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Optimality and Duality for Multiobjective Fractional Programming Involving n-Set Functions*

Do Sang Kim, Cheong Lai Jo, and Gue Myung Lee

Department of Applied Mathematics, Pukyong National University, 599-1

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We consider a multiobjective fractional programming problem (MFP) involving vector-valued objective n-set functions in which their numerators are different from each other, but their denominators are the same. By using the concept of proper efficiency, we establish optimality conditions and duality relations for our problem (MFP) under convexity assumptions on objective and constrained functions. © 1998 Academic Press

1. INTRODUCTION

The general theory for optimizing set functions was first developed by Morris [14]. This type of problem arises in various areas and has many interesting applications in mathematics, engineering, and statistics, for example, in fluid flow, electrical insulator design, and optimal plasma confinement [1, 5]. Many results of Morris [14] are confined only to set functions of a single set. Corley [6] started to give the concepts of partial derivatives and derivatives of real-valued n-set functions. In [10–12] and [18], the optimality and duality results for vector-valued n-set functions are studied. For details, the readers may consult [2–4, 9, 13, 15–17, 19].

In particular, Jo, Kim, and Lee [8] considered a multiobjective fractional programming problem involving vector-valued objective n-set functions in which their denominators are different from each other, and they established duality theorems by using the concepts of efficiency.

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Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable and let $F: \Gamma^n \to R^p$, $G: \Gamma^n \to R$, and $H: \Gamma^n \to R^m$ be differentiable n-set functions.

In this paper, we consider the multiobjective fractional problem

$$\begin{aligned} &\text{Minimize } \left(\frac{F_1(S)}{G(S)}, \dots, \frac{F_p(S)}{G(S)} \right) \\ &\text{subject to } S = (S_1, \dots, S_n) \in \Gamma^n,\ H(S) \leq 0, \end{aligned} \tag{MFP}$$

where $G(S) > 0$ for all $S \in \Gamma^n$.

To optimize (MFP) is to find properly efficient solutions.

We notice that in (MFP) each denominator of the objective function is the same single-valued n-set function. Geoffrion [7] introduced the definition of the properly efficient solution in order to eliminate the efficient solutions causing unbounded trade-offs between objective functions.

Corresponding to (MFP), we consider the parametric multiobjective problems

$$\begin{aligned} &\text{Minimize } \left(F_1(S) - \lambda_1 G(S), \dots, F_p(S) - \lambda_p G(S) \right) \\ &\text{subject to } S = (S_1, \dots, S_n) \in \Gamma^n,\ H(S) \leq 0, \end{aligned} \tag{MP_λ}$$

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{p} u_i [F_i(S) - \lambda_i G(S)] \\ &\text{subject to } S = (S_1, \dots, S_n) \in \Gamma^n,\ H(S) \leq 0, \end{aligned} \tag{$(\text{MP}_\lambda)_u$}$$

where $u_i > 0$, $i = 1, 2, \dots, p$, and $\sum_{i=1}^{p} u_i = 1$ are fixed.

In this paper, we prove that (MFP) and (MP_λ) have equivalent properly efficient solutions. For (MFP), necessary and sufficient conditions for a feasible solution to be properly efficient are established. These results are used to characterize proper efficient solutions for (MFP) by associated parametric problems (MP_λ) and scalar problems $(\text{MP}_\lambda)_u$ under convexity assumptions. Moreover, we establish the Mond–Weir type dual problem (MFD) of the program (MFP). We prove the weak and strong duality theorems by using the concept of proper efficiency.

2. PRELIMINARIES

We give some definitions and results from [6, 18] which are used in our later results. We define a pseudometric d on Γ^n as

$$d(S, T) = \left\{ \sum_{i=1}^{n} [\mu(S_i \,\Delta T_i)]^2 \right\}^{1/2},$$

where $S = (S_1, \dots, S_n)$, $T = (T_1, \dots, T_n) \in \Gamma^n$, and $S_i \,\Delta T_i$ denote the symmetric difference for S_i and T_i. For $f \in L_1(X, \Gamma, \mu)$ and $S_i \in \Gamma$, the integral $\int_{S_i} f \, d\mu$ will be denoted by $\langle f, \chi_{S_i} \rangle$, where χ_{S_i} denotes the characteristic function of S_i.

DEFINITION 2.1. A set function $F\colon \Gamma \to R$ is said to be differentiable at $S \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$, the derivative of F at S, such that

$$F(T) = F(S) + \langle f, \chi_T - \chi_S \rangle + \mu(S \,\Delta T) E(S, T) \quad \text{for all } T \in \Gamma,$$

where $\lim_{\mu(S \,\Delta T) \to 0} E(S, T) = 0$.

