lambda s_\mu^*$. Since a is arbitrary in a dense subset of $C^*(\Lambda)$, if we show that

$$||\pi(\Phi(a))|| \leqslant ||\pi(a)||,$$

then Theorem 4.5 will follow from Proposition 4.1.

For $n \in \mathbb{N}^k$, define $\mathscr{F}_n$ to be the C^*-subalgebra of $C^*(\Lambda)^\gamma$,

$$\mathscr{F}_n := \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda^{\leqslant n}, d(\lambda) = d(\mu)\}$$
$$\cong \bigoplus_{v \in \Lambda^0,\ m \leqslant n} \mathscr{K}(\ell^2(v\Lambda^{\leqslant n} \cap \Lambda^m)),$$

where the isomorphism follows from Lemma 2.7(ii).

Proposition 4.7. *There exists* $N_E \in \mathbb{N}^k$ *and a projection* P_E *such that* $b \mapsto P_E b$ *is an isomorphism of* $M^s_{\Pi E}$ *into* $\mathscr{F}_{N_E}$.

Proof. Recalling Notation 3.12 and Definition 3.14, let

$$N_E := \bigvee \{d(\lambda\xi_\lambda) : \lambda \in \Pi E, T(\lambda) \text{ nonexhaustive}\}.$$

Whenever $T^{\Pi E}(n, v)$ is nonexhaustive, $d(\xi^{\Pi E}(n, v)) \leqslant N_E - n$, so let $v^{\Pi E}(n, v) \in \Lambda^{\leqslant N_E - n}$ be an extension of $\xi^{\Pi E}(n, v)$. That is, for $\lambda \in \Pi E$, $v_\lambda := v^{\Pi E}(d(\lambda), s(\lambda))$ belongs to $\Lambda^{\leqslant N_E - d(\lambda)}$ and $v_\lambda(0, d(\xi_\lambda)) = \xi_\lambda$.

Let

$$P_E := \sum_{\substack{\lambda \in \Pi E \\ T(\lambda) \text{ nonexh.}}} s_{\lambda v_\lambda} s^*_{\lambda v_\lambda}.$$

For all $\lambda \in \Pi E$ with $T(\lambda)$ nonexhaustive,

$$s_{\lambda v_\lambda} s^*_{\lambda v_\lambda} \leqslant s_{\lambda\xi_\lambda} s^*_{\lambda\xi_\lambda} \leqslant Q(s)^{\Pi E}_\lambda$$

by Lemma 3.16. Since all the $Q(t)^{\Pi E}_\lambda$ are mutually orthogonal by Proposition 3.5, it follows that the $s_{\lambda\xi_\lambda} s^*_{\lambda\xi_\lambda}$ are mutually orthogonal, as are the $s_{\lambda v_\lambda} s^*_{\lambda v_\lambda}$. Hence, for all $\lambda \in \Pi E$ with $T(\lambda)$ nonexhaustive,

$$P_E s_{\lambda\xi_\lambda} s^*_{\lambda\xi_\lambda} = s_{\lambda v_\lambda} s^*_{\lambda v_\lambda}. \tag{4.2}$$

If λ, $\mu \in \Pi E$ with $d(\lambda) = d(\mu)$, $s(\lambda) = s(\mu)$ and $T(\lambda)$ nonexhaustive, then

$$\begin{aligned} P_E \Theta(s)^{\Pi E}_{\lambda,\mu} &= P_E \left(\sum_{\substack{\sigma \in \Pi E \\ T(\sigma) \text{ nonexh.}}} s_{\sigma\xi_\sigma} s^*_{\sigma\xi_\sigma} \right) \Theta(s)^{\Pi E}_{\lambda,\mu} \quad \text{by (4.2)} \\ &= P_E s_{\lambda\xi_\lambda} s^*_{\mu\xi_\lambda} \quad \text{by Lemma 3.16} \\ &= s_{\lambda v_\lambda} s^*_{\mu v_\lambda} \quad \text{by (4.2).} \end{aligned} \tag{4.3}$$

Lemma 3.6 of [13] says that if $\lambda \in \Lambda^{\leqslant n}$ and $\mu \in \Lambda^{\leqslant m}$ then $\lambda\mu \in \Lambda^{\leqslant n+m}$. Hence for all $\lambda \in \Pi E$ such that $T(\lambda)$ is nonexhaustive, $\lambda v_\lambda \in \Lambda^{\leqslant N_E}$. It follows from Proposition 3.13 that $b \mapsto P_E b$ sends nonzero matrix units in $M^s_{\Pi E}$ to nonzero matrix units in $\mathscr{F}_{N_E}$, proving that $b \mapsto P_E b$ is an isomorphism. $\square$

For $v \in s(\{v_\lambda : \lambda \in \Pi E, T(\lambda) \text{ nonexhaustive}\})$, define

$$P_v := \sum_{\substack{\lambda \in \Pi E, T(\lambda) \text{ nonexh.} \\ s(v_\lambda) = v}} s_{\lambda v_\lambda} s^*_{\lambda v_\lambda},$$

so $P_E = \sum_{v \in s(\{v_\lambda : \lambda \in \Pi E, T(\lambda) \text{ nonexh.}\})} P_v$. In particular $P_v = P_v P_E$, so Eq. (4.3) gives

$$P_v \Theta(s)^{\Pi E}_{\lambda,\mu} = P_v P_E \Theta(s)^{\Pi E}_{\lambda,\mu} = P_v s_{\lambda v_\lambda} s^*_{\mu v_\lambda} = \delta_{v,s(v_\lambda)} s_{\lambda v_\lambda} s^*_{\mu v_\lambda}.$$

for all $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$, $s(\lambda) = s(\mu)$, and $T(\lambda) = T(\mu)$ nonexhaustive. Hence

$$\begin{aligned} \Theta(s)^{\Pi E}_{\lambda,\mu} P_v = (P_v \Theta(s)^{\Pi E}_{\mu,\lambda})^* &= (\delta_{v,s(v_\mu)} s_{\mu v_\mu} s^*_{\lambda v_\mu})^* \\ &= \delta_{v,s(v_\lambda)} s_{\lambda v_\lambda} s^*_{\mu v_\lambda} = P_v \Theta(s)^{\Pi E}_{\lambda,\mu}, \end{aligned}$$

so each P_v is in the commutant of $M^s_{\Pi E}$. It follows that there exists a vertex v_0 such that

$$||P_{v_0} \Phi(a)|| = ||P_E \Phi(a)|| = ||\Phi(a)|| \tag{4.4}$$

where the second equality follows from Proposition 4.7.

