Idempotent lattices, Renner monoids and cross section lattices of the special orthogonal algebraic monoids

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Abstract

Let $M_{SO}(n)$ be the special orthogonal algebraic monoid, $T$ a maximal torus of the unit group, and $\bar{T}$ the Zariski closure of $T$ in the whole matrix monoid $M_n$. In this paper we explicitly determine the idempotent lattice $E(\bar{T})$, the Renner monoid $R$, and the cross section lattice $\Lambda$ of $M_{SO}$ in terms of the Weyl group and the concept of admissible sets (see Definition 3.1). It turns out that there is a one-to-one relationship between $E(\bar{T})$ and the admissible subsets, and that $R$ is a submonoid of $R_n$, the Renner monoid $M_n$. Also $\Lambda$ is a sublattice of $\Lambda_n$, the cross section lattice of $M_n$.

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1. Introduction

The purpose of this paper is to give an explicit description of the idempotent lattices, the Renner monoids and the cross section lattices of the special orthogonal algebraic monoids.

(∗) Throughout this paper, $K$ is an algebraically closed field.

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1.1. Idempotent lattices

Idempotents are very fundamental in the theory of a reductive algebraic monoid \( M \) [3, 4, 17]. In fact, \( M \) is pieced together from its unit group \( G \) and the set of idempotents \( E(M) \), since \( M = GE(M) = E(M)G \) [12, 15]. Let \( T \) be a maximal torus of \( G \) and \( \overline{T} \) the Zariski closure of \( T \) in \( M \). Let \( E(\overline{T}) \) be the set of idempotents in \( \overline{T} \). Then \( E(\overline{T}) \) is a lattice, called the idempotent lattice (of \( \overline{T} \)), and

\[
E(M) = \bigcup_{g \in G} g E(\overline{T}) g^{-1}
\]

[13, Corollary 1.6], [16, Corollary 6.10]. So, finding \( E(M) \) can be reduced to find \( E(\overline{T}) \). Using torus embeddings [2], Putcha has found that there is a lattice anti-isomorphism from the face lattice of a convex polyhedral cone to the idempotent lattice \( E(\overline{T}) \) in [13]. A detailed approach to finding \( E(\overline{T}) \) has been given by Solomon in [20, Section 5]. As an example, the idempotent lattice of the symplectic monoid was obtained by using the concept of admissible subsets [20, p. 336].

Inspired by Solomon’s work, we explicitly determine the idempotent lattice of the special orthogonal monoid \( MSO_n \) using the concept of admissible subsets in Section 3. The main result of this section is as follows (Theorem 3.2).

**Theorem A.** Let \( E_{ij} \) (\( i, j = 1, \ldots, n \)) be an elementary matrix. Then

(a) The map

\[ I \mapsto e_I = \sum_{j \in I} E_{jj} \]

is bijective from the admissible subsets of \( \{1, \ldots, n\} \) to \( E(\overline{T}) \), where \( e_I = 0 \) if \( I = \emptyset \).

(b) The set \( E(\overline{T}) \) of idempotents in \( \overline{T} \) is

\[ E(\overline{T}) = \{ e_I \mid I \subseteq \{1, \ldots, n\} \text{ is admissible} \}. \]

(c) \( e_{I_1} \cdot e_{I_2} = e_{I_1 \cup I_2} \) for any \( e_{I_1}, e_{I_2} \in E(\overline{T}) \).

1.2. Cross section lattices

The cross section lattice \( \Lambda \) of \( M \) was first introduced by M. Putcha [14]. Let \( B \subseteq G \) be a Borel subgroup with \( T \subseteq B \). The the cross section lattice is defined as follows:

\[
\Lambda = \Lambda(B) = \{ e \in E(\overline{T}) \mid Be = eBe \}.
\]

This is not the usual definition, but is equivalent to it for reductive algebraic monoids [16, p. 94], [18, p. 310]. It is a key concept. For example, \( M = \bigsqcup_{e \in \Lambda} GeG \) and \( R = \bigsqcup_{e \in \Lambda} WeW \).
We describe the cross section lattices of the special orthogonal monoids in Section 4. The following theorem is the main result (Proposition 4.2).

**Theorem B.** The cross section lattice of \( MSO_n \) is

\[
A = \left\{ I_n, \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \ddots \\ 0 \end{pmatrix}, \begin{pmatrix} I_{l-1} & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right\}
\]

\[= \{ e_I \in E(T) \mid I \text{ is a standard admissible subset of } \{1, \ldots, n\} \} \]

1.3. The Renner monoids

The Renner monoid \( R \) plays the same role for \( M \) that the Weyl group does for a reductive algebraic group. The Renner decomposition and Renner system in \( M \) are now central ideas in the structure theory. Many questions about \( M \) may be reduced to questions about \( R \) [8–11,20]. Let \( B \subseteq G \) a Borel subgroup with \( T \subseteq B \), \( N \) the normalizer of \( T \) in \( G \), \( \overline{N} \) the Zariski closure of \( N \) in \( M \). Then \( \overline{N} \) is a unit regular inverse monoid which normalizes \( T \). So, \( R = \overline{N}/T \) is a monoid (the so-called Renner monoid now) which contains the Weyl group \( W = N/T \) as its unit group. Renner [18] defined this concept, and found an analogue of the Bruhat decomposition for reductive algebraic monoids. Also he obtained a monoid version of the Tits System.

