Branching Flags, Branching Nets,
and Reverse Matchings

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The relation among three classes of combinatorial objects parametrized by partitions is discussed: the branching flag is a tree-type generalization of the flag in a vector space over a finite field; the branching nest is its finite set-theoretical counterpart; and the reverse matching, equinumerous to branching nests, is a dual concept of the complete matching. A mapping will be constructed of the set of branching flags into that of branching nests, which will give a decomposition of the variety of branching flags into cells parametrized by reverse matchings. Its Poincaré polynomial is related to a refinement of q-Stirling numbers.

1. INTRODUCTION

Let $GF(q)$ denote the finite field with $q$ elements. For a positive integer $k$, $V_k(q)$ is the $k$-dimensional vector space over $GF(q)$, $L_k(q)$ the lattice of subspaces of $V_k(q)$, $[1, k]$ the $k$-set $\{1, 2, \ldots, k\}$ and $B_k$ the lattice of subsets of $[1, k]$. Here, both $L_k(q)$ and $B_k$ are ordered by inclusion, and rank in $L_k(q)$ and $B_k$ is dimension and cardinality, respectively.

It is well known, and also easily seen from the arguments herein as a special case, that the number of complete flags in $V_k(q)$ is $(1 + q) (1 + q + q^2) \cdots (1 + q + \cdots + q^{k-1})$, and there is a mapping of the set of these flags into the set of permutations on the set $[1, k]$, such that the cardinality of the inverse image of a permutation $\sigma$ is just $q^{l(\sigma)}$, where $l(\sigma)$ denotes the number of inversions in $\sigma$.

This paper answers the question what would happen if we replace in these circumstances the set of permutations on a finite set by the set of injections or surjections between finite sets. This is done, as one expects, in terms of slightly generalizing the notion of flags. A further refinement is also made by imposing some Young diagrammatic condition to the map-
pings. This way, we reach the notion of a refinement, parametrized by partitions, of $q$-Stirling numbers of the second kind. An enumerative and geometric interpretation of the $q$-Fibonacci numbers is also obtained.

Let $k$ and $n$ be given positive integers.

**Definition 1.1.** A sequence $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ of elements of $\mathcal{Q}_k(q)$ is called a “branching flag” (of length $n$) if it satisfies:

1. $\dim U_n = 1$;
2. $\dim U_j - \dim U_{j+1} = 0$ or $1$ ($1 \leq j < n$);
3. if $\dim U_j = \dim U_{j+1} = \cdots = \dim U_{j+s-1} > \dim U_{j+s}$, then $U_j, U_{j+1}, \ldots, U_{j+s-1} \supsetneq U_{j+s}$.

We put $U_{n+1} = \{0\}$ for convenience’ sake, so that these elements form an order-preserving image of a rooted tree.

**Definition 1.2.** A branching flag $(U_1, U_2, \ldots, U_n)$ is called “surjective” if $\dim U_j = k$, and “injective” if $\dim U_j - \dim U_{j+1} = 1$ ($1 \leq j < n$).

We represent the elements of $\mathcal{Q}_k(q)$ as $k$-tuples $(x_1, \ldots, x_k)$ of elements of the field. A sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. For a partition $\lambda$ such that $\lambda_1 = n$, we put $\lambda'_j := \max \{i; \lambda_i < j\}$ for $1 \leq j \leq n$. Then $\lambda' := (\lambda'_1, \ldots, \lambda'_n)$ is the conjugate of $\lambda$, and $\lambda'' = \lambda$. We denote by $F_\lambda$ the $d$-dimensional subspace consisting of all the vectors $u = (u_1, \ldots, u_k)$ such that $u_i = 0$ for all $i > d$. The chain $F_{\lambda_1} \supset F_{\lambda_2} \supset \cdots \supset F_{\lambda_k}$ in $\mathcal{Q}_k(q)$ is called the reference flag w.r.t. $\lambda$.