Lemma 4.8. *Let $\lambda, \mu \in \Pi E$, suppose that $T(\lambda)$ is not exhaustive, and suppose that $\lambda \notin \mu\Lambda$. Then $\Lambda^{\min}(\lambda v_\lambda, \mu) = \emptyset$.*

Proof. Suppose for contradiction that $(\eta, \zeta) \in \Lambda^{\min}(\lambda v_\lambda, \mu)$. Then $\eta = s(v_\lambda)$ and $\lambda v_\lambda = \mu\zeta$ because $\lambda v_\lambda \in \Lambda^{\leqslant N_E}$ and $N_E \geqslant d(\mu)$ by definition. But then with

$$\alpha := v_\lambda(0, (d(\lambda) \vee d(\mu)) - d(\lambda)) \quad \text{and} \quad \beta := \zeta(0, (d(\lambda) \vee d(\mu)) - d(\mu)),$$

we have $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, and $\lambda \neq \mu\mu'$, so $d(\alpha) > 0$; hence $\alpha \in T(\lambda)$. Furthermore, $\Lambda^{\min}(\alpha, v_\lambda) \neq \emptyset$ by definition of α, and hence $\Lambda^{\min}(\xi_\lambda, \alpha) \neq \emptyset$, which contradicts the definition of ξ_λ. $\square$

Corollary 4.9. *If $\lambda, \mu, \sigma \in \Pi E$ and $T(\sigma)$ is nonexhaustive, then*

$$s_{\sigma v_\sigma} s^*_{\sigma v_\sigma} s_\lambda s^*_\mu = \begin{cases} s_{\sigma v_\sigma} s^*_{\mu\lambda' v_\sigma} & \text{if } \sigma = \lambda\lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The corollary follows from a straightforward calculation using Lemma 4.8 and Definition 2.5(iii). □

Lemma 4.10. *We have*

(1) $P_{v_0} a \in \operatorname{span}\{s_{\lambda\lambda' v_{\lambda\lambda'}} s^*_{\mu\lambda' v_{\lambda\lambda'}} : \lambda, \mu \in E, \lambda\lambda' \in \Pi E, T(\lambda\lambda')$ *nonexhaustive*, $s(v_{\lambda\lambda'}) = v_0\}$; *and*

(2) $\Phi(P_{v_0} a) = P_{v_0} \Phi(a)$.

In particular,

$$P_{v_0} \Phi(a) \in \operatorname{span}\{s_{\lambda v_\lambda} s^*_{\mu v_\lambda} : \lambda, \mu \in \Pi E, d(\lambda) = d(\mu),\ s(\lambda) = s(\mu), T(\lambda) \text{ nonexhaustive}\}.$$

Proof. First we use Corollary 4.9 to calculate

$$P_{v_0} a = \sum_{\lambda,\mu \in E} a_{\lambda,\mu} \left(\sum_{\substack{\lambda\lambda' \in \Pi E, T(\lambda\lambda') \text{ nonexh.} \\ s(v_{\lambda\lambda'}) = v_0}} s_{\lambda\lambda' v_{\lambda\lambda'}} s^*_{\mu\lambda' v_{\lambda\lambda'}} \right), \tag{4.5}$$

which proves (1). Furthermore, applying Φ to (4.5), we have

$$\begin{aligned} \Phi(P_{v_0} a) &= \sum_{\lambda,\mu \in E} a_{\lambda,\mu} \left(\sum_{\substack{\lambda\lambda' \in \Pi E, T(\lambda\lambda') \text{ nonexh.} \\ d(\lambda\lambda' v_{\lambda\lambda'}) = d(\mu\lambda' v_{\lambda\lambda'}) \\ s(v_{\lambda\lambda'}) = v_0}} s_{\lambda\lambda' v_{\lambda\lambda'}} s^*_{\mu\lambda' v_{\lambda\lambda'}} \right) \\ &= \sum_{\substack{\lambda,\mu \in E \\ d(\lambda) = d(\mu)}} \left(a_{\lambda,\mu} \sum_{\substack{\lambda\lambda' \in \Pi E, T(\lambda\lambda') \text{ nonexh.} \\ s(v_{\lambda\lambda'}) = v_0}} s_{\lambda\lambda' v_{\lambda\lambda'}} s^*_{\mu\lambda' v_{\lambda\lambda'}} \right) \\ &= P_{v_0} \Phi(a). \end{aligned}$$

The last statement of the lemma follows from (1) and (2) together with Remark 3.4(ii). □

We now modify the proof of [13, Theorem 4.3] to obtain a norm-decreasing map Q which will take $\pi(P_{v_0} a)$ into $\pi(C^*(\Lambda)^\gamma)$.

Lemma 4.11. *There exists a norm-decreasing map* $Q : \pi(C^*(\Lambda)) \to \pi(C^*(\Lambda)^\gamma)$ *such that*

$$||Q(\pi(\Phi(P_{v_0}a)))|| = ||\pi(\Phi(P_{v_0}a))|| \quad \textit{and} \quad Q(\pi(\Phi(P_{v_0}a))) = Q(\pi(P_{v_0}a)).$$

Proof. We follow the latter part of the proof of [13, Theorem 4.3] quite closely. Since Λ satisfies (B), there exists $x \in v_0\Lambda^{\leqslant\infty}$ such that $\lambda \neq \mu$ and $\lambda, \mu \in \Lambda v_0$ imply $\lambda x \neq \mu x$. Hence, for each $\lambda \neq \mu$ in Λv_0, there exists $M_{\lambda,\mu} \in \mathbb{N}^k$ such that $(\lambda x)(0,m) \neq (\mu x)(0,m)$ whenever $m \geqslant M_{\lambda,\mu}$; assume without loss of generality that $M_{\lambda,\mu} \geqslant d(\lambda) \vee d(\mu)$. Let

$$H := \{(\lambda\lambda' v_{\lambda\lambda'}, \mu\lambda' v_{\lambda\lambda'}) : \lambda, \mu, \lambda\lambda' \in \Pi E, T(\lambda\lambda') \text{ nonexhaustive}, \; s(v_{\lambda\lambda'}) = v_0\}.$$

By Lemma 4.10(1), $P_{v_0}a \in \operatorname{span}\{s_\sigma s_\tau^* : (\sigma,\tau) \in H\}$. Let

$$T := \{\rho \in \Lambda^{\leqslant N_E} : \rho = \sigma \text{ or } \rho = \tau \text{ for some } (\sigma,\tau) \in H\}.$$

Define

$$M := \bigvee \{M_{\rho,\tau} : \rho \in T, (\sigma,\tau) \in H \text{ for some } \sigma, \text{ and } \rho \neq \tau\} + n_x.$$

The idea is that M is "far enough out" along x to distinguish any pair of paths in H. By definition of M we have

$$(\tau x)(0,M) \neq (\rho x)(0,M) \tag{4.6}$$

when τ is the second coordinate of an element of H, ρ belongs to T, and $\tau \neq \rho$. Write x_M for $x(0,M)$.