Let \( M = M_n \). Then the Renner monoid \( R_n \) of \( M \) may be identified with the monoid of all zero–one matrices which have at most one entry equal to one in each row and column, i.e., \( R_n \) consists of all injective, partial maps on a set of \( n \) elements. The cardinality of \( R_n \) is \( |R_n| = \sum_{r=0}^n \binom{n}{r}^2 r! \). The unit group of \( R_n \) is the group \( P_n \) of permutation matrices.

Let \( SO_n (n = 2l) \) be the special orthogonal algebraic group over \( K \) (\( \text{char } K \neq 2 \)) (see Humphreys [5]), and let \( G = K^* SO_n \subseteq GL_n \). Then \( G \) is a connected reductive group and \( MSO_n = \overline{G} \), the Zariski closure of \( G \) in \( M_n \), is a reductive algebraic monoid called the special orthogonal monoid (see Definition 2.1). We determine the Renner monoid \( R \) of \( MSO_n \) and the cardinality of \( R \) in Section 5. It turns out that \( R \) is a submonoid of \( R_n \). The following theorem is a summary of Theorems 5.7 and 5.9 and Corollary 5.12.

**Theorem C.** Keeping the same notations above. Then

\[
(\text{a) } R = \left\{ \sum_{i \in I, w \in W} E_{i,w} \in R_n \mid I \subseteq \{1, \ldots, n\} \text{ is admissible} \right\}
\]

\[= \left\{ x \in R_n \mid x \text{ is singular, } D(x) \text{ and } R(x) \text{ are admissible, and of the same type if } |D(x)| = |R(x)| = l \right\} \cup W,
\]

where \( D(x) \) is the domain of \( x \) and \( R(x) \) is the range of \( x \);
\[ (b) \quad |\mathcal{R}| = \sum_{i=0}^{l-1} \left( \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i} \right) ! \right) + (2^l + 1) 2^{l-1} ! \quad \text{for } l \geq 1. \]

2. Preliminaries and basic concepts

The whole matrix monoid \( M_n \) is an algebraic monoid with the general linear group \( GL_n \) as its unit group, and \( GL_n = M_n \), the Zariski closure of \( GL_n \) in \( M_n \). Let
\[ B_n = B_n(K) = \{(a_{ij}) \in M_n \mid a_{ij} = 0 \text{ if } i < j \} \]
be a Borel subgroup of \( GL_n \). The monoid \( D_n = D_n(K) \) consists of diagonal matrices in \( M_n \). The subgroup \( T_n = T_n(K) \) of \( D_n \) consisting of all invertible diagonal matrices is a maximal torus of \( GL_n \), and \( T_n \) is the Zariski closure of \( T_n \) in \( M_n \). We use \( \mathcal{R}_n \) to denote the Renner monoid of \( M_n \). Then (see [18, p. 327])
\[ \mathcal{R}_n = \{(a_{ij}) \in M_n \mid a_{ij} \text{ is 0 or 1 and has at most one non-zero entry in each row and column} \}. \]
The set of the idempotents of \( \mathcal{R}_n \) is
\[ E(\mathcal{R}_n) = \{(a_{ij}) \in D_n \mid a_{ij} = 0 \text{ or 1 for all } i, j \}. \]
The cross section lattice of \( M_n \) is
\[ A_n = A(B_n) = \{ e \in E(\mathcal{T}_n) \mid B_n e = e B_n e \} = \left\{ \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, (0) \right\}. \]

It is well known that the unit group of \( \mathcal{R}_n \) is the Weyl group of \( GL_n \), which is isomorphic to the symmetric group \( S_n \) on \( n \) letters [6,7]. Let \( P_n \subseteq GL_n \) be the group of permutation matrices. Then \( S_n \) is isomorphic to \( P_n \) by the mapping \( \pi \mapsto \sum_{j=1}^{n} E_{\pi j, j} \), where \( \pi \in S_n \) and \( E_{\pi j, j} \) is an elementary matrix.

From now on, let \( n = 2l \) be even and \( J_l = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \in M_n \) be the symmetric matrix, where
\[ J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \]
is an \( l \times l \) matrix. The special orthogonal group is by definition
\[ G_0 = SO_n = \{ g \in SL_n \mid g^T J_l g = J_l \}, \]
which is connected and reductive.
Remark 1. The definition of \( \text{SO}_n \) here is from Humphreys [5, pp. 52–53].