**Definition 1.3.** A branching flag $(V_1, V_2, \ldots, V_n)$ is “subordinate to” $\lambda$ if $V_j \subset F_{\lambda_j}$ for $1 \leq j \leq n$.

We denote by $E_\lambda(q)$ (resp. $F_\lambda(q)$) the set of all surjective (resp. injective) branching flags subordinate to $\lambda$, and $e_\lambda(q)$ (resp. $f_\lambda(q)$) its cardinality.

It is easy to verify that $e_\lambda(q)$ and $f_\lambda(q)$ are nonzero if and only if $\lambda_i \geq k + 1 - i$ ($1 \leq i \leq k$), and $e_\lambda(q) = 1$ if $k = 1$. For $f_\lambda(q)$, we have a complete explicit formula

$$f_\lambda(q) = [\lambda_k][\lambda_{k-1} - 1] \cdots [\lambda_1 - k + 1],$$

where we put $[i] := (1 - q^i)/(1 - q)$ for $i$ integer, and we shall in the following mainly be interested in determining $e_\lambda(q)$, for the investigation for injective branching flags goes almost parallel to, and even much simpler than, that of surjective branching flags. For example, we have

**Theorem 1.4.** $e_\lambda(q)$ and $f_\lambda(q)$ are both monic polynomials in $q$ with non-negative integral coefficients.
Rather than proving 1.4 inductively by a recursive formula (see Section 3) we shall give a direct combinatorial proof using the notion of the branching nest, whose definition goes parallel to that of the branching flag as follows:

**Definition 1.5.** A sequence \( \mathcal{A} = (A_1, A_2, ..., A_n) \) of elements of \( B_k \) is called a “branching nest” (of length \( n \)) if it satisfies:

1. \( \#A_n = 1 \);
2. \( \#A_j - \#A_{j+1} = 0 \) or \( 1 \) (\( 1 \leq j < n \));
3. if \( \#A_i = \#A_{j+1} = \ldots = \#A_{j+s-1} > \#A_{j+s} \), then \( A_j, A_{j+1}, ..., A_{j+s-1} \supset A_{j+s} \).

We put \( A_{n+1} = \emptyset \) for convenience’ sake, so that these elements form an order-preserving image of a rooted tree.

**Definition 1.6.** A branching nest \( (A_1, A_2, ..., A_n) \) is called “surjective” if \( \#A_1 = k \), and “injective” if \( \#A_j - \#A_{j+1} = 1 \) (\( 1 \leq j < n \)).

Now, for given \( k, n \), and \( \lambda \) as above, the “reference nest” w.r.t. \( \lambda \) is by definition the chain \( N_{\lambda_1} \supset N_{\lambda_2} \supset \ldots \supset N_{\lambda_n} \) in \( B_k \), where \( N_\lambda := [1, d] \) (\( 1 \leq d \leq k \)).

**Definition 1.7.** A branching nest \( (A_1, A_2, ..., A_n) \) is “subordinate to” \( \lambda \) if \( A_j \subset N_{\lambda_j} \) for \( 1 \leq j \leq n \).

Let \( E_\lambda \) (resp. \( F_\lambda \)) denote the set of all surjective (resp. injective) branching nests subordinate to \( \lambda \), and \( e_\lambda \) (resp. \( f_\lambda \)) its cardinality. Then

**Theorem 1.8.** \( e_\lambda = e_\lambda(1) \) and \( f_\lambda = f_\lambda(1) \).

To prove 1.4 and 1.8 directly, we shall construct a mapping of \( E_\lambda(q) \) (resp. \( F_\lambda(q) \)) into \( E_\lambda \) (resp. \( F_\lambda \)), and show that the cardinality of the inverse image of each branching nest under this mapping is a power of \( q \).

Now surjections and injections between finite sets come into picture as follows:

**Definition 1.9.** A “reverse matching (resp. matching)” on \( \lambda \) is a surjection (resp. injection) \( \phi \) of \( [1, n] \) into \( [1, k] \) such that \( \phi(j) \leq \lambda_j' \) (\( 1 \leq j \leq n \)).

