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Some additional properties of elementary landscapes

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ABSTRACT

In this work, we derive a general class of multistep *composite* elementary landscapes and present the first non-trivial lower (upper) bounds on local minima (maxima) associated with elementary landscapes.

1. Introduction

In earlier work [1,2], authors of this work have developed a general theory for elementary landscapes with arbitrary neighborhood definitions. In this work, we extend that theory by deriving a general class of multistep *composite* elementary landscapes and presenting the first non-trivial lower (upper) bounds on local minima (maxima) associated with elementary landscapes.

For the reader's convenience, we now summarize the notation and nomenclature used in [1,2].

A landscape for a combinatorial optimization problem (COP) is defined by $\mathcal{L} = (X, f, \mathcal{N})$ [1,2], where $X = \{x_i\}$ is the finite solution space, $f = [f(x_i)] = [f_i]$ is the real objective function vector over X, and $\mathcal{N}$ is the search neighborhood defined by a digraph whose nodes are the $x_i \in X$. The neighborhood digraph has an associated adjacency matrix A and transition matrix T. For each $x_i \in X$, a non-zero a_{ij} designates x_j as a neighbor of x_i and t_{ij} gives the probability of moving to x_j in the next move. The transition matrix is defined to be $T = [t_{ij}] = [a_{ij}/d_i]$ where $d_i = \sum_{\forall j} a_{ij}$, the degree of node i, and the α-*normalized* objective function vector is defined to be $f_\alpha = [f(x_i) - \alpha] = [f_{\alpha i}]$. An elementary landscape, which may be *smooth* or *rough*, satisfies the Laplacian equation, $Lf_\alpha = \lambda f_\alpha$, where Laplacian $L = I - T$ and α is the expected value of f [1,2].

2. Composite elementary landscapes

Let $\mathcal{L}_1 = (X, f, \mathcal{N}_1)$ and $\mathcal{L}_2 = (X, f, \mathcal{N}_2)$ be landscapes differing *only* by their neighborhood definitions [3,4]. If $\mathcal{L}_2$ is the *composite* landscape generated by performing n sequential moves under $\mathcal{N}_1$, then $\mathcal{N}_2$ has associated transition matrix $T_2 = T_1^n$. Barnes et al. [1] termed such a neighborhood an "n-step neighborhood". However, from the perspective of $\mathcal{L}_2$, such an $\mathcal{N}_2$ is simply an alternative "one-step" neighborhood. Similarly, a new composite landscape, $\mathcal{L}_{21}$, could be generated by first performing a move according to $\mathcal{N}_2$ and then one according to $\mathcal{N}_1$, yielding $T_{21} = T_2 T_1$ [5].

Proposition 1. *For any connected elementary landscape $\mathcal{L}$ having transition matrix T and objective function f, the corresponding two-step landscape is smooth elementary.*

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Proof. For π the steady state of T, π is also that of T^2, the transition matrix of the two-step landscape. Thus, $\alpha = \pi f$ is the expected value of f for both landscapes. Since $\mathcal{L}$ is elementary, $Lf_\alpha = (I - T)f_\alpha = \lambda f_\alpha$ which implies

$$Tf_\alpha = (1 - \lambda)f_\alpha. \tag{1}$$

The two-step landscape is elementary because, using the above result,

$$(I - T^2)f_\alpha = f_\alpha - T^2 f_\alpha = f_\alpha - (1 - \lambda)^2 f_\alpha = (2\lambda - \lambda^2)f_\alpha.$$

The landscape is smooth because $2\lambda - \lambda^2$ cannot exceed 1. □

Lemma 1. *In an arbitrary landscape with transition matrix T, Laplacian L, and objective function f, $Lf = Lf_b$ for any real b.*

Proof. For a column vector, e, of ones, noting that each T-row sums to 1, we have

$$Lf_b = L(f - be) = Lf - bLe = Lf - b(I - T)e = Lf - be + bTe = Lf - be + be = Lf. \quad \square$$

Proposition 2. *The composite landscape formed by two elementary landscapes that share objective function f and expected value α is elementary.*

