JOURNAL OF COMBINATORIAL THEORY, Series A 53, 117–127 (1990)

Branching Flags, Branching Nets, and Reverse Matchings

YOSHIAKI UENO

Tokyo Institute of Polytechnics,
1583 Iiyama, Atsugi, Kanagawa 243-02 Japan

Communicated by the Managing Editors

Received December 1, 1985

DEDICATED TO PROFESSOR NAGAYOSHI IWAHORI ON HIS 60TH BIRTHDAY

The relation among three classes of combinatorial objects parametrized by partitions is discussed: the branching flag is a tree-type generalization of the flag in a vector space over a finite field; the branching nest is its finite set-theoretical counterpart; and the reverse matching, equinumerous to branching nests, is a dual concept of the complete matching. A mapping will be constructed of the set of branching flags into that of branching nests, which will give a decomposition of the variety of branching flags into cells parametrized by reverse matchings. Its Poincaré polynomial is related to a refinement of q-Stirling numbers. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let $GF(q)$ denote the finite field with q elements. For a positive integer k, $V_k(q)$ is the k-dimensional vector space over $GF(q)$, $\mathscr{L}_k(q)$ the lattice of subspaces of $V_k(q)$, $[1, k]$ the k-set $\{1, 2, ..., k\}$ and $\mathscr{B}_k$ the lattice of subsets of $[1, k]$. Here, both $\mathscr{L}_k(q)$ and $\mathscr{B}_k$ are ordered by inclusion, and rank in $\mathscr{L}_k(q)$ and $\mathscr{B}_k$ is dimension and cardinality, respectively.

It is well known, and also easily seen from the arguments herein as a special case, that the number of complete flags in $V_k(q)$ is $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})$, and there is a mapping of the set of these flags into the set of permutations on the set $[1, k]$, such that the cardinality of the inverse image of a permutation σ is just $q^{l(\sigma)}$, where $l(\sigma)$ denotes the number of inversions in σ.

This paper answers the question what would happen if we replace in these circumstances the set of permutations on a finite set by the set of injections or surjections between finite sets. This is done, as one expects, in terms of slightly generalizing the notion of flags. A further refinement is also made by imposing some Young diagrammatic condition to the map-

0097-3165/90 $3.00

pings. This way, we reach the notion of a refinement, parametrized by partitions, of q-Stirling numbers of the second kind. An enumerative and geometric interpretation of the q-Fibonacci numbers is also obtained.

Let k and n be given positive integers.

DEFINITION 1.1. A sequence $\mathscr{U} = (U_1, U_2, ..., U_n)$ of elements of $\mathscr{L}_k(q)$ is called a "branching flag" (of length n) if it satisfies:

(1) $\dim U_n = 1$;

(2) $\dim U_j - \dim U_{j+1} = 0$ or 1 $(1 \leqslant j < n)$;

(3) if $\dim U_j = \dim U_{j+1} = \cdots = \dim U_{j+s-1} > \dim U_{j+s}$, then U_j, U_{j+1} ..., $U_{j+s-1} \supset U_{j+s}$.

We put $U_{n+1} = \{0\}$ for convenience' sake, so that these elements form an order-preserving image of a rooted tree.

DEFINITION 1.2. A branching flag $(U_1, U_2, ..., U_n)$ is called "surjective" if $\dim U_1 = k$, and "injective" if $\dim U_j - \dim U_{j+1} = 1$ $(1 \leqslant j < n)$.

We represent the elements of $V_k(q)$ as k-tuples $(x_1, ..., x_k)$ of elements of the field. A sequence of integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is a partition if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$. For a partition λ such that $\lambda_1 = n$, we put $\lambda'_j := \max\{i; \lambda_i \leqslant j\}$ for $1 \leqslant j \leqslant n$. Then $\lambda' := (\lambda'_1, ..., \lambda'_n)$ is the conjugate of λ, and $\lambda'' = \lambda$. We denote by F_d the d-dimensional subspace consisting of all the vectors $\mathbf{u} = (u_1, ..., u_k)$ such that $u_i = 0$ for all $i > d$. The chain $F_{\lambda'_1} \supset F_{\lambda'_2} \supset \cdots \supset F_{\lambda'_n}$ in $\mathscr{L}_k(q)$ is called the reference flag w.r.t. λ.