A matching is sometimes called “complete matching” in the literature.

**Proposition 1.10.** There exists a 1–1 correspondence between \( E_\lambda \) (resp. \( F_\lambda \)) and the set of all reverse matchings (resp. matchings) on \( \lambda \).
Composing this bijection and the mapping of $E_\lambda(q)$ (resp. $F_\lambda(q)$) onto $E_\lambda$ (resp. $F_\lambda$), we have

**Theorem 1.11.** $e_\lambda(q) = \sum_\varphi q^{l(\varphi)}$ and $f_\lambda(q) = \sum_\psi q^{l(\psi)}$, where $\varphi$ (resp. $\psi$) runs through all reverse matchings (resp. matchings) on $\lambda$.

Here, $l(\varphi)$ is the natural generalization of the number of inversions in a permutation, defined as follows:

**Definition 1.12.** For a reverse matching (or matching) $\varphi$ on $\lambda$, $l(\varphi)$ is the number of nodes $(i, j)$ on the Young diagram $\lambda$ (i.e., $(i, j) \in \mathbb{Z}^2, 1 \leq i \leq k,$ and $1 \leq j \leq \lambda_i$) such that

1. $i > \varphi(j);$ 
2. $\varphi^{-1}(i) = \emptyset$ or $\min \varphi^{-1}(i) < j.$

Our first motivation in this work was to generalize T. Imai's pebble arranging puzzle [6].

**Definition 1.13.** $\lambda$ is "strict" if all parts $\lambda_i$ are distinct and positive. For a strict partition $\lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1 = n$, its "complementary" is the strict partition $\lambda^* := (\lambda_1^*, ..., \lambda_k^*)$, where $\lambda_i^* = n$ and $[1, n-1]$ is the disjoint union of $\{\lambda_i; 1 < i \leq k\}$ and $\{n - \lambda_i^*; 1 < i \leq l\}.$

Imai's puzzle, in our notation, is equivalent to proving $e_\lambda = e_{\lambda^*}$, where $\lambda$ is strict. By 1.8 this is a special case of

**Theorem 1.14.** If $\lambda$ is strict, then $e_\lambda(q) = e_{\lambda^*}(q)$.

A short proof was given to 1.14 by I. Amemiya [1] for $q = 1$, which survives to our case (Section 5). The proofs of Theorems 1.4, 1.8, and 1.11 are completed in Section 2.

**Notation.** We put $[i] := (1 - q^i)/(1 - q)$ for $i$ integer; $[i]! := [i][i-1] \cdots [1]$ for $i$ positive, and $[0]! := 1.$ For convenience' sake we put $e_\emptyset(q) = 1$, where $\emptyset$ is the void partition ($\lambda$). For a strict partition $\lambda = (\lambda_1, ..., \lambda_k)$, its Amemiya's notation is the 0–1 sequence $\alpha_1 \cdots \alpha_{n-1}$, where $\alpha_j := \lambda_j - \lambda_{j+1}$.

### 2. System of Basis Vectors

Let $\mathcal{U} = (U_1, ..., U_n)$ be a branching flag in $V_k(q)$ subordinate to $\lambda = (\lambda_1, ..., \lambda_k), n = \lambda_1 \geq \cdots \geq \lambda_k > 0$. We assume that all $U_j$ ($1 \leq j \leq n$) are included in $U_1$. This assumption holds when $\mathcal{U}$ is injective or surjective.
For each $j$, we define $a_j \in [1, k]$, $A_j \in \mathcal{B}_k$, $v_j \in V_k(q)$, and $B_j \subset V_k(q)$ by the following procedure:

(0) Put $A_{-1} = B_{-1} = \emptyset$ and $U_{-1} = \{0\}$.