We define the partial derivatives of n-set functions.

DEFINITION 2.2. Let $F\colon \Gamma^n \to R$ and $S \in \Gamma^n$. Then F is said to have the partial derivative with respect to T_i if the set function

$$H(T_i) \equiv F(S_1, \dots, S_{i-1}, T_i, S_{i+1}, \dots, S_n)$$

has derivative h_{S_i} at S_i.

In this case we define the ith partial derivative of F at S to be $F_S^i = H_{S_i}$. Using the partial derivative of the n-set function, we can define the derivative of the vector-valued n-set function.

DEFINITION 2.3 [18]. Let $F\colon \Gamma^n \to R^m$ and $S \in \Gamma^n$. Then F if said to be differentiable at S if the partial derivatives f_S^{ij}, $i = 1, 2, \dots, n$, of F_j exist for each $j \in \overline{m} = \{1, 2, \dots, m\}$ and satisfy

$$\begin{aligned} F(V) = F(S) &+ \left(\sum_{i=1}^{n} \langle f_S^{i1}, \chi_{V_i} - \chi_{S_i} \rangle, \dots, \sum_{i=1}^{n} \langle f_S^{im}, \chi_{V_i} - \chi_{S_i} \rangle \right) \\ &+ W_F(S, V) \end{aligned}$$

for all $V \in \Gamma^n$, where $(W_F(S, V))/(d(S, V)) \to 0$ as $d(S, V) \to 0$. If F is differentiable at each point S of Γ^n, we say that F is differentiable on Γ^n.

Throughout this paper, if $F\colon \Gamma^n \to R^p$, $G\colon \Gamma^n \to R$, and $H\colon \Gamma^n \to R^m$ are differentiable on Γ^n, we will denote the ith partial derivatives of F_j, G, and H_j at S by f_S^{ij}, g_S^i, and h_S^{ij}, respectively.

In the sequel we shall always denote the sets $\{1, 2, \dots, p\}$ and $\{1, 2,, \dots, m\}$ by $\overline{p}$ and $\overline{m}$, respectively.

The nonnegative orthant and the nonpositive orthant in R^p are denoted by $R_+^p = \{x \in R^p\colon\ x \geq 0\}$ and $R_-^p = \{x \in R^p\colon\ x \leq 0\}$, respectively. For a set E in R_+^p, the set of all interior points of E will be denoted by int E.

DEFINITION 2.4. A set function $F\colon \Gamma^n \to R$ is said to be convex if, for each $\lambda \in [0, 1]$ and $S, T \in \Gamma^n$,

$$\begin{aligned} &\overline{\lim_{k \to \infty}}\ F\big(S_1^k \cup T_1^k \cup (S_1 \cap T_1), \dots, S_n^k \cup T_n^k \cup (S_n \cap T_n)\big) \\ &\qquad \leq \lambda F(S) + (1 - \lambda) F(T) \end{aligned}$$

for any sequence of sets $S_i^k \subset S_i \setminus T_i$ and $T_i^k \subset T_i \setminus S_i$, $k = 1, 2, \dots,$ satisfying

$$\chi_{S_i^k} \overset{w*}{\to} \lambda \chi_{S_i \setminus T_i} \quad \text{and} \quad \chi_{T_i^k} \overset{w*}{\to} (1-\lambda)\chi_{T_i \setminus S_i}, \quad \text{for } i = 1, 2, \dots, n,$$

where $\overset{w*}{\to}$ stands for the w^*-convergence.

LEMMA 2.5 [6]. *Let $F\colon \Gamma^n \to R$ be a differentiable convex function. Then for each $S, T \in \Gamma^n$,*

$$F(S) \geq F(T) + \sum_{i=1}^{n} \langle f_T^i, \chi_{S_i} - \chi_{T_i} \rangle.$$

DEFINITION 2.6. A subset Φ of Γ^n is convex if for any $S, T \in \Phi$, $\lambda \in [0, 1]$ and sequences of sets $S_i^k \subset S_i \setminus T_i$ and $T_i^k \subset T_i \setminus S_i$, $k = 1, 2, \dots,$ satisfying $\chi_{S_i^k} \overset{w*}{\to} \lambda \chi_{S_i \setminus T_i}$ and $\chi_{T_i^k} \overset{w*}{\to} (1-\lambda)\chi_{T_i \setminus S_i}$, for $i = 1, 2, \dots, n$, there exists a subsequence $\{V_{k_n}\}$ of $\{V_k\}$ such that $V_{k_n} \in \Phi$ for all n, where $V_k = (S_1^k \cup T_1^k \cup (S_1 \cap T_1), \dots, S_n^k \cup T_n^k \cup (S_n \cap T_n))$.