Let \( G = K^*G_0 \subseteq \text{GL}_n \). Then \( G \) is a connected reductive group with rank \( r = l + 1 \) and semisimple rank \( l \) [19, 20].

Definition 2.1. The monoid \( \overline{G} \) (Zariski closure of \( G \) in \( \text{M}_n \)) is called the special orthogonal monoid and will be denoted by \( \text{MSO}_n \), where \( n = 2l \).

Let \( T_0 = G_0 \cap T_n \). Elements in \( T_0 \) have the shape

\[
t = \text{diag}(t_1, \ldots, t_l, t_1^{-1}, \ldots, t_l^{-1})
\]

where \( t_1, \ldots, t_l \) are arbitrary in \( K^* \). Thus \( T_0 \) is a maximal torus of dimension \( l \). Let us recall some facts about the Weyl group \( W(G, T) \). If \( \pi \in S_n \), let \( p_{\pi} = \sum_{i=1}^{n} E_{\pi,i,i} \in P_n \) be the corresponding permutation matrix. Then \( p_{\pi}(a_{ij})(a_{ij})^{-1}p_{\pi}^{-1} = (a_{\pi(i),\pi(j)}) \) where \( (a_{ij}) \) is any \( n \times n \) matrix. It follows that \( p_{\pi}^{-1}(a_{ij})p_{\pi} = p_{\pi^{-1}}(a_{ij})p_{\pi} = (a_{\pi(i),\pi(j)}) \).

Define an involution \( \theta : i \mapsto l + 1 - i \) of \( \{1, 2, \ldots, n\} \) by

\[
\theta(i) = n + 1 - i \quad \text{for} \quad 1 \leq i \leq n.
\]

Let \( C \) denote the centralizer of \( \theta \) in \( S_n \). Then \( p_{\pi} \) normalizes \( T_0 \) if and only if \( \pi \in C \). The group \( C \) is a semidirect product \( C = C_1C_2 \) where \( C_1 \) is a normal abelian subgroup of order \( 2^l \) generated by the transpositions \((11), \ldots, (l l)\) and \( C_2 \simeq S_l \) consists of all permutations \( \pi \in S_n \) which stabilize \( \{1, \ldots, l\} \) and act on the complement \( \{l + 1, \ldots, n\} \) in the unique manner consistent with the assertion that \( \pi \in C \). Note that permutation matrices in \( C \) need not be in \( \text{SO}_n \). So, let \( C'_1 \) be a subgroup of \( C_1 \) generated by \((11)(22), (22)(33), \ldots, (l - 1 l - 1)\). Then \( C'_1 \) consists of even permutations in \( C_1 \). Let \( C'_2 \simeq C_2 \) and \( C' = C'_1C'_2 \). It follows that \( C' \subseteq \text{SO}_n \) and \( |C'| = 2^{l-1}l! \). But \( \omega_1T_0 = \omega_2T_0 \) if and only if \( \omega_1 = \omega_2 \) for any \( \omega_1, \omega_2 \in C' \). Thus \( W \) is isomorphic to \( C' \subseteq S_n \). Also, \( W \) is isomorphic to \( (Z_2)^{l-1} \rtimes S_l \).

Let \( T = K^*T_0 \). Then \( T \) is a maximal torus of \( G \) and the Weyl group \( W(G, T) \) is isomorphic to \( W(G_0, T_0) \). We let \( W \) denote either of them in what follows. If \( n = 4 \), then \( \theta = (11)(22) = (14)(23) \), and \( C'_1 \) is a subgroup of \( C_{S_4}(\theta) \) generated by \( (14)(23) \). So

\[
C'_1 = \{1, (14)(23)\}.
\]

Taking \( \pi = (12)(34) \), we see that \( \theta\pi = \pi\theta \) which means that \( \pi \in C_{S_4}(\theta) \). It is clear that \( \pi \) stabilizes \( \{1, \ldots, l\} = \{1, 2\} \) and \( \pi \not\in C_1 \). Let \( C'_2 \) be a subgroup of \( C_{S_4}(\theta) \) generated by \( \pi \). Then

\[
C'_2 = \{1, (12)(34)\}.
\]
Thus the Weyl group $W = C'_1C'_2 = \{1, (14)(23), (12)(34), (13)(24)\}$. The corresponding matrix form of the Weyl group is

$$W = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$ 

3. Idempotent lattice $E(\mathcal{T})$ of $MSO_n$

Just as is used in the case of symplectic monoid, we need the following definition due to [20, p. 336] to determine the idempotent lattice $E(\mathcal{T})$ of $MSO_n$.

**Definition 3.1.** A subset $I \subseteq \{1, \ldots, n\}$ is called admissible if $j \in I$ implies $\bar{j} \notin I$, where $\bar{j} = \theta(j)$ as above; the empty set $\phi$ and $\{1, \ldots, n\}$ are also considered to be admissible.