DEFINITION 1.3. A branching flag $(V_1, V_2, ..., V_n)$ is "subordinate to" λ if $V_j \subset F_{\lambda'_j}$ for $1 \leqslant j \leqslant n$.

We denote by $E_\lambda(q)$ (resp. $F_\lambda(q)$) the set of all surjective (resp. injective) branching flags subordinate to λ, and $e_\lambda(q)$ (resp. $f_\lambda(q)$) its cardinality.

It is easy to verify that $e_\lambda(q)$ and $f_{\lambda'}(q)$ are nonzero if and only if $\lambda_i \geqslant k + 1 - i$ $(1 \leqslant i \leqslant k)$, and $e_\lambda(q) = 1$ if $k = 1$. For $f_{\lambda'}(q)$, we have a complete explicit formula

$$f_{\lambda'}(q) = [\lambda_k][\lambda_{k-1} - 1] \cdots [\lambda_1 - k + 1],$$

where we put $[i] := (1 - q^i)/(1 - q)$ for i integer, and we shall in the following mainly be interested in determining $e_\lambda(q)$, for the investigation for injective branching flags goes almost parellel to, and even much simpler than, that of surjective branching flags. For example, we have

THEOREM 1.4. *$e_\lambda(q)$ and $f_{\lambda'}(q)$ are both monic polynomials in q with nonnegative integral coefficients.*

Rather than proving 1.4 inductively by a recursive formula (see Section 3) we shall give a direct combinatorial proof using the notion of the branching nest, whose definition goes parallel to that of the branching flag as follows:

DEFINITION 1.5. A sequence $\mathscr{A} = (A_1, A_2, ..., A_n)$ of elements of $\mathscr{B}_k$ is called a "branching nest" (of length n) if it satisfies:

(1) $\# A_n = 1$;

(2) $\# A_j - \# A_{j+1} = 0$ or 1 $(1 \leqslant j < n)$;

(3) if $\# A_j = \# A_{j+1} = \cdots = \# A_{j+s-1} > \# A_{j+s}$, then $A_j, A_{j+1}, ..., A_{j+s-1} \supset A_{j+s}$.

We put $A_{n+1} = \varnothing$ for convenience' sake, so that these elements form an order-preserving image of a rooted tree.

DEFINITION 1.6. A branching nest $(A_1, A_2, ..., A_n)$ is called "surjective" if $\# A_1 = k$, and "injective" if $\# A_j - \# A_{j+1} = 1$ $(1 \leqslant j < n)$.

Now, for given k, n, and λ as above, the "reference nest" w.r.t. λ is by definition the chain $N_{\lambda'_1} \supset N_{\lambda'_2} \supset \cdots \supset N_{\lambda'_n}$ in $\mathscr{B}_k$, where $N_d := [1, d]$ $(1 \leqslant d \leqslant k)$.

DEFINITION 1.7. A branching nest $(A_1, A_2, ..., A_n)$ is "subordinate to" λ if $A_j \subset N_{\lambda'_j}$ for $1 \leqslant j \leqslant n$.

Let E_λ (resp. F_λ) denote the set of all surjective (resp. injective) branching nests subordinate to λ, and e_λ (resp. f_λ) its cardinality. Then

THEOREM 1.8. *$e_\lambda = e_\lambda(1)$ and $f_\lambda = f_\lambda(1)$.*

To prove 1.4 and 1.8 directly, we shall construct a mapping of $E_\lambda(q)$ (resp. $F_\lambda(q)$) into E_λ (resp. F_λ), and show that the cardinality of the inverse image of each branching nest under this mapping is a power of q.

Now surjections and injections between finite sets come into picture as follows:

DEFINITION 1.9. A "reverse matching (resp. matching)" on λ is a surjection (resp. injection) φ of $[1, n]$ into $[1, k]$ such that $\varphi(j) \leqslant \lambda'_j$ $(1 \leqslant j \leqslant n)$.