(1) Let $1 \leq j \leq n$, and let $s$ be the smallest index such that $\dim U_j > \dim U_{j+s}$. Then $U_{j+s}$ is a subspace of $U_j$ of codimension 1. We put

$$a_j := \min \{ i \mid \exists (u_1, \ldots, u_k) \in U_j, u_i = 0 \text{ for all } i \in A_{j+s}, u_i \neq 0 \};$$

$$A_j := A_{j+s} \cup \{ a_j \} (\text{disjoint union});$$

$$v_j := (u_{a_1}, \ldots, u_{a_k}) \text{ is the unique vector in } U_j \text{ such that } u_{a_i} = 0 \text{ for all } i \in A_{j+s}, \text{ and } u_{a_j} = 1;$$

$$B_j := B_{j+s} \cup \{ v_j \}.$$

Then $B_j$ is a basis for $U_j$. We call $(v_1, \ldots, v_n)$ the "system of basis vectors" (SBV) of the branching flag $U$. The matrix

$$\begin{bmatrix}
 v_{11} & v_{12} & \cdots & v_{1n} \\
 v_{21} & v_{22} & \cdots & v_{2n} \\
 \vdots \\
 v_{k1} & v_{k2} & \cdots & v_{kn}
\end{bmatrix},$$

where $v_j = (v_{ij}, \ldots, v_{kj})$, is called the "SBV-matrix" of the branching flag $U$. Note that its component $v_{ij} = 0$ unless $j \leq \lambda_i$, so that it can be seen as a Young diagram whose boxes $(i, j)$ are filled in with elements $v_{ij}$ of $GF(q)$.

The graph of the mapping $\varphi : [1, n] \to [1, k], j \mapsto a_j$ lies on $\lambda$, while $(A_1, \ldots, A_n)$ is a branching nest subordinate to $\lambda$, with $\# A_j = \dim U_j$, and they correspond with each other via

1. $A_1 = \{ a_1, \ldots, a_n \};$
2. $A_j = (A_{j-1} \setminus \{ a_{j-1} \}) \cup \{ a_j \} \ (1 < j \leq n);$
3. $a_j$ is the unique element of $A_{j} \setminus A_{j+s}$ (1 $\leq j \leq n$), where $s$ is the smallest index such that $\# A_j > \# A_{j+s}$.

It is a routine work to check that by these relations mappings $\varphi : [1, n] \to [1, k]$ whose graph lies on $\lambda$ and branching nests subordinate to $\lambda$ correspond bijectively, and that $\varphi$ is surjective (resp. injective) iff the corresponding branching nest is surjective (resp. injective).

Thus, to each surjective (resp. injective) branching flag $U$ subordinate to $\lambda$, we have assigned a reverse matching (resp. matching) $\varphi$ on $\lambda$ and a surjective (resp. injective) branching nest $\mathcal{A}$ subordinate to $\lambda$, which correspond with each other. Note also that the branching flag $U$ is
completely determined by its SBV-matrix, for, from the SBV-matrix, one can read off successively the data \( a_1, \ldots, a_n; A_1, \ldots, A_n; \dim U_1, \ldots, \dim U_n; \) and finally \( B_1, \ldots, B_n. \)

We now consider the number of ways one can construct a sequence of vectors \( v_1, \ldots, v_n \) which is an SBV of some branching flag which maps into a given (reverse) matching \( \varphi \) on \( \lambda. \) This is done by the following procedure:

**Definition 2.1.** \( j \in [1, n] \) is "special" for \( \varphi \) if \( j \) is the minimum element of \( \varphi^{-1}(\varphi(j)). \)

**Proposition 2.2.** Let \( \mathcal{U}, \mathcal{A}, \) and \( \varphi \) as above. Then the following three conditions are equivalent:

1. \( \dim U_j > \dim U_{j+1} \)
2. \( \# A_j > \# A_{j+1} \)
3. \( j \) is special for \( \varphi. \)