Notice that $W$ maps admissible sets to admissible sets and $w^{-1}e_w = e_{wI}$ for any $w \in W$.

If $n = 4$, then the admissible subsets of $\{1, 2, 3, 4\}$ are

$\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3, 4\}$.

A similar discussion to [20, p. 336] gives the following theorem describing the relationship between admissible subsets and idempotent lattice $E(\mathcal{T})$ of $MSO_n$. We omit the similar argument to that of [20].

**Theorem 3.2.** (a) The map

$$I \mapsto e_I = \sum_{j \in I} E_{jj}$$

is bijective from the admissible subsets of $\{1, \ldots, n\}$ to $E(\mathcal{T})$, where $e_I = 0$, if $I = \phi$.

(b) The idempotent lattice $E(\mathcal{T})$ is

$$E(\mathcal{T}) = \{ e_I \mid I \text{ is admissible} \}.$$ 

(c) $e_{I_1} \cdot e_{I_2} = e_{I_1 \cup I_2}$ for any $e_{I_1}, e_{I_2} \in E(\mathcal{T})$.

If $n = 4$, the set of idempotent lattice $E(\mathcal{T})$ of $MSO_4$ is

$$E(\mathcal{T}) = \{ 0, 1, E_{11}, E_{22}, E_{33}, E_{44}, E_{11} + E_{22}, E_{33} + E_{44}, E_{11} + E_{33}, E_{22} + E_{44} \}.$$ 

Let $E_i(\mathcal{T}) \subseteq E(\mathcal{T})$ denote the set of rank $i$ idempotent elements in $\mathcal{T}$, where $i = 0, 1, \ldots, l, n$. Then by Theorem 3.2 we have the following corollary.
Corollary 3.3. For $i = 0, 1, \ldots, l, n$,

(a) $E_i(T) = \{ e_I \mid I \subseteq \{1, \ldots, n\} \text{ is admissible and } |I| = i \}$;

(b) $|E_i(T)| = \left\{ \sum_{j=0}^{i} \binom{l}{j} \binom{l-j}{i-j} \right\} 1$, if $i = 0, 1, \ldots, l$,

$\sum_{j=0}^{i} \binom{l-j}{i-j}$, if $i = n$.

Corollary 3.4. (a) $E_1(T) = \{ E_{ii} \mid i = 1, \ldots, n \}$.

(b) $|E_1(T)| = n$.

Remark 2. There are no admissible subsets with size $k$ ($l < k < n$). The rank one elements in $E(T)$ are in one-to-one correspondence with the admissible subsets containing exactly one element of $\{1, \ldots, n\}$.

Corollary 3.5.

$|E(T)| = \sum_{i=0}^{l} \sum_{j=0}^{i} \binom{l}{j} \binom{l-j}{i-j} + 1$.

Proof. By Corollary 3.3 and $|E(T)| = \sum_{i=0}^{l} |E_i(T)| + 1$. \hfill \Box

4. The cross section lattice of $MSO_n$

An admissible subset $I$ is referred to as standard if there is an integer $i \in \{1, \ldots, l, n\}$ such that $I = \{1, \ldots, i\}$; the empty set and the set $\{1, \ldots, l-1, l+1\}$ are also considered to be standard. For example, the standard admissible subsets of $\{1, 2, 3, 4\}$ are $\emptyset, \{1\}, \{2\}, \{1, 3\}, \{1, 2, 3, 4\}$.

To define the cross section lattice, we need some notations: $M$ is a reductive monoid, $G$ is its unit group, $T$ is a maximal torus of $G$, and $B \subseteq G$ is a Borel subgroup with $T \subseteq B$.

Definition 4.1. The cross section lattice $\Lambda$ of $M$ is defined by

$\Lambda = \Lambda(B) = \{ e \in E(T) \mid Be = e Be \}$.

In case $M = MSO_n$, it follows that $G = K^*SO_n$ and that $B = G \cap B_n$ is a Borel subgroup of $G$.

Remark 3. If $G$ is any algebraic group, then it is not always the case that $G \cap B_n$ is a Borel subgroup of $G$ [1,21].
Proposition 4.2. The cross section lattice of $\text{MSO}_n$ is

$$\Lambda = \{ e_I \in E(\mathcal{T}) \mid I \text{ is a standard admissible subset of } \{1, \ldots, n\} \}$$

$$= \left\{ I_n, \left( \begin{array}{c} I_l \ 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{array} \right), \left( \begin{array}{c} I_{l-1} \ 0 \end{array} \right), \ldots, \left( \begin{array}{c} I_1 \ 0 \end{array} \right), 0 \right\}.$$ 

Proof. It is true that $\Lambda$ is as stated by calculating directly from the definition. \(\square\)

We now consider some examples. If $n = 2$, then all the admissible subsets of $\{1, 2\}$ are $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$. They are all standard. So, the cross section lattice of $\text{MSO}_2$ is

$$\Lambda = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right\}.$$ 

The cross section lattice of $\text{MSO}_4$ is given by

$$\Lambda = \left\{ (0), \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \right\}.$$ 

5. The Renner monoid of $\text{MSO}_n$

The main purpose of this section is to determine the Renner monoid of the special orthogonal algebraic monoid $\text{MSO}_n$, even case. We get some by-products as well, such as the cardinalities of the Renner monoids.