For $n \leqslant N_E$ we set

$$Q_n := \sum_{\rho \in T,\, d(\rho)=n} \pi(s_{\rho x_M} s_{\rho x_M}^*)$$

and we define $Q : \pi(C^*(\Lambda)) \to \pi(C^*(\Lambda))$ by

$$Q(b) := \sum_{n \leqslant N_E} Q_n b Q_n.$$

As in [13], Q is norm-decreasing because the Q_n are mutually orthogonal projections. Also as in [13], $||Q(\pi(\Phi(P_{v_0}a)))|| = ||\pi(\Phi(P_{v_0}a))||$ because Q maps the nonzero matrix units in $\pi(P_{v_0}M^s_{\Pi E})$ to nonzero matrix units in $\pi(\mathscr{F}_{N_E+M})$ (see the proof of [13, Theorem 4.3] for details).

To establish that $Q(\pi(P_{v_0}a)) = Q(\pi(\Phi(P_{v_0}a)))$, let $(\sigma,\tau) \in H$ with $d(\sigma) \neq d(\tau)$. As in the proof of [13, Theorem 4.3], $Q(\pi(s_\sigma s_\tau^*))$ is nonzero only if there exist $\rho \in T \cap \Lambda^{d(\sigma)}$

and α, β such that

$$(\tau x_M \alpha)(0, M) = (\rho x_M \beta)(0, M). \tag{4.7}$$

We claim that $(\tau x_M \alpha)(0, M) = (\tau x_M)(0, M)$ for all $\alpha \in s(x_M)\Lambda$: suppose otherwise for contradiction. Then there exists i such that $d(\alpha)_i > 0$ and $d(\tau x_M)_i < M_i$ so $d(x_M)_i < (M - d(\tau))_i$. But $s((\tau x_M)(0, M)) = s(x_M(0, M - d(\tau)))$, and since $M \geqslant d(\tau) + n_x$, we have $M - d(\tau) \geqslant n_x$. It follows that $\Lambda^{e_i}(x(M - d(\tau))) = \emptyset$ by (2.1). The factorisation property now gives $s(x_M)\Lambda^{e_i} = \emptyset$, contradicting $d(\alpha)_i > 0$. The same argument gives $(\rho x_M \beta)(0, M) = (\rho x_M)(0, M)$ for all β. So (4.7) is equivalent to $(\tau x_M)(0, M) = (\rho x_M)(0, M)$ which is impossible by (4.6). Hence $Q(\pi(s_\sigma s_\tau^*)) = 0$ as required. $\square$

Proof of Theorem 4.5. By (4.4), we have $||\Phi(a)|| = ||P_{v_0}\Phi(a)||$, and Lemma 4.10 gives

$$P_{v_0}\Phi(a) \in \text{ span}\{s_{\lambda v_\lambda} s^*_{\mu v_\lambda} : \lambda, \mu \in \Pi E, d(\lambda) = d(\mu),\ s(\lambda) = s(\mu), T(\lambda) \text{ nonexhaustive}\}.$$

Since π is injective on the core by Theorem 3.1, we therefore have

$$||\pi(\Phi(a))|| = ||\Phi(a)|| = ||P_{v_0}\Phi(a)|| = ||\pi(P_{v_0}\Phi(a))||. \tag{4.8}$$

Using (4.8), Lemma 4.10(2), and Lemma 4.11, we therefore have

$$\begin{aligned} ||\pi(\Phi(a))|| &= ||\pi(P_{v_0}\Phi(a))|| = ||\pi(\Phi(P_{v_0}a))|| \\ &= ||Q(\pi(\Phi(P_{v_0}a)))|| = ||Q(\pi(P_{v_0}a))|| \\ &\leqslant ||\pi(P_{v_0})\pi(a)|| \leqslant ||\pi(a)||. \end{aligned}$$

The result then follows from Proposition 4.1. $\square$

Appendix A. The Cuntz–Krieger relations

The objective of the Cuntz–Krieger relations is to associate to each finitely aligned k-graph Λ a universal C^*-algebra $C^*(\Lambda)$ generated by partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ which has the following properties:

(a) The partial isometries s_λ are all nonzero.
(b) Connectivity in Λ is modelled by multiplication in $C^*(\Lambda)$.
(c) $C^*(\Lambda)$ is spanned by the elements $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.
(d) The *core* subalgebra $\overline{\text{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}$ is AF.
(e) A representation π of $C^*(\Lambda)$ is faithful on the core if and only if $\pi(s_v) \neq 0$ for every vertex v.

Relations (i) and (ii) of Definition 2.5 address property (b). Definition 2.5(iii) ensures that property (c) is satisfied. Definition 2.5(iii) has not appeared explicitly in previous analyses of Cuntz–Krieger algebras, but it has always been a consequence of the Cuntz–Krieger relations (see, for example, [13, Proposition 3.5]). Proposition 6.4 of [12] indicates why we have to impose Definition 2.5(iii) explicitly to deal with k-graphs that are not row-finite. The analysis of Section 3 shows that relations (i)–(iii) of Definition 2.5 also guarantee property (d).

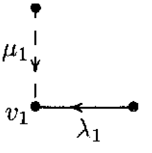
We must now produce a fourth Cuntz–Krieger relation which guarantees that $C^*(\Lambda)$ satisfies (a) and (e); in the following discussion, therefore, we assume that Definition 2.5(i)–(iii) hold. We describe examples of k-graphs using their 1-*skeletons* as in [13, Section 2].

The analyses of [6,13] suggest that a suitable relation might be

$$t_v = \sum_{\lambda \in v\Lambda^{\leqslant n}} t_\lambda t_\lambda^* \quad \text{whenever } v\Lambda^{\leqslant n} \text{ is finite.} \tag{A.1}$$

However, this relation fails to guarantee (a), even for row-finite k-graphs, as can be seen from the following example.

Example A.1. Consider the row-finite 2-graph Λ_1 with 1-skeleton

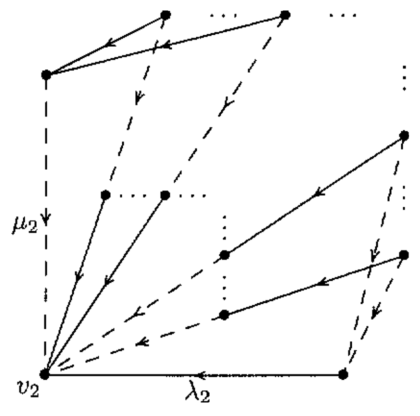


where $d(\lambda_1) = (1, 0)$ and $d(\mu_1) = (0, 1)$. The range projections $s_{\lambda_1} s_{\lambda_1}^*$ and $s_{\mu_1} s_{\mu_1}^*$ are orthogonal by (A.1) for $n = (1, 1)$, but must both be equal to s_{v_1} by (A.1) with $n = (0, 1)$ and $n = (1, 0)$. Consequently $s_{v_1} = 0$, so (A.1) falls to ensure condition (a) for $C^*(\Lambda_1)$.