Proof. For T_i the transition matrix of elementary landscape i,

$$L_i f_\alpha = (I - T_i)f_\alpha = (I - T_i)f = \lambda_i f_\alpha$$

which implies that

$$T_i f = f - \lambda_i f_\alpha. \tag{2}$$

T_2T_1 is the transition matrix of the composite landscape, which, if elementary, is defined by $L_{21}f_\alpha = \lambda_{21}f_\alpha$. Consider the following derivation in which e is a column vector of ones:

$$\begin{aligned}
L_{21}f_\alpha = (I - T_2T_1)f &= f - T_2T_1f = f - T_2(f - \lambda_1 f_\alpha) = f - T_2f + \lambda_1 T_2 f_\alpha \\
&= f - T_2f + \lambda_1 T_2(f - \alpha e) \\
&= f - T_2f + \lambda_1 T_2 f - \alpha\lambda_1 e \\
&= f - (f - \lambda_2 f_\alpha) + \lambda_1(f - \lambda_2 f_\alpha) - \alpha\lambda_1 e \\
&= \lambda_2 f_\alpha + \lambda_1 f - \lambda_1\lambda_2 f_\alpha - \alpha\lambda_1 e \\
&= \lambda_2(f - \alpha e) + \lambda_1 f - \lambda_1\lambda_2(f - \alpha e) - \alpha\lambda_1 e \\
&= \lambda_2 f - \alpha\lambda_2 e + \lambda_1\lambda_2 f + \alpha\lambda_1\lambda_2 e - \alpha\lambda_1 e \\
&= (\lambda_2 + \lambda_1 - \lambda_2\lambda_1)f - (\lambda_2 + \lambda_1 - \lambda_2\lambda_1)\alpha e \\
&= (\lambda_2 + \lambda_1 - \lambda_2\lambda_1)f_\alpha \\
&= \lambda_{21} f_\alpha.
\end{aligned}$$

Thus, the composite landscape is elementary. □

Corollary 1. *The composite landscape formed by two elementary landscapes that share an objective function is elementary.*

Proof. By allowing the landscapes to have different expected values, the above proof is slightly modified to yield this result. □

This corollary may be used to determine the required properties of elementary landscapes $\mathcal{L}_1$ and $\mathcal{L}_2$ that cause $\mathcal{L}_{21}$ to be smooth or rugged:

(1) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are smooth then $\mathcal{L}_{21}$ is smooth ($0 \le \lambda_{21} \le 1$).
(2) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are rugged then $\mathcal{L}_{21}$ is smooth ($0 \le \lambda_{21} \le 1$).
(3) If $\mathcal{L}_1$ is smooth and $\mathcal{L}_2$ is rugged (or vice versa) then $\mathcal{L}_{21}$ is rugged ($1 \le \lambda_{21} \le 2$).

Corollary 2. *The composite landscape formed by any number of elementary landscapes is elementary.*

Proof. Follows from Corollary 1. □

Alternate proofs for Propositions 1 and 2, making use of the relationships between elementary landscapes and AR(1) processes [3], are presented in [2].

3. Lower bounds on local minima and upper bounds on local maxima

In [1,3], it is proven that all local minima (maxima) of an elementary landscape are bounded above (below) by α. Using results from Vassilev et al. [6], we now derive a simple *lower* (*upper*) bound on local minima (maxima) for any elementary landscape. We also identify time series concepts that may enhance ongoing research into elementary landscapes.

Vassilev et al. [6] use information theory to analyze the time series generated from a random walk [3] and first transform the time series using the mapping $S(\varepsilon) = \{s_1, s_2, \ldots, s_n\}$:

$$s_i = \begin{cases} \bar{1} & \text{if } f_\alpha(x_i) - f_\alpha(x_{i-1}) < -\varepsilon \\ 0 & \text{if } |f_\alpha(x_i) - f_\alpha(x_{i-1})| \le \varepsilon \\ 1 & \text{if } f_\alpha(x_i) - f_\alpha(x_{i-1}) > \varepsilon. \end{cases}$$

The constant ε is a non-negative number taken from the interval $[0, \max_{\forall i} f_\alpha(x_i)]$. The idea behind this transformation is to extract information from the landscape by ignoring some non-essential features. The value of ε measures the accuracy of the calculations of the string, $S(\varepsilon) = \{s_1, s_2, \ldots, s_n\}$.