Let $R(i)$ be the set of rank $i$ elements, where $i = 0, 1, \ldots, n$. Then we have

Lemma 5.1. $R(1) = \{ E_{ij} \mid i, j = 1, \ldots, n \}$ and $|R(1)| = n^2$.

Proof. It suffices to show that $\{ E_{ij} \mid i, j = 1, \ldots, n \} \subseteq R(1)$.

Firstly, we prove that $\{ E_{1,j} \mid j = 1, \ldots, n \} \subseteq R(1)$. There are three cases:

(a) If $j \in \{1, \ldots, l\}$, let $w = (1j)(\bar{j} \bar{l})$. Then $w$ stabilizes $\{1, \ldots, l\}$ and $w\theta = \theta w$. It follows that $w \in W_2 \subseteq W$ and $w(j) = 1$.

(b) If $j = \bar{l} = n \in \{l + 1, \ldots, n\}$, let $w = (1j)(\bar{j} \bar{l}) = (1\bar{l})(\bar{j} \bar{l})$. Then $w \in W_1 \subseteq W$ and $w(j) = 1$.

(c) If $j \in \{l + 1, \ldots, n\}$ but $j \neq \bar{l} = n$, let $w_2 = (j \bar{l})(\bar{j} \bar{l})$. Then $w_2$ stabilizes $\{1, \ldots, l\}$ and $w_2\theta = \theta w_2$. So $w_2 \in W_2 \subseteq W$ and $w_2(j) = 1$. Let $w_1 = (1\bar{l})(\bar{j} \bar{l}) \in W_1$ and $w = w_1w_2$. Then $w \in W$ and $w(j) = w_1(w_2(j)) = w_1(\bar{1}) = 1$. 


So, $E_{ij} = E_{i,w_1} = E_{11} w \in E_{11} W$ for $j = 1, \ldots, n$. Thus $\{E_{ij} \mid j = 1, \ldots, n\} = E_{11} W$, which is a subset of $\mathcal{R}(1)$.

Similarly, $\{E_{ij} \mid j = 1, \ldots, n\} = E_{ii} W \subseteq \mathcal{R}(1)$ for $i = 2, \ldots, n$.

Therefore, $\mathcal{R}(1) = \{E_{ij} \mid i, j = 1, \ldots, n\}$ with size $n^2$. □

**Remark 4.** Lemma 5.1 shows that $\mathcal{R}(1) = \mathcal{R}_n(1)$, the set of rank one elements in $\mathcal{R}_n$.

However, $\mathcal{R}(2) \neq \mathcal{R}_n(2)$ since $[1, n]$ is not an admissible subset of $\{1, \ldots, n\}$, and so $E_{11} + E_{n,n} \notin \mathcal{R}(2)$, but $E_{11} + E_{n,n} \in \mathcal{R}_n(2)$. For the same reason, we know that $\mathcal{R}(i) \neq \mathcal{R}_n(i)$, for $i = 3, \ldots, n$.

**Lemma 5.2.** For any admissible subset $I \subseteq \{1, \ldots, n\}$ with $|I| = i$, where $i = 1, \ldots, l - 1, n$, there exist $w \in W$ and a unique standard admissible subset $I_0 = \{1, \ldots, i\}$ such that $wI = I_0$.

**Proof.** If $I = \{1, \ldots, n\}$, then $I_0 = I$ and $w = 1 \in W$, and we are done. Now let $I$ be admissible and $I \neq \emptyset$. Use induction on the size $i$ of the admissible subset $I$. If $i = 1$, then $I = \{j\}$ for some $j \in \{1, \ldots, n\}$ and $I_0 = \emptyset$. By Lemma 5.1, we know there exists $w \in W$ such that $w(I) = I_0$.