For the *row-finite* k-graphs of [13] ($v\Lambda^{e_i}$ is always finite), we avoided the problem illustrated by this example by assuming that our k-graphs (Λ, d) were *locally convex*: the k-graph (Λ, d) is locally convex if for all $v \in \Lambda^0$, $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$, both $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are nonempty [13, Definition 3.9].

For locally convex row-finite k-graphs, the Cuntz–Krieger relations used in [13] are equivalent to Definition 2.5(i)–(iii) and (A.2). It is shown in [13, Theorem 3.15] that these relations imply (a), and the discussion of [13, page 109] shows that they imply (e). However, Example A.2 demonstrates that for non-row-finite k-graphs, local convexity is not enough to ensure that (A.1) implies (e).

Example A.2. Consider the locally convex finitely aligned 2-graph Λ_2 with 1-skeleton



where solid edges have degree (1,0) and dashed edges have degree (0,1). Relation (A.1) does not impose any equalities at v_2 because $v_2\Lambda_2^{\leqslant n}$ is infinite for all $n\neq 0$. The Cuntz–Krieger family $\{S_\lambda : \lambda\in\Lambda_2\}$ provided by the boundary-path representation satisfies $S_{v_2} - (S_{\lambda_2}S_{\lambda_2}^* + S_{\mu_2}S_{\mu_2}^*) = 0$. However, for any nontrivial projection P, taking $T_{v_2} := S_{v_2}\oplus P$ and $T_\sigma = S_\sigma \oplus 0$ for $\sigma\in\Lambda_2\backslash\{v_2\}$ gives a Cuntz–Krieger Λ_2-family satisfying Definition 2.5(i)–(iii) and (A.1) in which $T_{v_2} - (T_{\lambda_2}T_{\lambda_2}^* + T_{\mu_2}T_{\mu_2}^*)\neq 0$. In particular, $\{S_\lambda : \lambda\in\Lambda_2\}$ satisfies Definition 2.5(i)–(iii) and (A.1), but the representation determined by $\{S_\lambda : \lambda\in\Lambda_2\}$ is not faithful on the core, even though $S_v\neq 0$ for all $v\in\Lambda_2^0$.

The key property of Λ_2 which causes the problems with relation (A.1) is that there exists a finite subset of $v_2\Lambda_2$ (namely $\{\lambda_2,\mu_2\}$) whose range projections together dominate all the range projections associated to paths in $v_2\Lambda_2\backslash\{v\}$, but no such subset of the form $v_2\Lambda_2^{\leqslant n}$. For a finitely aligned k-graph Λ and $v\in\Lambda^0$, we can use Definition 2.5(iii) to characterise the finite subsets of $v\Lambda$ whose range projections together dominate all the range projections associated to nontrivial paths with range v: they are precisely the finite exhaustive sets of Definition 2.4.

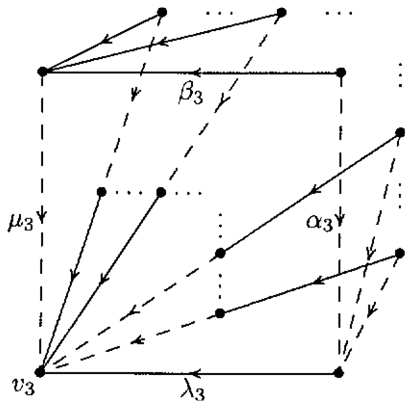
Example A.2 therefore suggests that Cuntz–Krieger relation (iv) should be

$$t_v = \sum_{\lambda\in E} t_\lambda t_\lambda^* \quad \text{for every } v\in\Lambda^0 \text{ and finite exhaustive } E\subset v\Lambda\backslash\{v\}. \tag{A.2}$$

Example (Example A.1 continued). The only finite exhaustive subset of $v_1\Lambda_1$ which does not contain v_1 is the set $\{\lambda_1,\mu_1\}$. In particular, (A.2) does not insist that either $t_{\lambda_1}t_{\lambda_1}^*$ or $t_{\mu_1}t_{\mu_1}^*$ is equal to t_{v_1}, and so replacing (A.1) with (A.2) eliminates the pathology associated to the non-local-convexity of Λ_1.

The only problem with (A.2) is that it is predicated on the notion that the range projections associated to paths in a finite exhaustive subset of $v\Lambda\backslash\{v\}$ are mutually orthogonal. The following example shows that this is not true.

Example A.3. Consider the locally convex 2-graph Λ_3 with 1-skeleton



where solid edges have degree (1,0) and dashed edges have degree (0,1). As in Example A.2, the fourth Cuntz–Krieger relation must insist that the range projections associated to λ_3 and μ_3 together fill up t_{v_3}, or else (e) will fail because $\{\lambda_3, \mu_3\}$ is finite and exhaustive. However, the range projections $t_{\lambda_3}t_{\lambda_3}^*$ and $t_{\mu_3}t_{\mu_3}^*$ are not orthogonal: by Lemma 2.7(i), $t_{\lambda_3}t_{\lambda_3}^*t_{\mu_3}t_{\mu_3}^* = t_{\lambda_3\alpha_3}t_{\lambda_3\alpha_3}^*$. Indeed there is no finite exhaustive subset of $v\Lambda$ whose range projections are orthogonal.

The solution to the problem illustrated in Example A.3 is to use products rather than sums to express the fourth Cuntz–Krieger relation.

Example (Example A.3 continued). Lemma 2.7(i) says that in any family satisfying Definition 2.5(i)–(iii), the projections $t_{\lambda_3}t_{\lambda_3}^*$ and $t_{\mu_3}t_{\mu_3}^*$ commute. Consequently, it makes sense to express the requirement that the range projections associated to λ_3 and μ_3 fill up t_{v_3} with the formula

$$(t_{v_3} - t_{\lambda_3}t_{\lambda_3}^*)(t_{v_3} - t_{\mu_3}t_{\mu_3}^*) = 0. \tag{A.3}$$

Relation (iv) of Definition 2.5, namely

$$\prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0 \quad \text{for every } v \in \Lambda^0 \text{ and finite exhaustive } E \subset v\Lambda \tag{A.4}$$

is the generalisation of (A.3) to arbitrary finite exhaustive sets in an arbitrary finitely aligned k-graph. Note that (A.4) reduces to (A.2) when the range projections associated to paths in E are mutually orthogonal (as in Λ_2). Proposition 2.12 together with Theorem 3.1 show that (A.4) ensures (a) and (e).