A matching is sometimes called "complete matching" in the literature.

PROPOSITION 1.10. *There exists a 1–1 correspondence between E_λ (resp. F_λ) and the set of all reverse matchings (resp. matchings) on λ.*

Composing this bijection and the mapping of $E_\lambda(q)$ (resp. $F_\lambda(q)$) onto E_λ (resp. F_λ), we have

THEOREM 1.11. *$e_\lambda(q) = \sum_\varphi q^{l(\varphi)}$ and $f_\lambda(q) = \sum_\psi q^{l(\psi)}$, where φ (resp. ψ) runs through all reverse matchings (resp. matchings) on λ.*

Here, $l(\varphi)$ is the natural generalization of the number of inversions in a permutation, defined as follows:

DEFINITION 1.12. For a reverse matching (or matching) φ on λ, $l(\varphi)$ is the number of *nodes* (i, j) on the Young diagram λ (i.e., $(i, j) \in \mathbb{Z}^2$, $1 \leqslant i \leqslant k$, and $1 \leqslant j \leqslant \lambda_i$) such that

(1) $i > \varphi(j)$;

(2) $\varphi^{-1}(i) = \varnothing$ or $\min \varphi^{-1}(i) < j$.

Our first motivation in this work was to generalize T. Imai's pebble arranging puzzle [6].

DEFINITION 1.13. λ is "strict" if all parts λ_i are distinct and positive. For a strict partition $\lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1 = n$, its "complementary" is the strict partition $\lambda^* := (\lambda_1^*, ..., \lambda_l^*)$, where $\lambda_1^* = n$ and $[1, n-1]$ is the disjoint union of $\{\lambda_i; 1 < i \leqslant k\}$ and $\{n - \lambda_i^*; 1 < i \leqslant l\}$.

Imai's puzzle, in our notation, is equivalent to proving $e_\lambda = e_{\lambda^*}$, where λ is strict. By 1.8 this is a special case of

THEOREM 1.14. *If λ is strict, then $e_\lambda(q) = e_{\lambda^*}(q)$.*

A short proof was given to 1.14 by I. Amemiya [1] for $q = 1$, which survives to our case (Section 5). The proofs of Theorems 1.4, 1.8, and 1.11 are completed in Section 2.

Notation. We put $[i] := (1 - q^i)/(1 - q)$ for i integer; $[i]! := [i][i-1] \cdots [1]$ for i positive, and $[0]! := 1$. For convenience' sake we put $e_\phi(q) = 1$, where ϕ is the void partition ( ). For a strict partition $\lambda = (\lambda_1, ..., \lambda_k)$, its Amemiya's notation is the 0–1 sequence $\alpha_1 \cdots \alpha_{n-1}$, where $\alpha_j := \lambda_j' - \lambda_{j+1}'$.

2. SYSTEM OF BASIS VECTORS

Let $\mathscr{U} = (U_1, ..., U_n)$ be a branching flag in $V_k(q)$ subordinate to $\lambda = (\lambda_1, ..., \lambda_k)$, $n = \lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$. We assume that all U_j $(1 \leqslant j \leqslant n)$ are included in U_1. This assumption holds when $\mathscr{U}$ is injective or surjective.

For each j, we define $a_j \in [1, k]$, $A_j \in \mathscr{B}_k$, $\mathbf{v}_j \in V_k(q)$, and $B_j \subset V_k(q)$ by the following procedure:

(0) Put $A_{n+1} = B_{n+1} = \varnothing$ and $U_{n+1} = \{0\}$.