Now suppose that $I \subseteq \{1, \ldots, n\}$ is any admissible subset with $1 < |I| = i \leq l - 1$. Let $I = J \cup \{k\}$ where $J$ is a subset of $I$ with $|J| = i - 1$ and $k \in I \setminus J$. It follows that $J$ is admissible. By the induction hypothesis, there exist $w' \in W$ and a unique standard admissible subset $I' = \{1, \ldots, i - 1\}$ such that $w'J = I'$. Then $wI = I' \cup \{p\}$ where $p = w'(k) \notin I'$. There are four cases for $p$:

1. If $p = i$, then $I_0 = I' \cup \{i\}$ and $w = w'$ are what we want.
2. If $p \in \{1, \ldots, l\}$, and $p \neq i$, let $w_1 = (p)(\tilde{p}i)$. Then $w_1 \theta = \theta w_1$ and $w_1$ stabilizes $\{1, \ldots, l\}$. Thus, $w_1 \in W_2 \subseteq W$. Note that $w_1(j) = j$ for $j \in I'$. Taking $w = w_1 w'$, we obtain that $w \in W$ and $w(I) = w_1(I') = I' \cup \{w_1(p)\} = I_0$.
3. If $p = i = n + 1 - i$, then $w_1 = (i)(\tilde{i}l)$ with $i \leq l - 1$. Then $w_1 \in W_1 \subseteq W$ and $w_1I' = 1$. Let $w = w_1 w'$. We obtain that $w \in W$ and $w(I) = w_1(I') = I_0$.
4. If $p \in \{l + 1, \ldots, n\}$ but $p \neq i = n + 1 - i$, Let $w_1 = (p)(\tilde{p}i)$. Then $w_1 \in W_2 \subseteq W$ and $w_1(j) = j$ for $j \in I'$. Taking $w = (i)(\tilde{i}l)w_1 w'$, we get $w \in W$ and $w(I) = (i)(\tilde{i}l)w_1(I' \cup \{p\}) = (i)(\tilde{i}l)(I' \cup \{i\}) = I' \cup \{i\} = I_0$.

This proves the theorem. □

**Corollary 5.3.** The Weyl group $W$ acts transitively on $E_1(\overline{T})$ by $w^{-1} e_i w = e_{w_i}$ for $i = 1, \ldots, l - 1$.

**Lemma 5.4.** Let $I \subseteq \{1, \ldots, n\}$ be admissible with $|I| = l$. Then there exists $w \in W$ such that either $w(I) = \{1, \ldots, l\}$ or $w(I) = \{1, \ldots, l - 1, l + 1\}$.

**Proof.** Since $I \subseteq \{1, \ldots, n\}$ is an admissible subset with $|I| = l$, then $I = J \cup \{k\}$ where $J$ is a subset of $I$ with $|J| = l - 1$ and $k \in I \setminus J$. It follows that $J$ is admissible. By
Lemma 5.2, there exist \( w \in W \) and a unique standard admissible subset \( I' = \{1, \ldots, l - 1\} \) such that \( wJ = I' \). Then \( wI = I' \cup \{p\} \) where \( p = w(k) \notin I' \). We claim that \( p = l \) or \( l + 1 \). Otherwise, \( p \in \{l + 2, l + 3, \ldots, n\} \). It follows that \( \theta(p) = \bar{p} = n + 1 - p \in I' \subseteq w'(I) \), i.e., \( p \) and \( \bar{p} \) are both in \( w'(I) \), which is impossible since \( w'(I) \) is admissible. This proves the theorem. \( \Box \)

**Corollary 5.5.** Under the action, by conjugation, of \( W \) on \( E_1(\mathcal{T}) \), there are exactly two orbits. One is \( WE_1W \) and the other is \( WE_2W \), where \( J_1 = \{1, \ldots, l\} \) and \( J_2 = \{1, \ldots, l - 1, l + 1\} \).

We will use the following definition soon.

**Definition 5.6.** An admissible subset \( I \) of size \( l \) is called type I if there exists \( w \in W \) such that \( wI = J_1 = \{1, \ldots, l - 1, l\} \); type II if \( wI = J_2 = \{1, \ldots, l - 1, l + 1\} \).

**Theorem 5.7.** With the notation above, the Renner monoid of the special orthogonal monoid \( MSO_n \) is as follows:

\[
\mathcal{R} = \left\{ \sum_{i \in I, w \in W} E_{i,wi} \in \mathcal{R}_n \ \middle| \ I \subseteq \{1, \ldots, n\} \text{ is admissible} \right\}.
\]

**Proof.** Since \( \mathcal{R} = E(\mathcal{T})W \) by [20, Proposition 3.2.1], it suffices to compute \( e_Iw \) for every \( e_I \in E(\mathcal{T}) \), \( w \in W \), where \( I \) is admissible. From Theorem 4.2(a) we know that \( e_I = \sum_{i \in I} E_{ii} \). Thus \( e_Iw = \sum_{i \in I} E_{ii}w = \sum_{i \in I} E_{i,wi} \), and so the theorem follows. \( \Box \)

**Corollary 5.8.**

\[
\mathcal{R} = \left\{ \sum_{i \in I, w \in W} E_{wi,i} \in \mathcal{R}_n \ \middle| \ I \subseteq \{1, \ldots, n\} \text{ is admissible} \right\}.
\]

**Proof.** This result comes from the fact that \( \mathcal{R} = WE(\mathcal{T}) \), and \( w^{-1}e_j = \sum_{i \in I} E_{wi,j} \). \( \Box \)

**Theorem 5.9.**

\[
\mathcal{R} = \left\{ x \in \mathcal{R}_n \ \middle| \ x \text{ is singular}, D(x) \text{ and } R(x) \text{ are admissible} \right. \\
\left. \text{and of the same type if } |D(x)| = |R(x)| = l \right\} \cup W,
\]

where \( D(x) \) is the domain of \( x \) and \( R(x) \) is the range of \( x \).