**Procedure.** Let \( \Sigma = \{0, 1, *\} \) be a set of symbols. For a given (reverse) matching \( \varphi \) on \( \lambda \) we construct a Young tableau of shape \( \lambda, \) called the "SBV-tableau," with its entries in \( \Sigma \) as follows:

1. Fill in with 1 all boxes \((\varphi(j), j), 1 \leq j \leq n.\)
2. For each \( j, \) fill in with 0 all boxes \((i, j)\) such that \( i < \varphi(j).\) If \( j \) is special, fill in with 0 all boxes \((\varphi(j), l)\) such that \( l < j.\)
3. Fill in with * all the remaining boxes of \( \lambda.\)

Now, an SBV-matrix is obtained by replacing each symbol * with an arbitrary element of \( GF(q). \) Thus the number of ways one can construct an SBV for a given (reverse) matching \( \varphi \) is \( q^{l(\varphi)} \), where \( l(\varphi) \) is the number of *'s in the SBV-tableau. This concludes the proof of 1.11, and thus also 1.4 and 1.8.

**Example.** If \( \lambda = (5, 5, 4, 2) \) and \( \varphi = (\frac{1}{4}, \frac{2}{3}, \frac{3}{1}, \frac{4}{2}), \) then the SBV-tableau of \( \varphi \) is

\[
\begin{array}{ccccc}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & * & 1 \\
0 & 0 & 1 & * & \\
1 & * & \\
\end{array}
\]

so that \( l(\varphi) = 3. \)
3. Recursive Formulae

We shall prove three recursive formulae (inductive on $\lambda_1 + \cdots + \lambda_k$) for $e_\lambda(q)$. The first proof of each formula depends on the geometric construction of new branching flags from an old one, while the second proof uses the combinatorics of SBV-tableaux. These two styles of proof translate into each other via the discussions in Section 2, especially Proposition 2.2.

**Lemma 3.1.** Let $\lambda$, $n$, and $k$ be as in Section 2 with $k = \lambda'_1 > 1$, and let $\theta$ and $\psi$ be partitions given by $\theta' = (\lambda'_1, \ldots, \lambda'_n)$ and $\psi' = (\lambda'_1 - 1, \ldots, \lambda'_{n-1} - 1)$. Then

\[
e_\lambda(q) = \begin{cases} \left[ \lambda'_n \right] \{ e_\theta(q) + e_\psi(q) \}, & \text{if } \lambda'_n > 1; \\ e_\theta(q), & \text{if } \lambda'_{n-1} = 1. \end{cases} \tag{3.1a}
\]

**First Proof.** Let $(U_1, U_2, \ldots, U_n)$ be an element of $E_\lambda(q)$. $U_n$ can be chosen to be any 1-dimensional subspace of $F_{\lambda''}$, accounting for the first factor on the right-hand side of (3.1a). There are two possibilities for $U_n$.

Case 1. $\dim U_{n-1} = 1$. If so, then $(U_1, \ldots, U_{n-1})$ is again a surjective branching flag, accounting for the first term on the right-hand side of (3.1a). If $\lambda'_{n-1} = 1$, then this is the only case, and this accounts for (3.1b).

Case 2. $\dim U_{n-1} = 2$. But then, $(U_1/U_n, U_2/U_n, \ldots, U_{n-1}/U_n)$ is a surjective branching flag in the quotient space $V_k(q)/U_n$ of dimension $k - 1$. This accounts for the second term on the right-hand side of (3.1a).

**Second Proof.** Consider the SBV-tableau of a reverse matching $\varphi$ on $\lambda$. Then the $n$th column is of the form $(0, \ldots, 0, 1, \ast, \ldots, \ast)$, the number of $\ast$'s being between 0 and $\lambda'_{n-1}$. This accounts for the first factor $1 + q + \cdots + q^{\lambda'_{n-1}}$ in (3.1a). If $n$ is not special for $\varphi$, delete the $n$th column to obtain an SBV-tableau of shape $\theta$. If $n$ is special, then delete the $n$th column and the $\varphi(n)$th row to obtain an SBV-tableau of shape $\psi$. Finally, note that if $\lambda'_{n-1} = 1$ then $n$ cannot be special for $\varphi$.