DEFINITION 2.7. A feasible solution S^0 of (MFP) is a regular solution of (MP$_\lambda$) if there exists a feasible solution S for (MFP) such that $H_j(S^0) + \sum_{i=1}^n \langle h_{S^0}^{ij}, \chi_{S_i} + \chi_{S_i^0} \rangle < 0$, $j \in \overline{m}$.

3. OPTIMALITY

The vector minimum problem (MFP) is the problem of finding all (properly) efficient solutions.

DEFINITION 3.1. A feasible solution S^0 of (MFP) is an efficient solution of (MFP) if there is no other feasible S for (MFP) such that

$$\frac{F_i(S)}{G(S)} \leq \frac{F_i(S^0)}{G(S^0)} \quad \text{for all } i \in \overline{p}, \tag{1}$$

and

$$\frac{F_j(S)}{G(S)} < \frac{F_j(S^0)}{G(S^0)} \quad \text{for some } j \in \overline{p}. \tag{2}$$

By eliminating efficient solutions causing unbounded trade-off between objective functions, we can define the properly efficient solutions as follows.

DEFINITION 3.2 [7]. A feasible solution S^0 of (MFP) is a properly efficient solution of (MFP) if it is efficient and if there exists a scalar

$M > 0$ such that, for each i,

$$\frac{\left(F_i(S^0)\right)/\left(G(S^0)\right) - \left(F_i(S)\right)/\left(G(S)\right)}{\left(F_j(S)\right)/\left(G(S)\right) - \left(F_j(S^0)\right)/\left(G(S^0)\right)} \leq M$$

for some j such that $(F_j(S))/(G(S)) > (F_j(S^0))/(G(S^0))$ whenever S is feasible for (MFP) and

$$\frac{F_i(S)}{G(S)} < \frac{F_i(S^0)}{G(S^0)}.$$

An efficient point that is not properly efficient is said to be improperly efficient. Thus for S^0 to be improperly efficient means that for every scalar $M > 0$ (no matter how large) there is feasible point S and an i such that

$$\frac{F_i(S)}{G(S)} < \frac{F_i(S^0)}{G(S^0)} \quad \text{and} \quad \frac{\left(F_i(S^0)\right)/\left(G(S^0)\right) - \left(F_i(S)\right)/\left(G(S)\right)}{\left(F_j(S)\right)/\left(G(S)\right) - \left(F_j(S^0)\right)/\left(G(S^0)\right)} > M$$

for all j such that $(F_j(S))/(G(S)) > (F_j(S^0))/(G(S^0))$.

The following theorem connects (MFP) and (MP_λ).

THEOREM 3.3. *S^0 is a properly efficient solution of (MFP) if and only if S^0 is a properly efficient solution of $(\mathrm{MP}_{\lambda^0})$, where $\lambda_j^0 = (F_j(S^0))/(G(S^0))$ for all $j \in \bar{p}$.*

Proof. Let S^0 be a properly efficient solution of (MFP) and let

$$\lambda_j^0 = \frac{F_j(S^0)}{G(S^0)} \quad \text{for all } j \in \bar{p}. \tag{3}$$

If S^0 is not an efficient solution of $(\mathrm{MP}_{\lambda^0})$, then there exists a feasible solution S of $(\mathrm{MP}_{\lambda^0})$ such that

$$F_i(S) - \lambda_i^0 G(S) \leq F_i(S^0) - \lambda_i^0 G(S^0) \quad \text{for all } i \in \bar{p}$$

and

$$F_j(S) - \lambda_j^0 G(S) < F_j(S^0) - \lambda_j^0 G(S^0) \quad \text{for some } j \in \bar{p}.$$

It follows that

$$\frac{F_i(S)}{G(S)} \leq \frac{F_i(S^0)}{G(S^0)} \quad \text{for all } i \in \bar{p} \tag{4}$$

and

$$\frac{F_j(S)}{G(S)} < \frac{F_j(S^0)}{G(S^0)} \quad \text{for some } j \in \bar{p}, \tag{5}$$

contradicting the efficiency of S^0 in (MFP). Hence S^0 is an efficient solution of $(\mathrm{MP}_{\lambda^0})$.