Appendix B. 1-Graphs and locally convex row-finite k-graphs

Recall from [13] that a k-graph (Λ, d) is *row-finite* if $v\Lambda^{e_i}$ is finite for all $i \in \{1, \dots, k\}$ and $v \in \Lambda^0$. Recall also from [13] that (Λ, d) is *locally convex* if $\lambda \in v\Lambda^{e_i}$ and $v\Lambda^{e_j} \neq \emptyset$ for $i \neq j$ implies $s(\lambda)\Lambda^{e_j} \neq \emptyset$.

Proposition B.1. *For* 1*-graphs, the Cuntz–Krieger families of Definition* 2.5 *coincide with those of* [6]. *For locally convex row-finite k-graphs, the Cuntz–Krieger families of Definition* 2.5 *coincide with those of* [13].

We prove Proposition B.1 with three lemmas.

Lemma B.2. *Let* (Λ, d) *be a k-graph. If* $k > 1$, *suppose that* Λ *is locally convex and row-finite. Let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a Cuntz–Krieger* Λ*-family. Then* $\{t_\lambda : \lambda \in \Lambda\}$ *is a Cuntz–Krieger* Λ*-family in the sense of* [6] *if* $k = 1$, *and is a Cuntz–Krieger* Λ*-family in the sense of* [13] *if* $k > 1$.

Proof. By Lemma 2.7(iii), we know that $t_v \geqslant \sum_{\lambda \in E} t_\lambda t_\lambda^*$ whenever $E \subset v\Lambda^{e_i}$ is finite. By [13, Proportion 3.11], it suffices to show that for every $v \in \Lambda^0$ and $1 \leqslant i \leqslant k$ such that $0 < |v\Lambda^{e_i}| < \infty$, we have

$$t_v = \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*.$$

By Definition 2.5(iv), we need only show that $v\Lambda^{e_i}$ is exhaustive whenever $0 < |v\Lambda^{e_i}| < \infty$. This is trivial for $k = 1$: every path with range v is either equal to v, in which case it is extended by every path in $v\Lambda^{e_1}$, or has an initial segment of length 1, and hence must extend an edge in Λ^{e_1}. Now suppose $k > 1$ and Λ is locally convex and row-finite, fix v, i with $v\Lambda^{e_i} \neq \emptyset$, and let $\lambda \in v\Lambda$. We must show that there exists $\mu \in v\Lambda^{e_i}$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. If $\lambda = v$, then $\Lambda^{\min}(\lambda, \mu) = \{(\mu, s(\mu))\}$ for all $\mu \in v\Lambda^{e_i}$. If $d(\lambda) \geqslant e_i$, then with $\mu = \lambda(0, e_i) \in v\Lambda^{e_i}$, we have $\Lambda^{\min}(\lambda, \mu) = \{(s(\lambda), \lambda(e_i, d(\lambda)))\} \neq \emptyset$. Finally, if $\lambda \neq v$ and $d(\lambda)_i = 0$, then since $v\Lambda^{e_i}$ is non-empty, $|d(\lambda)|$ applications of the local convexity condition show that there exists $\alpha \in s(\lambda)\Lambda^{e_i}$. With $\mu := (\lambda\alpha)(0, e_i)$ and $\beta := (\lambda\alpha)(e_i, d(\lambda\alpha))$ we have $\mu \in v\Lambda^{e_i}$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. $\square$

Lemma B.3. *Let* Λ *be a* 1*-graph and suppose that* $\{t_\lambda : \lambda \in \Lambda\}$ *is a Cuntz–Krieger* Λ*-family in the sense of* [6]. *Then* $\{t_\lambda : \lambda \in \Lambda\}$ *satisfies* (iv) *of Definition* 2.5.

Proof. Let $v \in \Lambda^0$ and let E be a finite exhaustive subset of $v\Lambda$. We proceed by induction on $L(E) := |\{i \in \mathbb{N} : E \cap \Lambda^i \neq \emptyset\}|$. For a basis case, suppose that $L(E) = 1$, so $E \subset \Lambda^i$ for some i. Then $\{\lambda(0,j) : \lambda \in E\} = v\Lambda^j$ for $1 \leqslant j \leqslant i$, and then i applications of [6, Eq. (1.3)] give

$$\prod_{\lambda \in E} (s_v - s_\lambda s_\lambda^*) = s_v - \sum_{\lambda \in E} s_\lambda s_\lambda^* = 0.$$

Now fix $l \geqslant 1$ and suppose that Definition 2.5(iv) holds whenever $L(E) \leqslant l$, and suppose that $L(E) = l+1$. Let $I := \max\{i : E \cap \Lambda^i \neq \emptyset\}$. Since $L(E) \geqslant 2$, $\{\lambda \in E : d(\lambda) < I\}$ is nonempty, so let $J := \max\{j < I : E \cap \Lambda^j \neq \emptyset\}$. Fix $\lambda \in E$ with $d(\lambda) = I$. Since E is exhaustive, we have either $\lambda(0,j) \in E$ for some $j \leqslant J$ or $\{\lambda(0,J)v : v \in s(\lambda(0,J))\Lambda^{I-J}\} \subset E$. If $\lambda(0,j) \in E$ for some $j \leqslant J$, then $t_v - t_\lambda t_\lambda^* \geqslant t_v - t_{\lambda(0,j)} t_{\lambda(0,j)}^*$, and $E' := E \backslash \{\lambda\}$ is exhaustive with $\prod_{\mu \in E'} (s_v - s_\mu s_\mu^*) = \prod_{\mu \in E} (s_v - s_\mu s_\mu^*)$. On the other hand, if $\{\lambda(0,J)v : v \in s(\lambda(0,J))\Lambda^{I-J}\} \subset E$, then

$$E' := (E \backslash \{\lambda(0,J)v : v \in s(\lambda(0,J))\Lambda^{I-J}\}) \cup \{\lambda(0,J)\}$$

is also exhaustive, and $\prod_{\mu \in E'} (s_v - s_\mu s_\mu^*) = \prod_{\mu \in E} (s_v - s_\mu s_\mu^*)$. Repeating this process for each $\lambda \in E \cap \Lambda^I$, we obtain a finite exhaustive $E'' \in v\Lambda$ which satisfies

(1) $\{i \in \mathbb{N} : E'' \cap \Lambda^i \neq \emptyset\} = \{i \in \mathbb{N} : E \cap \Lambda^i \neq \emptyset\} \backslash \{I\}$, so $L(E'') = L(E) - 1 = l$; and
(2) $\prod_{\mu \in E''} (s_v - s_\mu s_\mu^*) = \prod_{\mu \in E} (s_v - s_\mu s_\mu^*)$.