The next lemma is needed in Section 5 for the proof of Theorem 1.14.

**Lemma 3.2.** Let $\lambda$ be as above, its diagram having a corner box $(i, j)$, with $i, j > 1$, and let $\mu, \theta, \psi$ be partitions given by $\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_k)$, $\theta = (\lambda_1 - 1, \ldots, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_k)$, and $\psi = (\lambda_1 - 1, \ldots, \lambda_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_k)$. Then

\[
e_\lambda(q) = q e_\mu(q) + e_\theta(q) + e_\psi(q). \tag{3.2}
\]

**First Proof.** Let $(U_1, U_2, \ldots, U_n)$ be an element of $E_\lambda(q)$ and $r$ be the dimension of $U_j$.

First suppose that $U_j$ includes $x_i := (0, \ldots, 1, \ldots, 0)$ (1 at the $i$th position).
There are two possibilities for $U_{j+1}$:

Case 1. $\dim U_{j+1} = r$. If so, then $(U_1, \ldots, U_j, \ldots, U_k)$ is again a surjective branching flag, accounting for the second term on the right-hand side of (3.2).

Case 2. $\dim U_{j+1} = r - 1$. But then, project the spaces $U_1, \ldots, U_j, \ldots, U_k$ onto $\langle x_1, \ldots, x_i, \ldots, x_k \rangle$ by sending $(u_1, \ldots, u_k)$ to $(u_1, \ldots, u_i, \ldots, u_k)$. Then the images of these spaces form a surjective branching flag subordinate to $\psi$ in the target space. This accounts for the third term on the right-hand side of (3.2).

Now consider the case that $U_j$ does not include the whole line spanned by $x_i$. Then by replacing $U_j$ by its projection, call it $W_j$, onto $F_{i-1}$ along the line $x_i$ we obtain a branching flag subordinate to $\mu$. There are $q$ ways of obtaining $U_j$ from $W_j$, accounting for the first term on the right-hand side of (3.2).

**Second Proof.** If the symbol in the corner box $(i, j)$ of the SBV-tableau is $\ast$, then just delete the box. Otherwise the entry is 1. If $j$ is not special, delete the $j$th column; if $j$ is special, delete the $j$th column and the $i$th row.

**Lemma 3.3.** Let $\lambda$ be as above, and let $\lambda^{(j)}$ be partitions given by $\lambda^{(j)} := (\lambda_1 - j, \ldots, \lambda_{k-1} - j)$, $j = 1, 2, \ldots, \lambda_k$. Then

$$e_\lambda(q) = \sum_{i=1}^{\lambda_k} \sum_{j=0}^{\lambda_k - i} \lambda_k - i C_j q^{\lambda_k - i - j} e_{\lambda^{(j)}}(q). \tag{3.3}$$

**First Proof.** Let $i$ be the smallest index such that $\dim U_{j+1} = k - 1$. Then $1 \leq i \leq \lambda_k$ and $U_i = \ldots = U_j = V_k(q)$. Let $j$ be the number of spaces among $U_{i+1}, \ldots, U_{\lambda_k}$ which contain $x_k$. There are $\lambda_k - i C_j$ ways of choosing such spaces. Forgetting these $j$ spaces, and projecting the remaining $\lambda_k - i - j$ spaces onto $F_{k-1}$ along $x_k$, one obtains a branching flag subordinate to $\lambda^{(j)}$. There are $q$ ways to recover each of these spaces, or in total, $q^{\lambda_k - i - j}$ ways. This accounts for (3.3).

**Second Proof.** The $k$th row of the SBV-tableau of a reverse matching $\varphi$ is of the form $(0, \ldots, 0, 1, \#, \ldots, \#)$, $\#$ being either 1 or $\ast$. Let $i$ be special with $\varphi(i) = k$, and let there be $j$ 1's among the $\lambda_k - i$ 's. Delete the $k$th row and the $j$ columns containing these 1's. Details are omitted.