Now we shall show S^0 is a properly efficient solution of $(\mathrm{MP}_{\lambda^0})$. If S^0 is not properly efficient for $(\mathrm{MP}_{\lambda^0})$, then, for every sufficiently large scalar $M > 0$, there is an $S \in \Gamma^n$ and an i such that

$$F_i(S) - \lambda_i^0 G(S) < 0 \tag{6}$$

and

$$\frac{\left[F_i(S^0) - \lambda_i^0 G(S^0) - F_i(S) + \lambda_i^0 G(S)\right]}{\left[F_j(S) - \lambda_j^0 G(S) - F_j(S^0) + \lambda_j^0 G(S^0)\right]} > M \tag{7}$$

for all j such that

$$F_j(S) - \lambda_j^0 G(S) > 0 \tag{8}$$

i.e.,

$$\frac{F_i(S)}{G(S)} < \frac{F_i(S^0)}{G(S^0)} \tag{6$'$}$$

$$\frac{-F_i(S) + \lambda_i^0 G(S)}{F_j(S) - \lambda_j^0 G(S)} > M \tag{7$'$}$$

for all j such that

$$\frac{F_j(S)}{G(S)} > \frac{F_j(S^0)}{G(S^0)}. \tag{8$'$}$$

Now (7$'$) can be rewritten as

$$\frac{\left(F_i(S^0)\right)/\left(G(S^0)\right) - \left(F_i(S)\right)/\left(G(S)\right)}{\left(F_j(S)\right)/\left(G(S)\right) - \left(F_j(S^0)\right)/\left(G(S^0)\right)} > M.$$

So (6$'$), (7$'$), and (8$'$) imply that S^0 is not properly efficient for (MFP). Hence S^0 is properly efficient for $(\mathrm{MP}_{\lambda^0})$.

Conversely, let S^0 be a properly efficient solution of $(\mathrm{MP}_{\lambda^0})$, where

$$\lambda_j^0 = \frac{F_j(S^0)}{G(S^0)}, \qquad j \in \bar{p}. \tag{3}$$

Then we shall show that S^0 is properly efficient for (MFP). If S^0 is not an efficient solution of (MFP), then there exists a feasible solution S for

(MFP) such that

$$\frac{F_i(S)}{G(S)} \leq \frac{F_i(S^0)}{G(S^0)} \quad \text{for all } i \in \bar{p} \tag{4}$$

and

$$\frac{F_j(S)}{G(S)} < \frac{F_j(S^0)}{G(S^0)} \quad \text{for some } j \in \bar{p}. \tag{5}$$

Now (3) together with (4) and (5) contradict the efficiency of S^0 in $(\mathrm{MP}_{\lambda^0})$. Thus S^0 is an efficient solution of (MFP).

Now we shall show that S^0 is properly efficient for (MFP). If S^0 is not properly efficient for (MFP), then for every sufficiently large $M > 0$, there is an $S \in \Gamma^n$ and an $i \in \bar{p}$ such that

$$\frac{F_i(S)}{G(S)} < \frac{F_i(S^0)}{G(S^0)} \tag{9}$$

and

$$\frac{\big(F_i(S^0)\big)/\big(G(S^0)\big) - \big(F_i(S)\big)/\big(G(S)\big)}{\big(F_j(S)\big)/\big(G(S)\big) - \big(F_j(S^0)\big)/\big(G(S^0)\big)} > M \tag{10}$$

for all j such that

$$\frac{F_j(S)}{G(S)} > \frac{F_j(S^0)}{G(S^0)} \tag{11}$$

i.e.,

$$F_i(S) - \lambda_i^0 G(S) < 0 \tag{9$'$}$$

and

$$\frac{-F_i(S) + \lambda_i^0 G(S)}{F_j(S) - \lambda_j^0 G(S)} > M \tag{10$'$}$$

for all j such that

$$F_j(S) - \lambda_j^0 G(S) > 0. \tag{11$'$}$$

So (9$'$), (10$'$), and (11$'$) imply that S^0 is not properly efficient for $(\mathrm{MP}_{\lambda^0})$. Hence S^0 is properly efficient in (MFP).

THEOREM 3.4. (a) *Let* $u \in \operatorname{int} R_+^p$ *be fixed. If* S^0 *is an optimal solution of* $(\mathrm{MP}_{\lambda_0})_u$, *where* $\lambda_j^0 = (F_j(S^0))/(G(S^0))$ *for all* $j \in \bar{p}$, *then* S^0 *is a properly efficient solution of* $(\mathrm{MP}_{\lambda^0})$.

(b) *Let* $F_i - \lambda_i^0 G$, $i \in \bar{p}$, *and* H_j, $j \in \overline{m}$, *be convex set functions, where* $\lambda_j^0 = (F_j(S^0))/(G(S^0))$ *for all* $j \in \bar{p}$. *If* S^0 *is a properly efficient solution of* $(\mathrm{MP}_{\lambda^0})$, *then* S^0 *is an optimal solution of* $(\mathrm{MP}_{\lambda^0})_u$ *for some* $u \in \operatorname{int} R_+^p$.