The result now follows from the inductive hypothesis applied to E''. □

Lemma B.4. *Let (Λ, d) be a locally convex row-finite k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family in the sense of* [13, *Definition* 3.3]. *Then $\{t_\lambda : \lambda \in \Lambda\}$ satisfies* (iv) *of Definition* 2.5.

Proof. Let $v \in \Lambda^0$, let E be a finite exhaustive subset of $v\Lambda$, and let $N := \bigvee_{\lambda \in E} d(\lambda)$. Now let $E' := \{\lambda v : \lambda \in E, v \in s(\lambda)\Lambda^{\leqslant N - d(\lambda)}\}$. By [13, Lemma 3.6], and since E is exhaustive, we have $E' = v\Lambda^{\leqslant N}$. Hence relation (4) of [13, Definition 3.3] ensures that $s_v = \sum_{\mu \in E'} s_\mu s_\mu^*$, so

$$\prod_{\lambda \in E} (s_v - s_\lambda s_\lambda^*) \leqslant \prod_{\mu \in E'} (s_v - s_\mu s_\mu^*) = s_v - \sum_{\mu \in v\Lambda^{\leqslant N}} s_\mu s_\mu^* = 0. \qquad \square$$

Proof of Proposition B.1. Lemma B.2 shows that the Cuntz–Krieger families of Definition 2.5 give Cuntz–Krieger families as defined in [6,13]. Relations (i) and (ii) of Definition 2.5 are obviously satisfied by the Cuntz–Krieger families of both [6,13]. In a 1-graph, $\Lambda^{\min}(\lambda, \mu)$ equals $\{(\lambda', s(\mu))\}$ if $\mu = \lambda\lambda'$, $\{(s(\lambda), \mu')\}$ if $\lambda = \mu\mu'$, and $\emptyset$ otherwise. It follows that relation (iii) of Definition 2.5 is satisfied by the Cuntz–Krieger families of [6]. Proposition 3.5 of [13] shows that for locally convex row-finite k-graphs, Relation (iii) of Definition 2.5 is satisfied by the Cuntz–Krieger families of [13]. The result now follows from Lemmas B.3 and B.4. □

Appendix C. Checking the relations in terms of generators

Theorem C.1. *Let (Λ, d) be a finitely aligned k-graph. Let*

$$\left\{ t_\lambda : \lambda \in \left(\bigcup_{i=1}^{k} \Lambda^{e_i} \right) \cup \Lambda^0 \right\}$$

be a family of partial isometries in a C^*-algebra. Then there is at most one Cuntz–Krieger Λ-family $\{t'_\lambda : \lambda \in \Lambda\}$ such that $t'_\lambda = t_\lambda$ for all $\lambda \in (\bigcup_{i=1}^{k} \Lambda^{e_i}) \cup \Lambda^0$. Furthermore, such a Cuntz–Krieger Λ-family exists if and only if

(i) *$\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections.*
(ii) *$t_\lambda t_\alpha = t_\mu t_\beta$ when $\lambda, \mu, \alpha, \beta \in (\bigcup_{i=1}^{k} \Lambda^{e_i}) \cup \Lambda^0$ satisfy $\lambda\alpha = \mu\beta$.*
(iii) *$t_\lambda^* t_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \bigcup_{i=1}^{k} \Lambda^{e_i}$.*
(iv) *for every $v \in \Lambda^0$ and every finite exhaustive $E \subset \bigcup_{i=1}^{k} v\Lambda^{e_i}$,*

$$\prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0.$$

Before proving Theorem C.1, we establish a number of preliminary results.

Lemma C.2. *Let (Λ, d) be a finitely aligned k-graph. Suppose that $\{t_\lambda : \lambda \in \Lambda\}$ is a collection of partial isometries satisfying Definition* 2.5(i) *and* (ii). *Then $\{t_\lambda : \lambda \in \Lambda\}$ satisfies Definition* 2.5(iii) *if and only if*

$$t_\lambda^* t_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^* \quad \textit{for all } \lambda, \mu \in \bigcup_{i=1}^{k} \Lambda^{e_i}. \tag{C.1}$$

Proof. Since (C.1) is a special case of Definition 2.5(iii), we need only show the "if" direction. This in turn will follow from [12, Lemma 9.2] if we can show that Definition 2.5(i) and (ii) together with (C.1) imply relations (3) and (4) of [12, Definition 7.1], namely that

$$t_\lambda^* t_\lambda = t_{s(\lambda)} \quad \text{for all } \lambda \in \Lambda; \text{ and} \tag{C.2}$$

$$t_v \geqslant \sum_{\lambda \in F} t_\lambda t_\lambda^* \quad \text{whenever } F \subset \Lambda^n v \text{ is finite.} \tag{C.3}$$

An inductive argument on the length of λ establishes (C.2). With this in hand, (C.3) then follows from (C.1) together with Definition 2.5(ii) as in Lemma 2.7(iii). $\square$

Proposition C.3. *Let (Λ, d) be a finitely aligned k-graph. A family $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying Definition* 2.5(i)–(iii) *is a Cuntz–Krieger*

Λ-family if and only if for every $v \in \Lambda^0$ and every finite exhaustive subset $E \subset \bigcup_{i=1}^k v\Lambda^{e_i}$,

$$\prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0. \tag{C.4}$$

Notation C.4. In this section, we make use of the following notation:

- Given a set $E \subset \Lambda$, define $I(E) := \bigcup_{i=1}^k \{\lambda(0, e_i) : \lambda \in E, d(\lambda)_i > 0\}$.
- Given $E \subset \Lambda$ and $\mu \in \Lambda$, let $\mathrm{Ext}(\mu; E) := \bigcup_{\lambda \in E} \{\alpha : (\alpha, \beta) \in \Lambda^{\min}(\mu, \lambda)\}$.
- Given $E \subset \Lambda$, let $L(E) := \sum_{i=1}^k \max_{\lambda \in E} d(\lambda)_i$.