(1) Let $1 \leqslant j \leqslant n$, and let s be the smallest index such that $\dim U_j > \dim U_{j+s}$. Then U_{j+s} is a subspace of U_j of codimension 1. We put

$$a_j := \min\{i \mid \exists (u_1, ..., u_k) \in U_j,$$
$$u_l = 0 \text{ for all } l \in A_{j+s}, u_i \neq 0\};$$
$$A_j := A_{j+s} \cup \{a_j\} \text{ (disjoint union)};$$
$$\mathbf{v}_j := (u_1, ..., u_k) \text{ is the unique vector in } U_j \text{ such that}$$
$$u_l = 0 \text{ for all } l \in A_{j+s} \text{ and } u_{a_j} = 1;$$
$$B_j := B_{j+s} \cup \{\mathbf{v}_j\}.$$

Then B_j is a basis for U_j. We call $(\mathbf{v}_1, ..., \mathbf{v}_n)$ the "system of basis vectors" (SBV) of the branching flag U. The matrix

$$\begin{matrix} v_{11} v_{12} \cdots v_{1n} \\ v_{21} v_{22} \cdots v_{2n} \\ \cdots \\ v_{k1} v_{k2} \cdots v_{kn}, \end{matrix}$$

where $\mathbf{v}_j = (v_{1j}, ..., v_{kj})$, is called the "SBV-matrix" of the branching flag $\mathscr{U}$. Note that its component $v_{ij} = 0$ unless $j \leqslant \lambda_i$, so that it can be seen as a Young diagram whose boxes (i, j) are filled in with elements v_{ij} of $GF(q)$.

The graph of the mapping $\varphi: [1, n] \to [1, k]$, $j \mapsto a_j$ lies on λ, while $(A_1, ..., A_n)$ is a branching nest subordinate to λ, with $\# A_j = \dim U_j$, and they correspond with each other via

(1) $A_1 = \{a_1, ..., a_n\}$;

(2) $A_j = (A_{j-1} \backslash \{a_{j-1}\}) \cup \{a_j\}$ $(1 < j \leqslant n)$;

(3) a_j is the unique element of $A_j \backslash A_{j+s}$ $(1 \leqslant j \leqslant n)$, where s is the smallest index such that $\# A_j > \# A_{j+s}$.

It is a routine work to check that by these relations mappings $\varphi: [1, n] \to [1, k]$ whose graph lies on λ and branching nests subordinate to λ correspond bijectively, and that φ is surjective (resp. injective) iff the corresponding branching nest is surjective (resp. injective).

Thus, to each surjective (resp. injective) branching flag $\mathscr{U}$ subordinate to λ, we have assigned a reverse matching (resp. matching) φ on λ and a surjective (resp. injective) branching nest $\mathscr{A}$ subordinate to λ, which correspond with each other. Note also that the branching flag $\mathscr{U}$ is completely determined by its SBV-matrix, for, from the SBV-matrix, one can read off successively the data $a_1, ..., a_n$; $A_1, ..., A_n$; dim U_1, ..., dim U_n; and finally $B_1, ..., B_n$.

We now consider the number of ways one can construct a sequence of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ which is an SBV of some branching flag which maps into a given (reverse) matching φ on λ. This is done by the following procedure:

DEFINITION 2.1. $j \in [1, n]$ is "special" for φ if j is the minimum element of $\varphi^{-1}(\varphi(j))$.

PROPOSITION 2.2. *Let $\mathscr{U}$, $\mathscr{A}$, and φ as above. Then the following three conditions are equivalent*:

(1) $\dim U_j > \dim U_{j+1}$

(2) $\# A_j > \# A_{j+1}$

(3) *j is special for φ.*

Procedure. Let $\Sigma = \{0, 1, *\}$ be a set of symbols. For a given (reverse) matching φ on λ we construct a Young tableau of shape λ, called the "SBV-tableau," with its entries in Σ as follows:

(1) Fill in with 1 all boxes $(\varphi(j), j)$, $1 \leqslant j \leqslant n$.

(2) For each j, fill in with 0 all boxes (i, j) such that $i < \varphi(j)$. If j is special, fill in with 0 all boxes $(\varphi(j), l)$ such that $l < j$.

(3) Fill in with * all the remaining boxes of λ.