**Proof.** Let \( \mathcal{R}' \) denote the set of the right-hand side in the theorem. It follows from Theorem 5.7 that \( \mathcal{R} \subseteq \mathcal{R}' \) since \( W \) maps admissible sets to admissible sets and both \( wI \) and \( I \) are of the same type if \( |I| = l \).

We now prove the other inclusion. For any \( x \in \mathcal{R}' \), if \( x \in W \), then \( x \in \mathcal{R} \) since \( W \subseteq \mathcal{R} \). If \( x \notin W \), then \( x \) is singular and both \( D(x) \) and \( R(x) \) are admissible. So, \( |D(x)| = |R(x)| \), denoted by \( i \). Then \( i \leq l \).
(a) If \( i \neq l \), it follows from Lemma 5.2 that there exist \( w_1, w_2 \in W \) and a unique standard admissible set \( I_0 = \{1, \ldots, i \} \) (\( i \leq l - 1 \)) such that

\[
D(x) = w_2 R(x) = w_1 D(x) = w_2 R(x) = I_0.
\]

Thus \( w_1^{-1}xw_2 = e_{I_0} \in A \subseteq \mathcal{R} \), and hence \( x = w_1 e_{I_0}w_2^{-1} \in \mathcal{R} \setminus W \), since \( \mathcal{R} = WE(\mathcal{T}) = E(\mathcal{T})W \) and \( x \) is singular.

(b) If \( i = l \), then \( D(x) \) and \( R(x) \) are of the same type because of \( x \in \mathcal{R}' \). By Lemma 5.4, there are \( w_1 \) and \( w_2 \) in \( W \) such that

\[
w_1 D(x) = w_2 R(x) = \begin{cases} J_1, & \text{if } D(x) \text{ and } R(x) \text{ are of type I}, \\ J_2, & \text{if } D(x) \text{ and } R(x) \text{ are of type II}, \end{cases}
\]

where \( J_1 = \{1, \ldots, l - 1, l\} \) and \( J_2 = \{1, \ldots, l - 1, l + 1\} \). It follows that \( w_1^{-1}xw_2 = e_{J_1} \in A \subseteq E(\mathcal{T}) \) or \( w_1^{-1}xw_2 = e_{J_2} \in A \subseteq E(\mathcal{T}) \). That is, \( x = w_1 e_{J_1}w_2^{-1} \) or \( x = w_1 e_{J_2}w_2^{-1} \). Hence \( x \in \mathcal{R} \) since \( \mathcal{R} = WE(\mathcal{T}) = E(\mathcal{T})W \).

Therefore \( \mathcal{R} = \mathcal{R}' \), i.e., the theorem is true. \( \square \)

**Remark 5.** In the proof above, we obtained \( \mathcal{R} = W Aw \) as well.

The Renner monoid of \( MSO_2 \) is

\[
\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
\]

Let us now consider the Renner monoid of \( MSO_4 \). The idempotent set \( E(\mathcal{R}) = E(\mathcal{T}) \) of \( MSO_4 \) is a union of sets of rank \( i \) idempotent elements in \( \mathcal{T} \), for \( i = 0, 1, 2, 4 \):

\[
E(\mathcal{T}) = E_0(\mathcal{T}) \cup E_1(\mathcal{T}) \cup E_2(\mathcal{T}) \cup E_4(\mathcal{T}),
\]

where

\[
E_0(\mathcal{T}) = \{0\}, \quad E_1(\mathcal{T}) = \{E_{11}, E_{22}, E_{33}, E_{44}\},
\]

\[
E_2(\mathcal{T}) = \{E_{11} + E_{22}, E_{33} + E_{44}, E_{11} + E_{33}, E_{22} + E_{44}\},
\]

\[
E_4(\mathcal{T}) = \{E_{11} + E_{22} + E_{33} + E_{44}\}.
\]

Since \( \mathcal{R} = E_0(\mathcal{T})W \cup E_1(\mathcal{T})W \cup E_2(\mathcal{T})W \cup E_4(\mathcal{T})W \), we get

\[
\mathcal{R} = \{0, E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}, E_{41}, E_{42}, E_{43}, E_{44}, E_{41} + E_{42}, E_{41} + E_{43}, E_{42} + E_{43}, E_{11} + E_{22}, E_{14} + E_{23}, E_{12} + E_{21}, E_{13} + E_{24}, E_{33} + E_{44}, E_{32} + E_{41}, E_{34} + E_{43}, E_{31} + E_{42}, E_{11} + E_{33}, E_{14} + E_{32}, E_{12} + E_{34}, E_{13} + E_{31}, E_{22} + E_{44}\},
\]
\[ E_{23} + E_{41}, E_{21} + E_{43}, E_{24} + E_{42}, E_{11} + E_{22} + E_{33} + E_{44}, E_{14} + E_{23} + E_{32} + E_{41}, E_{12} + E_{21} + E_{34} + E_{43}, E_{13} + E_{24} + E_{31} + E_{42} \].