Lemma C.5. *Let (Λ, d) be a finitely aligned k-graph and let $v \in \Lambda^0$. Suppose $E \subset v\Lambda$ is finite and exhaustive, and let $\mu \in v\Lambda$. Then* $\mathrm{Ext}(\mu; E)$ *is a finite exhaustive subset of $s(\mu)\Lambda$.*

Proof. Since E is finite and Λ is finitely aligned we know that $\mathrm{Ext}(\mu; E)$ is finite, so we need only check that $\mathrm{Ext}(\mu; E)$ is exhaustive. Let $\sigma \in s(\mu)\Lambda$. Since E is exhaustive, there exists $\lambda \in E$ with $\Lambda^{\min}(\lambda, \mu\sigma) \neq \emptyset$, say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu\sigma)$. So $\lambda\alpha = \mu\sigma\beta$, and hence

$$(\alpha(0, (d(\lambda) \vee d(\mu)) - d(\lambda)), (\sigma\beta)(0, (d(\lambda) \vee d(\mu)) - d(\mu))) \in \Lambda^{\min}(\lambda, \mu).$$

Hence $\tau := (\sigma\beta)(0, (d(\lambda) \vee d(\mu)) - d(\mu))$ belongs to $\mathrm{Ext}(\mu; E)$, and then

$$((\sigma\beta)(d(\sigma), d(\sigma) \vee d(\tau)), (\sigma\beta)(d(\tau), d(\sigma) \vee d(\tau))) \in \Lambda^{\min}(\sigma, \tau). \qquad \square$$

Lemma C.6. *Let (Λ, d) be a finitely aligned k-graph, let $v \in \Lambda^0$, and suppose that $E \subset v\Lambda \backslash \{v\}$ is finite and exhaustive. Then $I(E)$ is also finite and exhaustive.*

Proof. We have $I(E)$ is finite because E is finite, so we just need to show that $I(E)$ is exhaustive. Let $\mu \in v\Lambda$. Since E is exhaustive, there exists $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$, say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. Since $\lambda \in E$, we have $d(\lambda) \neq 0$, so fix i such that $d(\lambda)_i \neq 0$; then $\lambda(0, e_i) \in I(E)$. Let $\rho := (\lambda\alpha)(0, d(\mu) \vee e_i)$, let $\eta := \rho(e_i, d(\rho))$, and let $\zeta := \rho(d(\mu), d(\rho))$. Then $\lambda(0, e_i)\eta = \rho = \mu\zeta$, so $(\eta, \zeta) \in \Lambda^{\min}(\lambda(0, e_i), \mu)$. Since $\mu \in v\Lambda$ was arbitrary, it follows that $I(E)$ is exhaustive. $\square$

Lemma C.7. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a family of partial isometries satisfying Definition* 2.5(i)–(iii). *Let $v \in \Lambda^0$, let $\lambda \in v\Lambda$ and suppose that $E \subset s(\lambda)\Lambda$ is finite and satisfies $\prod_{\nu \in E} (t_{s(\lambda)} - t_\nu t_\nu^*) = 0$. Then*

$$t_v - t_\lambda t_\lambda^* = \prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*).$$

Proof. Since $t_{\lambda\mu} t_{\lambda\mu}^* \leqslant t_\lambda t_\lambda^*$ for all $\mu \in s(\lambda)\Lambda$, we have

$$(t_v - t_\lambda t_\lambda^*)(t_v - t_{\lambda\nu} t_{\lambda\nu}^*) = t_v - t_\lambda t_\lambda^*$$

for all $\nu \in E$. It follows that

$$(t_v - t_\lambda t_\lambda^*) \prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*) = t_v - t_\lambda t_\lambda^*. \tag{C.5}$$

On the other hand,

$$\begin{aligned}
(t_v - t_\lambda t_\lambda^*)&\left(\prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\right) \\
&= t_v\left(\prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\right) - t_\lambda t_\lambda^*\left(\prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\right) \\
&= \left(\prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\right) - \left(\prod_{\nu \in E}(t_\lambda t_\lambda^* - t_{\lambda\nu} t_{\lambda\nu}^*)\right) \\
&= \left(\prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\right) - t_\lambda\left(\prod_{\nu \in E} (t_{s(\lambda)} - t_\nu t_\nu^*)\right) t_\lambda^* \\
&= \prod_{\nu \in E} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*)
\end{aligned}$$

because $\prod_{\nu \in E} (t_{s(\lambda)} - t_\nu t_\nu^*) = 0$ by hypothesis. □

Lemma C.8. *Let (Λ, d) be a finitely aligned k-graph. Let $v \in \Lambda^0$ and suppose $E \subset v\Lambda$ is finite. Suppose $\lambda \in I(E)$. Then $L(\mathrm{Ext}(\lambda; E)) < L(E)$.*

Proof. Since $\lambda \in I(E)$, we have $d(\lambda) = e_i$ and $\lambda\lambda' \in E$ for some i, λ'. For $j \in \{1, \ldots, k\}$, we have

$$\max_{\nu \in \mathrm{Ext}(\lambda; E)} d(\nu)_j = \max_{\mu \in E, \Lambda^{\min}(\lambda,\mu) \neq \emptyset} ((d(\lambda) \vee d(\mu)) - e_i)_j. \tag{C.6}$$

If $i \neq j$, then (C.6) becomes

$$\max_{\nu \in \operatorname{Ext}(\lambda;E)} d(\nu)_j = \max_{\mu \in E, \Lambda^{\min}(\lambda,\mu) \neq \emptyset} d(\mu)_j \leqslant \max_{\mu \in E} d(\mu)_j.$$

On the other hand, if $i = j$, then we use (C.6) to calculate

$$\begin{aligned}\max_{\nu \in \operatorname{Ext}(\lambda;E)} d(\nu)_j &= \max_{\mu \in E, \Lambda^{\min}(\lambda,\mu) \neq \emptyset} ((d(\lambda) \vee d(\mu)) - e_i)_i \\ &\leqslant \max_{\mu \in E} ((d(\lambda) \vee d(\mu)) - e_i)_i \\ &= \left(\max_{\mu \in E} d(\mu)_i\right) - 1 \\ &\quad \text{since } \lambda\lambda' \in E \text{ so there exist } \mu \in E \text{ with } d(\mu)_i \geqslant 1.\end{aligned}$$

We therefore have

$$\begin{aligned}L(\operatorname{Ext}(\lambda;E)) &= \sum_{j=1}^{k} \max_{\nu \in \operatorname{Ext}(\lambda;E)} d(\nu)_j \\ &\leqslant \left(\sum_{j \in \{1,\dots,k\} \setminus \{i\}} \max_{\mu \in E} d(\mu)_j\right) + \left(\max_{\mu \in E} d(\mu)_i\right) - 1 \\ &< \sum_{j=1}^{k} \max_{\mu \in E} d(\mu)_j \\ &= L(E). \qquad \square\end{aligned}$$

Proof of Proposition C.3. We must show that for every $v \in \Lambda^0$ and every finite exhaustive $F \subset v\Lambda$, we have

$$\prod_{\mu \in F} (t_v - t_\mu t_\mu^*) = 0. \tag{C.7}$$

We proceed by induction on $L(F)$. If $L(F) = 1$, then $F \subset \bigcup_{i=1}^{k} v\Lambda^{e_i}$, and (C.7) is an instance of (C.4).