Now, an SBV-matrix is obtained by replacing each symbol * with an arbitrary element of $GF(q)$. Thus the number of ways one can construct an SBV for a given (reverse) matching φ is $q^{l(\varphi)}$, where $l(\varphi)$ is the number of *'s in the SBV-tableau. This concludes the proof of 1.11, and thus also 1.4 and 1.8.

EXAMPLE. If $\lambda = (5\ 5\ 4\ 2)$ and $\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 2 \end{pmatrix}$, then the SBV-tableau of φ is

0	0	0	1	0
0	1	0	*	1
0	0	1	*	
1	*			

so that $l(\varphi) = 3$.

3. Recursive Formulae

We shall prove three recursive formulae (inductive on $\lambda_1 + \cdots + \lambda_k$) for $e_\lambda(q)$. The first proof of each formula depends on the geometric construction of new branching flags from an old one, while the second proof uses the combinatorics of SBV-tableaux. These two styles of proof translate into each other via the discussions in Section 2, especially Proposition 2.2.

Lemma 3.1. *Let λ, n, and k be as in Section 2 with $k = \lambda'_1 > 1$, and let θ and ψ be partitions given by $\theta' = (\lambda'_1, ..., \lambda'_{n-1})$ and $\psi' = (\lambda'_1 - 1, ..., \lambda'_{n-1} - 1)$. Then*

$$e_\lambda(q) = \begin{cases} [\lambda'_n]\{e_\theta(q) + e_\psi(q)\}, & \text{if } \lambda'_{n-1} > 1; \quad (3.1a) \\ e_\theta(q), & \text{if } \lambda'_{n-1} = 1. \quad (3.1b) \end{cases}$$

First Proof. Let $(U_1, U_2, ..., U_n)$ be an element of $E_\lambda(q)$. U_n can be chosen to be any 1-dimensional subspace of $F_{\lambda'_n}$, accounting for the first factor on the right-hand side of (3.1a). There are two possibilities for U_{n-1}.

Case 1. dim $U_{n-1} = 1$. If so, then $(U_1, ..., U_{n-1})$ is again a surjective branching flag, accounting for the first term on the right-hand side of (3.1a). If $\lambda'_{n-1} = 1$, then this is the only case, and this accounts for (3.1b).

Case 2. dim $U_{n-1} = 2$. But then, $(U_1/U_n, U_2/U_n, ..., U_{n-1}/U_n)$ is a surjective branching flag in the quotient space $V_k(q)/U_n$ of dimension $k-1$. This accounts for the second term on the right-hand side of (3.1a). ∎

Second Proof. Consider the SBV-tableau of a reverse matching φ on λ. Then the nth column is of the form ${}^t(0, ..., 0, 1, *, ..., *)$, the number of $*$'s being between 0 and $\lambda'_n - 1$. This accounts for the first factor $1 + q + \cdots + q^{\lambda'_n - 1}$ in (3.1a). If n is not special for φ, delete the nth column to obtain an SBV-tableau of shape θ. If n *is* special, then delete the nth column and the $\varphi(n)$th row to obtain an SBV-tableau of shape ψ. Finally, note that if $\lambda'_{n-1} = 1$ then n cannot be special for φ. ∎

The next lemma is needed in Section 5 for the proof of Theorem 1.14.

Lemma 3.2. *Let λ be as above, its diagram having a corner box (i, j), with $i, j > 1$, and let μ, θ, and ψ be partitions given by $\mu = (\lambda_1, ..., \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, ..., \lambda_k)$, $\theta = (\lambda_1 - 1, ..., \lambda_i - 1, \lambda_{i+1}, ..., \lambda_k)$, and $\psi = (\lambda_1 - 1, ..., \lambda_{i-1} - 1, \lambda_{i+1}, ..., \lambda_k)$. Then*

$$e_\lambda(q) = q e_\mu(q) + e_\theta(q) + e_\psi(q). \quad (3.2)$$

First Proof. Let $(U_1, U_2, ..., U_n)$ be an element of $E_\lambda(q)$ and r be the dimension of U_j.

First suppose that U_j includes $\mathbf{x}_i := (0, ..., 1, ..., 0)$ (1 at the ith position).