The following result is an analogue of [18, Proposition 7.3].

**Proposition 5.10.** For any \( e_I \in A \) with \( |I| = i \), where \( i = 0, 1, \ldots, l - 1 \),

\[ W_{e_I} W = \{ x \in \mathcal{R} \mid \text{rank}(x) = i \} = \{ x \in \mathcal{R} \mid x \text{ has } i \text{ nonzero rows} \} = \{ x \in \mathcal{R}_n \mid \text{D}(x) \text{ and } R(x) \text{ are admissible with } |D(x)| = |R(x)| = i \} \].

Furthermore,

\[ |W_{e_I} W| = \left[ \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right]^{2} i! \],

where \( D(x) \) is the domain of \( x \) and \( R(x) \) the range of \( x \).

**Proof.** Observe that \( Ge_I G = \bigsqcup_{x \in W_{e_I} W} \text{BxB} \) consists of \( n \times n \) matrices of rank \( i \) in \( MSO_n \) where \( i = |I| = 0, 1, \ldots, l - 1 \). One gets the first part of the proposition.

Now, there are \( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \) ways to choose \( i \) of the \( n \) rows making \( D(x) \) admissible. There are the same number of ways to choose \( i \) of the \( n \) columns such that \( R(x) \) is admissible. For each pair of the choices of the rows and columns, there are \( i! \) elements of \( \mathcal{R} \) of rank \( i \) with a nonzero entry in each of the \( i \) rows and each of the \( i \) columns chosen. Thus, there are \( \left[ \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right]^{2} i! \) possibilities. \( \square \)

Similarly, we get the following

**Proposition 5.11.** Let \( J_1 = \{1, \ldots, l\} \) and \( J_2 = \{1, \ldots, l - 1, l + 1\} \). Then

\[ W_{e_{J_1}} W \cup W_{e_{J_2}} W = \{ x \in \mathcal{R} \mid \text{rank}(x) = l \} = \{ x \in \mathcal{R} \mid x \text{ has } l \text{ nonzero rows} \} = \{ x \in \mathcal{R}_n \mid \text{D}(x) \text{ and } R(x) \text{ are admissible and of the same type with } |D(x)| = |R(x)| = l \} \].

Furthermore,

\[ |W_{e_{J_1}} W \cup W_{e_{J_2}} W| = \frac{1}{2} \left[ \sum_{j=0}^{l} \binom{l}{j} \right]^{2} l! \],

where \( D(x) \) is the domain of \( x \) and \( R(x) \) the range of \( x \).
Proof. The first part follows from Theorem 7.15 above.

Now, there are \( \frac{1}{2} \sum_{j=0}^{l} \binom{l}{j} \) ways to choose \( l \) of the \( n \) rows such that the resulting subsets are of type I (respectively II). There are the same number of ways to choose \( l \) of the \( n \) columns. For each pair of the choices of the rows and columns there are \( l! \) elements of \( R \) of rank \( l \) with a nonzero entry in each of the \( l \) rows and each of the \( l \) columns chosen. Thus there are \( \frac{1}{4} \left[ \sum_{j=0}^{l} \binom{l}{j} \right]^2 l! \) possibilities for elements on \( W e_{\lambda_1} W \) (respectively \( W e_{\lambda_2} W \)). Hence, the number of elements in \( W e_{\lambda_1} W \cup W e_{\lambda_2} W \) is as stated. \( \square \)

Corollary 5.12. \( |R| = \sum_{i=0}^{l-1} \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right)^2 i! \) for \( l \geq 1 \).

Proof. It is clear that

\[
|R| = \sum_{i=0}^{l-1} \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right)^2 i! + |W e_{\lambda_1} W \cup W e_{\lambda_2} W| + |W|
\]

\[
= \sum_{i=0}^{l-1} \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right)^2 i! + \frac{1}{2} \left[ \sum_{j=0}^{l} \binom{l}{j} \right]^2 l! + 2^{l-1} l!
\]

\[
= \sum_{i=0}^{l-1} \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right)^2 i! + \frac{1}{2} 2^i l! + 2^{l-1} l!
\]

\[
= \sum_{i=0}^{l-1} \left( \sum_{j=0}^{i} \binom{i}{j} \binom{l-j}{i-j} \right)^2 i! + (2^i + 1) 2^{l-1} l!.
\]

For instance, the Renner monoid of \( MSO_4 \) has 37 elements. \( \square \)

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References