Now suppose that (C.7) holds whenever $L(F) \leqslant n$, and fix $v \in \Lambda^0$ and $F \subset v\Lambda$ finite exhaustive with $L(F) = n + 1$. If $v \in F$, there is nothing to prove, so assume without loss of generality that $v \notin F$. Then $I(F)$ is finite and exhaustive by Lemma C.6. Fix $\lambda \in I(F)$. By Lemma C.5, we know that $\operatorname{Ext}(\lambda;F)$ is finite and exhaustive. By Lemma C.8, we know that $L(\operatorname{Ext}(\lambda;F)) \leqslant n$, so the inductive hypothesis ensures that $\prod_{\nu \in \operatorname{Ext}(\lambda;F)} (t_{s(\lambda)} - t_\nu t_\nu^*) = 0$. It then follows from Lemma C.7 that

$$\prod_{\nu \in \operatorname{Ext}(\lambda;F)} (t_v - t_{\lambda\nu} t_{\lambda\nu}^*) = t_v - t_\lambda t_\lambda^*. \tag{C.8}$$

For each $\nu \in \mathrm{Ext}(\lambda; F)$, there exists $\mu \in F$ with $\lambda\nu = \mu\mu'$, so $t_{\lambda\nu}t_{\lambda\nu}^* \leqslant t_\mu t_\mu^*$, and hence

$$\prod_{\nu \in \mathrm{Ext}(\lambda;F)} (t_v - t_{\lambda\nu}t_{\lambda\nu}^*) \geqslant \prod_{\mu \in F} (t_v - t_\mu t_\mu^*). \tag{C.9}$$

We can therefore calculate

$$\begin{aligned}\prod_{\mu \in F} (t_v - t_\mu t_\mu^*) &\leqslant \prod_{\lambda \in I(F)} \left(\prod_{\nu \in \mathrm{Ext}(\lambda;F)} (t_v - t_{\lambda\nu}t_{\lambda\nu}^*) \right) \quad \text{by (C.9)} \\ &= \prod_{\lambda \in I(F)} (t_v - t_\lambda t_\lambda^*) \quad \text{by (C.8)} \\ &= 0 \quad \text{by (C.4).} \qquad \square\end{aligned}$$

Proof of Theorem C.1. The factorisation property and Definition 2.5(ii) show that any Cuntz–Krieger Λ-family $\{t'_\lambda : \lambda \in \Lambda\}$ satisfying $t'_\lambda = t_\lambda$ for all $\lambda \in (\bigcup_{i=1}^k \Lambda^{e_i}) \cup \Lambda^0$ must satisfy

$$t'_\lambda = t_{\lambda_1} t_{\lambda_2} \cdots t_{\lambda_{|d(\lambda)|}} \tag{C.10}$$

for each $\lambda \in \Lambda$ and each factorisation $\lambda = \lambda_1 \cdots \lambda_{|d(\lambda)|}$ where the λ_i belong to $(\bigcup_{i=1}^k \Lambda^{e_i}) \cup \Lambda^0$. This proves that there is at most one such Cuntz–Krieger Λ-family.

Suppose that such a Cuntz–Krieger Λ-family $\{t'_\lambda : \lambda \in \Lambda\}$ exists. Then conditions (i)–(iv) of Theorem C.1 are immediate consequences of the Cuntz–Krieger relations.

Now suppose that $\{t_\lambda : \lambda \in (\bigcup_{i=1}^k \Lambda^{e_i}) \cup \Lambda^0\}$ satisfy (i)–(iv) of Theorem C.1. An inductive argument using condition (ii) of Theorem C.1 shows that (C.10) gives a well-defined family of partial isometries $\{t'_\lambda : \lambda \in \Lambda\}$.

We have that $\{t'_\lambda : \lambda \in \Lambda\}$ satisfies Definition 2.5(i) because this is precisely condition (i) of Theorem C.1. Eq. (C.10) and the factorisation property for Λ ensure that $\{t'_\lambda : \lambda \in \Lambda\}$ satisfies Definition 2.5(ii). Condition (iii) of Theorem C.1 and Lemma C.2 then imply that $\{t'_\lambda : \lambda \in \Lambda\}$ satisfies Definition 2.5(iii). We can now use Proposition C.3 and condition (iv) of Theorem C.1 to show that $\{t'_\lambda : \lambda \in \Lambda\}$ satisfies Definition 2.5(iv). $\square$

References

[1] S. Adji, M. Laca, M. Nilsen, I. Raeburn, Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups, Proc. Amer. Math. Soc. 122 (1994) 1133–1141.
[2] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C^*-algebras of row-finite graphs, New York J. Math. 6 (2000) 307–324.
[3] J. Cuntz, W. Krieger, A class of C^*-algebras and topological Markov chains, Invent. Math. 56 (1980) 251–268.
[4] M. Enomoto, Y. Watatani, A graph theory for C^*-algebras, Math. Japan. 25 (1980) 435–442.
[5] N.J. Fowler, Discrete product systems of Hilbert bimodules, Pacific J. Math. 204 (2002) 335–375.

[6] N.J. Fowler, M. Laca, I. Raeburn, The C^*-algebras of infinite graphs, Proc. Amer. Math. Soc. 128 (2000) 2319–2327.
[7] N.J. Fowler, I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, Indiana Univ. Math. J. 48 (1999) 155–181.
[8] N.J. Fowler, A. Sims, Product systems over right-angled Artin semigroups, Trans. Amer. Math. Soc. 354 (2002) 1487–1509.
[9] A. Kumjian, D. Pask, Higher rank graph C^*-algebras, New York J. Math. 6 (2000) 1–20.
[10] A. Kumjian, D. Pask, I. Raeburn, Cuntz–Krieger algebras of directed graphs, Pacific J. Math. 184 (1998) 161–174.
[11] A.L.T. Paterson, Graph inverse semigroups, goupoids and their C^*-algebras, J. Operator Theory 48 (2002) 645–662.
[12] I. Raeburn, A. Sims, Product systems of graphs and the Toeplitz algebras of higher-rank graphs, J. Operator Theory, to appear.
[13] I. Raeburn, A. Sims, T. Yeend, Higher-rank graphs and their C^*-algebras, Proc. Edinburgh Math. Soc. 46 (2003) 99–115.
[14] J. Spielberg, A functorial approach to the C^*-algebras of a graph, Internat. J. Math. 13 (2002) 245–277.
[15] W. Szymański, The range of K-invariants for C^*-algebras of infinite graphs, Indiana Univ. Math. J. 51 (2002) 239–249.