There are two possibilities for U_{j+1}:

Case 1. $\dim U_{j+1}=r$. If so, then $(U_1, ..., \hat{U}_j, ..., U_k)$ is again a surjective branching flag, accounting for the second term on the right-hand side of (3.2).

Case 2. $\dim U_{j+1}=r-1$. But then, project the spaces $U_1, ..., \hat{U}_j, ..., U_k$ onto $\langle \mathbf{x}_1, ..., \hat{\mathbf{x}}_i, ..., \mathbf{x}_k \rangle$ by sending $(u_1, ..., u_k)$ to $(u_1, ..., \hat{u}_i, ..., u_k)$. Then the images of these spaces form a surjective branching flag subordinate to ψ in the target space. This accounts for the third term on the right-hand side of (3.2).

Now consider the case that U_j does not include the whole line spanned by $\mathbf{x}_i$. Then by replacing U_j by its projection, call it W, onto F_{i-1} along the line $\mathbf{x}_i$ we obtain a branching flag subordinate to μ. There are q ways of obtaining U_j from W, accounting for the first term on the right-hand side of (3.2). ∎

Second Proof. If the symbol in the corner box (i, j) of the SBV-tableau is *, then just delete the box. Otherwise the entry is 1. If j is not special, delete the jth column; if j *is* special, delete the jth column and the ith row. ∎

LEMMA 3.3. *Let λ be as above, and let $\lambda^{(j)}$ be partitions given by $\lambda^{(j)} := (\lambda_1 - j, ..., \lambda_{k-1} - j)$, $j = 1, 2, ..., \lambda_k$. Then*

$$e_\lambda(q) = \sum_{i=1}^{\lambda_k} \sum_{j=0}^{\lambda_k - i} {}_{\lambda_k - i}C_j q^{\lambda_k - i - j} e_{\lambda^{(j)}}(q). \tag{3.3}$$

First Proof. Let i be the smallest index such that $\dim U_{j+1} = k-1$. Then $1 \leqslant i \leqslant \lambda_k$ and $U_1 = \cdots = U_i = V_k(q)$. Let j be the number of spaces among $U_{i+1}, ..., U_{\lambda_k}$ which contain $\mathbf{x}_k$. There are ${}_{\lambda_k - i}C_j$ ways of choosing such spaces. Forgetting these j spaces, and projecting the remaining $\lambda_k - i - j$ spaces onto F_{k-1} along $\mathbf{x}_k$, one obtains a branching flag subordinate to $\lambda^{(j)}$. There are q ways to recover each of these spaces, or in total, $q^{\lambda_k - i - j}$ ways. This accounts for (3.3). ∎

Second Proof. The kth row of the SBV-tableau of a reverse matching φ is of the form (0, ..., 0, 1, #, ..., #), # being either 1 or *. Let i be special with $\varphi(i) = k$, and let there be j 1's among the $\lambda_k - i$ #'s. Delete the kth row and the j columns containing these 1's. Details are omitted. ∎

4. ALGORITHM

We give in this section an algorithm to compute $e_\lambda(q)$ using the q-difference operator.

DEFINITION 4.1. We define operators H and Δ on $\mathbb{Z}[q, t]$ as

$$Hf(q, t) := f(q, 1+qt); \tag{4.1}$$

$$\Delta f(q, t) := (f(q, 1+qt) - f(q, t))/(1+qt-t). \tag{4.2}$$

Then it is easy to verify by direct calculation that

LEMMA 4.2. (1) $H = (1+qt-t)\,\Delta + 1;$ (4.3)

(2) $\Delta H = qH\Delta;$ (4.4)

(3) $\Delta t = H + t\Delta;$ (4.5)

(4) $\Delta^a t = [a]\,H\Delta^{a-1} + t\Delta^a \ (a \geqslant 1).$ (4.6)

THEOREM 4.3.

$$e_\lambda(q) = (\Delta^{\lambda'_n} t \Delta^{\lambda'_{n-1} - \lambda'_n} t \cdots \Delta^{\lambda'_1 - \lambda'_2} t)\,|_{t=0}. \tag{4.7}$$