Vassilev et al. [6] characterize the ruggedness or "information content" of the landscape by introducing an entropy measure of the ensemble associated with sub-blocks of length 2 of the string $S(\varepsilon) = \{s_1, s_2, \ldots, s_n\}$. In addition, they measure the ruggedness of the landscape via the modality of the time series path using the following construction. Consider a compression of $S(\varepsilon)$ deleting all 0 values and all the elements whose right adjacent element has equal value. This yields a new set whose elements alternate between 1 and $\bar{1}$ (the set can begin with either) and is the shortest string that represents the slopes of the neighboring landscape path. The length of the compressed string is the modality, μ. The "partial information content" is defined as $M(\varepsilon) = \frac{\mu}{n}$, $0 \le M(\varepsilon) \le 1$, where $M(\varepsilon) = 0$ implies a flat landscape and $M(\varepsilon) = 1$ implies maximal modality. The relative accuracy of the estimation of the information content and partial information content is inversely proportional to ε. *Information stability* is characterized by the smallest value of ε, ε^*, such that $S(\varepsilon)$ is a string of zeros.

Proposition 3. *Local minima of non-flat elementary landscapes are bounded below by* $\alpha - \frac{\varepsilon^*}{\lambda}$, *where* λ *is the eigenvalue of the associated L.*

Proof. For a local minimum x_i^*, let $f_{\alpha i,\max}$ be the maximum value of f_α in $\mathcal{N}(x_i^*)$ and $\text{Avg}_{y\in\mathcal{N}(x_i^*)} f_\alpha(y)$ be the average value of f_α in $\mathcal{N}(x_i^*)$. Thus, $\left|f_\alpha(x_i^*) - f_{\alpha i,\max}\right| \le \varepsilon^*$.

Hence,

$$\operatorname*{Avg}_{y\in\mathcal{N}(x_i^*)} f_\alpha(y) \le f_{\alpha i,\max} \tag{11}$$

and

$$f_\alpha(x_i^*) - f_{\alpha i,\max} \le f_\alpha(x_i^*) - \operatorname*{Avg}_{y\in\mathcal{N}(x_i^*)} f_\alpha(y)$$

which implies

$$-\varepsilon^* \le f_\alpha(x_i^*) - f_{\alpha i,\max} \le f_\alpha(x_i^*) - \operatorname*{Avg}_{y\in\mathcal{N}(x_i^*)} f_\alpha(y) \tag{12}$$

and, therefore,

$$-\varepsilon^* \le f_\alpha(x_i^*) - T_i f_\alpha. \tag{13}$$

Since the landscape is elementary, $T_i f_\alpha = (1-\lambda) f_\alpha(x_i)$. Substitution into Eq. (13) yields $-\varepsilon^* \le f_\alpha(x_i^*) - (1-\lambda) f_\alpha(x_i^*)$, where $0 < \lambda \le 2$. Therefore $-\frac{\varepsilon^*}{\lambda} \le f_\alpha(x_i^*)$ which implies $\alpha - \frac{\varepsilon^*}{\lambda} \le f(x_i^*)$. □

Corollary 4. *Local maxima for elementary landscapes are bounded above by* $\alpha + \frac{\varepsilon^*}{\lambda}$.

Proof. For any solution x, let g_x be the minimum value of f_α over $N(x)$, and let $\bar{x} = T_x f_\alpha$ be the average of f_α over $N(x)$, where T_x is the T-row corresponding to x. Thus, $|f_\alpha(x) - g_x| \le \varepsilon^*$ and since $\bar{x} \ge g_x$, it follows that $f_\alpha(x) - \bar{x} \le f_\alpha(x) - g_x \le \varepsilon^*$. Thus,

$$\varepsilon^* \ge f_\alpha(x) - T_x f_\alpha = f_\alpha(x) - (1-\lambda) f_\alpha = \lambda f_\alpha(x) = \lambda(f(x) - \alpha)$$

and so $f(x) \le \alpha + \frac{\varepsilon^*}{\lambda}$ for all solutions x (and in particular, for any local maximum). □

4. Concluding remarks

In this work, we have described a general class of composite elementary landscapes and have presented the first non-trivial lower (upper) bounds on local minima (maxima) associated with elementary landscapes.

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