**4. Algorithm**

We give in this section an algorithm to compute $e_\lambda(q)$ using the $q$-difference operator.
**Definition 4.1.** We define operators $H$ and $A$ on $\mathbb{Z}[q, t]$ as

\[
Hf(q, t) := f(q, 1 + qt);
\]

\[
Af(q, t) := \frac{(f(q, 1 + qt) - f(q, t))}{(1 + qt - t)}.
\]

Then it is easy to verify by direct calculation that

**Lemma 4.2.**

1. $H = (1 + qt - t)A + 1$;

2. $AH = qHA$;

3. $At = H + tA$;

4. $A^at = [a] HA^{-1} + tA^a$ ($a \geq 1$).

**Theorem 4.3.**

\[
e_d(q) = (A^\lambda tA^\lambda_{a-1} - t\cdots A^\lambda_1 - t)\bigg|_{t=0}.
\]

**Proof.** Put $a = \lambda_n$ and $b = \lambda_n - 1$. When $n = 1$, both sides are equal to 1 (if $a = 1$) or 0 (if $a > 1$), and the equality holds. We proceed to the case $n > 1$ and use induction on $n$. Then, by the inductive formula (3.1) and the induction hypothesis, it suffices to show that

**Claim.** For any integers $a$ and $b$ such that $0 < a \leq b$, and any polynomial $h(q, t)$,

\[
(A^a t A^b - ah)\bigg|_{t=0} = [a] (A^b + A^{b-1}h)\bigg|_{t=0}.
\]

By using (4.6), and then using (4.3), we have

\[
\text{LHS} = ([a] HA^{a-1} + tA^a) A^{b-a}h\bigg|_{t=0}
\]

\[
= [a] HA^{b-1}h\bigg|_{t=0}
\]

\[
= [a] ((1 + qt - t)A + 1) A^{b-1}h\bigg|_{t=0}
\]

\[
= [a] (A^bh + A^{b-1}h)\bigg|_{t=0}.
\]

Q.E.D.

Now put $f[i] := f(q, [i])$ for $f(q, t) \in \mathbb{Z}[q, t]$ and consider the sequence $f[0], f[1], \cdots$ in $\mathbb{Z}[q]$. Then we have $Af[i] = q^{-i}(f[i+1] - f[i])$, so that the correspondence $(f[i]) \mapsto (Af[i])$ is the $q$-difference operator. One then can interpret (4.7) as indicating a Young-diagrammatic algorithm to obtain $e_d(q)$ as follows:

**Algorithm.** Fill in the first column of $\lambda$ with polynomials $[1], [2], \ldots, [\lambda_1]$. Then multiply each entry by $[i]$, $i$ being the row number, and fill in the next column with these products. If $\lambda_j > \lambda_{j+1}$, then apply the
$q$-difference operator $\lambda'_j - \lambda'_{j+1}$ times before multiplication. Proceed with these operations up to the last column, then adjunct an extra box with zero entry on top of the last column and take the $q$-difference $\lambda'_n$ times.

5. Duality

We shall prove 1.14. For a 0–1 sequence $\alpha = \alpha_1 \cdots \alpha_{n-1}$, let $\tilde{\alpha}$ denote the sequence $\beta_{n-1} \cdots \beta_1$, where $\beta_j := 1 - \alpha_j$ ($1 \leq j \leq n - 1$). Then it is easily seen that

**Lemma 5.1 (Amemiya).** If $\lambda$ is denoted by $\alpha$ in Amemiya's notation (Section 1), then $\lambda^*$ is denoted by $\tilde{\alpha}$.