Proof. Put $a = \lambda'_n$ and $b = \lambda'_{n-1}$. When $n = 1$, both sides are equal to 1 (if $a = 1$) or 0 (if $a > 1$), and the equality holds. We proceed to the case $n > 1$ and use induction on n. Then, by the inductive formula (3.1) and the induction hypothesis, it suffices to show that

Claim. For any integers a and b such that $0 < a \leqslant b$, and any polynomial $h(q, t)$,

$$(\Delta^a t \Delta^{b-a} h)\,|_{t=0} = [a](\Delta^b h + \Delta^{b-1} h)\,|_{t=0}. \tag{4.8}$$

By using (4.6), and then using (4.3), we have

$$\begin{aligned}\text{LHS} &= ([a]\,H\Delta^{a-1} + t\Delta^a)\,\Delta^{b-a} h\,|_{t=0}\\ &= [a]\,H\Delta^{b-1} h\,|_{t=0}\\ &= [a]((1+qt-t)\,\Delta + 1)\,\Delta^{b-1} h\,|_{t=0}\\ &= [a](\Delta^b h + \Delta^{b-1} h)\,|_{t=0}.\end{aligned}$$

Q.E.D.

Now put $f[i] := f(q, [i])$ for $f(q, t) \in \mathbb{Z}[q, t]$ and consider the sequence $f[0], f[1], \cdots$ in $\mathbb{Z}[q]$. Then we have $\Delta f[i] = q^{-i}(f[i+1] - f[i])$, so that the correspondence $(f[i]) \mapsto (\Delta f[i])$ is the q-difference operator. One then can interpret (4.7) as indicating a Young-diagrammatic algorithm to obtain $e_\lambda(q)$ as follows:

ALGORITHM. Fill in the first column of λ with polynomials $[1]$, $[2]$, ..., $[\lambda'_1]$. Then multiply each entry by $[i]$, i being the row number, and fill in the next column with these products. If $\lambda'_j > \lambda'_{j+1}$, then apply the

q-difference operator $\lambda'_j - \lambda'_{j+1}$ times before multiplication. Proceed with these operations up to the last column, then adjunct an extra box with zero entry on top of the last column and take the q-difference λ'_n times.

5. Duality

We shall prove 1.14. For a 0–1 sequence $\alpha = \alpha_1 \cdots \alpha_{n-1}$, let $\bar{\alpha}$ denote the sequence $\beta_{n-1} \cdots \beta_1$, where $\beta_j := 1 - \alpha_j$ $(1 \leqslant j \leqslant n-1)$. Then it is easily seen that

Lemma 5.1 (Amemiya). *If λ is denoted by α in Amemiya's notation (Section 1), then λ^* is denoted by $\bar{\alpha}$.*

Proof of 1.14. Let $[\alpha]$ denote the polynomial $e_\lambda(q)$, λ being the strict partition whose Amemiya's notation is α. Then, by the above lemma, it suffices to show that $[\alpha] = [\bar{\alpha}]$. But writing $\alpha = \beta 10 \gamma$, one has $[\beta 10 \gamma] = q[\beta 01 \gamma] + [\beta 0 \gamma] + [\beta 1 \gamma]$ by Lemma 3.2, and similarly, $[\bar{\gamma} 10 \bar{\beta}] = q[\bar{\gamma} 01 \bar{\beta}] + [\bar{\gamma} 0 \bar{\beta}] + [\bar{\gamma} 1 \bar{\beta}]$ for $\bar{\alpha}$. It is also obvious that $[\alpha 1] = [0 \alpha] = [\alpha]$ and $[1] = [0] = 1$. Now the equality $[\alpha] = [\bar{\alpha}]$ is proved by induction on the length of α and on the number binarily expressed by α, as required. Q.E.D.

6. q-Stirling Numbers of the Second Kind

We put $S_{n,k}(q) := e_\lambda(q)/[k]!$ for $\lambda = (n, ..., n)$ (k times). Then by the recursion (3.1) we have

$$S_{n,k}(q) = [k]\, S_{n-1,k}(q) + S_{n-1,k-1}(q) \qquad (n, k > 1); \tag{6.1}$$

$$S_{1,k}(q) = \begin{cases} 1 & (k = 1); \\ 0 & (k > 1), \end{cases} \tag{6.2}$$

so that $S_{n,k}(q)$ is the q-Stirling number of the second kind [3, 8]. Further, by a routine induction based on (3.1) or (4.7) one can show that, for an arbitrary partition λ, $e_\lambda(q)$ is divisible by $[\alpha_1]!\,[\alpha_2]! \cdots [\alpha_n]!$ in $\mathbb{Z}[q]$, where α_j is the multiplicity of j as a part in λ, so that the quotient polynomial generalizes the q-Stirling number of the second kind.

7. q-Fibonacci Numbers

A skew diagram λ/μ is a cloud of cells (i, j) with $\mu_i < j \leqslant \lambda_i$, where $\lambda = (\lambda_1, ..., \lambda_k)$ and $\mu = (\mu_1, ..., \mu_k)$, with $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$, $\mu_1 \geqslant \cdots \geqslant \mu_k \geqslant 0$, and $\lambda_i \geqslant \mu_i$ $(1 \leqslant i \leqslant k)$. The notion of the SBV-tableau of skew shape λ/μ is defined similarly.

For an integer $n \geqslant 1$, let $u_n(q)$ be the number of SBV-matrices of skew shape λ/μ, where $\lambda = (n-1, n-1, n-2, ..., 3, 2)$ and $\mu = (n-3, n-4, ..., 2, 1, 0, 0)$. Then by an analogous argument as in Sections 2 and 3, we have

(1) $u_n(q)$ is the number of complete flags $U_1 \supset \cdots \supset U_{n-1}$ in $V_{n-1}(q)$ such that the inequality

$$\dim(U_i \cap V_j(q)) > \dim(U_{i+1} \cap V_j(q))$$

implies $|i-j| \leqslant 1$, for all $i, j \in [1, n-1]$, where we put $U_n := \{0\}$;

(2) $u_n(q) = \sum_\sigma q^{l(\sigma)}$, where σ runs through all permutations on $[1, n-1]$ satisfying $|i-\sigma(i)| \leqslant 1$ for all $i \in [1, n-1]$, and $l(\sigma)$ is the number of inversions in σ;

(3) $u_n(q)$ satisfies the recursion: $u_1(q) = u_2(q) = 1$; $u_n(q) = u_{n-1}(q) + qu_{n-2}(q)$, for $n \geqslant 3$; hence $u_n(q)$ is the q-Fibonacci number [5].

Acknowledgments

The author finds it a pleasure to thank Professor Teizo Imai of Yamanashi University, for his helpful correspondences, and Professor Ichiro Amemiya of Saitama Medical School, for his contributions in shortening the proofs in Sections 4 and 5. We also thank Professor Nagayoshi Iwahori for encouraging us with many stimulating suggestions.

References

1. I. Amemiya, a letter to T. Imai, April 1982.
2. I. Amemiya, a letter to T. Imai, October 1982.
3. M. L. Comtet, Nombres de Stirling généraux et fonctions symétrique, *C.R. Acad. Sci. Paris Sér. A* **275** (1972), 747–750.
4. I. P. Goulden and D. M. Jackson, An inversion model for q-identities, *European J. Combin.* **4** (1983), 225–230.
5. V. E. Hogatt, Jr. and M. Bicknell, Roots of Fibonacci polynomials, *Fibonacci Quart.* **11**, No. 3 (1973), 271–274.
6. T. Imai, Sangaku shimatsuki (A thorough story of a divine problem), *Sugaku Sem.* **21**, No. 10 (1982), 46–49. [Japanese]
7. S. C. Milne, Mappings of subspaces into subsets, *J. Combin. Theory Ser. A* **33** (1982), 36–47.
8. S. C. Milne, Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers, *Adv. in Math.* **43** (1982), 173–196.