**Proof of 1.14.** Let $[\alpha]$ denote the polynomial $e_\lambda(q)$, $\lambda$ being the strict partition whose Amemiya's notation is $\alpha$. Then, by the above lemma, it suffices to show that $[\alpha] = [\tilde{\alpha}]$. But writing $\alpha = \beta 10\gamma$, one has $[\beta 10\gamma] = q^k [\beta 01\gamma] + [\beta 0\gamma] + [\beta 1\gamma]$ by Lemma 3.2, and similarly, $[\tilde{\gamma} 10\tilde{\beta}] = q^k [\tilde{\gamma} 01\tilde{\beta}] + [\tilde{\gamma} 0\tilde{\beta}] + [\tilde{\gamma} 1\tilde{\beta}]$ for $\tilde{\alpha}$. It is also obvious that $[\alpha 1] = [\tilde{\alpha} 1] = [\alpha]$ and $[1] = [0] = 1$. Now the equality $[\alpha] = [\tilde{\alpha}]$ is proved by induction on the length of $\alpha$ and on the number binarily expressed by $\alpha$, as required. Q.E.D.

6. $q$-Stirling Numbers of the Second Kind

We put $S_{n,k}(q) := e_\lambda(q)/[k]!$ for $\lambda = (n, \ldots, n)$ ($k$ times). Then by the recursion (3.1) we have

$$S_{n,k}(q) = [k] S_{n-1,k}(q) + S_{n-1,k-1}(q) \quad (n, k > 1);$$

$$S_{1,k}(q) = \begin{cases} 1 & (k = 1); \\ 0 & (k > 1), \end{cases}$$

so that $S_{n,k}(q)$ is the $q$-Stirling number of the second kind [3, 8]. Further, by a routine induction based on (3.1) or (4.7) one can show that, for an arbitrary partition $\lambda$, $e_\lambda(q)$ is divisible by $[\alpha_1]! [\alpha_2]! \cdots [\alpha_n]!$ in $\mathbb{Z}[q]$, where $\alpha_j$ is the multiplicity of $j$ as a part in $\lambda$, so that the quotient polynomial generalizes the $q$-Stirling number of the second kind.

7. $q$-Fibonacci Numbers

A skew diagram $\lambda/\mu$ is a cloud of cells $(i, j)$ with $\mu_i < j \leq \lambda_i$, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$, with $\lambda_1 \geq \cdots \geq \lambda_k > 0$, $\mu_1 \geq \cdots \geq \mu_k \geq 0$. 
and \( \lambda_i \geq \mu_i \) (\( 1 \leq i \leq k \)). The notion of the SBV-tableau of skew shape \( \lambda/\mu \) is defined similarly.

For an integer \( n \geq 1 \), let \( u_n(q) \) be the number of SBV-matrices of skew shape \( \lambda/\mu \), where \( \lambda = (n-1, n-1, n-2, \ldots, 3, 2) \) and \( \mu = (n-3, n-4, \ldots, 2, 1, 0, 0) \). Then by an analogous argument as in Sections 2 and 3, we have

1. \( u_n(q) \) is the number of complete flags \( U_1 \supseteq \cdots \supseteq U_{n-1} \) in \( V_n(q) \) such that the inequality
   \[
   \dim(U_i \cap V_j(q)) > \dim(U_{i+1} \cap V_j(q))
   \]
   implies \( |i-j| \leq 1 \), for all \( i, j \in [1, n-1] \), where we put \( U_n := \{0\} \);

2. \( u_n(q) = \sum_\sigma q^{|\sigma|} \), where \( \sigma \) runs through all permutations on \([1, n-1]\) satisfying \( |i - \sigma(i)| \leq 1 \) for all \( i \in [1, n-1] \), and \( l(\sigma) \) is the number of inversions in \( \sigma \);

3. \( u_n(q) \) satisfies the recursion: \( u_1(q) = u_2(q) = 1; \ u_n(q) = u_{n-1}(q) + qu_{n-2}(q), \) for \( n \geq 3 \); hence \( u_n(q) \) is the \( q \)-Fibonacci number [5].

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