Proof. The proof of (a) follows from Theorem 3.1 [7]. For part (b), if S^0 is a properly efficient solution for $(\mathrm{MP}_{\lambda^0})$, there exists $M > 0$ such that for each $i \in \bar{p}$, the system

$$F_i(S^0) - \lambda_i^0 G(S^0) > F_i(S) - \lambda_i^0 G(S)$$

and

$$\begin{aligned} &F_i(S^0) - \lambda_i^0 G(S^0) - F_i(S) + \lambda_i^0 G(S) \\ &\qquad > M\big[F_j(S) - \lambda_j^0 G(S) - F_j(S^0) + \lambda_j^0 G(S^0)\big] \end{aligned}$$

for $j \neq i$ has no solution in Φ, where $\Phi = \{S \in \Gamma^n \mid H(S) \leq 0\}$. We can check that Φ is a convex subset of Γ^n. By Farkas–Minkowski theorem [4], for the ith system, there exist $u_j^i \geq 0$, $\sum_{j=1}^p u_j^i = 1$ such that

$$\begin{aligned} &u_i^i\big[F_i(S) - \lambda_i^0 G(S) - F_i(S^0) + \lambda_i^0 G(S^0)\big] \\ &\qquad + \sum_{j \neq i} u_j^i\big[F_i(S) - \lambda_i^0 G(S) - F_i(S^0) + \lambda_i^0 G(S^0) \\ &\qquad\qquad + M\big(F_j(S) - \lambda_j^0 G(S)\big) - M\big(F_j(S^0) - \lambda_j^0 G(S^0)\big)\big] \\ &\quad \geq 0 \quad \text{for all } S \in \Phi. \end{aligned}$$

Rearranging terms of this inequality and summing the p-inequalities, we have

$$\begin{aligned} &\sum_{j=1}^p \Big(1 + M \sum_{i \neq j} u_j^i\Big)\big[F_j(S) - \lambda_j^0 G(S)\big] \\ &\qquad \geq \sum_{j=1}^p \Big(1 + M \sum_{i \neq j} u_j^i\Big)\big[F_j(S^0) - \lambda_j^0 G(S)\big] \end{aligned}$$

for all $S \in \Phi$. Let $u = (u_1, \ldots, u_p)$, where $u_j = 1 + M \sum_{i \neq j} u_j^i$, $j \in \bar{p}$. Then S^0 is an optimal solution of $(\mathrm{MP}_{\lambda^0})_u$.

THEOREM 3.5. *Let* S^0 *be a regular properly efficient solution of* (*MFP*), *and let* $F_i - \lambda_i^0 G$, $i \in \bar{p}$, *and* H_j, $j \in \overline{m}$, *be convex set functions, where* $\lambda_j^0 = (F_j(S^0))/(G(S^0))$ *for all* $j \in \bar{p}$. *Then there exist* $u^0 \in \operatorname{int} R_+^p$, $\sum_{j=1}^p u_j^0$

$= 1$, *and* $v^0 \in R^m_+$ *such that*

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j^0 \Big(f_{S^0}^{ij} - \lambda_j^0 g_{S^0}^{i} \Big) + \sum_{j=1}^{m} v_j^0 h_{S^0}^{ij}, \chi_{S_i} - \chi_{S_i^0} \right\rangle \geq 0 \quad \textit{for any } S \in \Gamma^n,$$

$$\sum_{j=1}^{p} u_j^0 \big[F_j(S^0) - \lambda_j^0 G(S^0) \big] = 0,$$

and

$$\sum_{j=1}^{p} v_j^0 H_j(S^0) \geq 0.$$

Proof. By Theorem 3.3, S^0 is a properly efficient solution of $(\mathrm{MP}_{\lambda^0})$. By Theorem 3.4(b), there exists $u^0 \in \operatorname{int} R^p_+$ such that S^0 is an optimal solution of $(\mathrm{MP}_{\lambda^0})_{u^0}$. By Corollary 3.9 in [6], we can obtain the result.

THEOREM 3.6. *Suppose that there exists a feasible solution S^0 of (MFP) and there exist $u^0 \in \operatorname{int} R^p_+$, $\sum_{j=1}^{p} u_j^0 = 1$, $\lambda^0 \in R^p$, and $v^0 \in R^m_+$ such that*

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j^0 \Big(f_{S^0}^{ij} - \lambda_j^0 g_{S^0}^{i} \Big) + \sum_{j=1}^{m} v_j^0 h_{S^0}^{ij}, \chi_{S_i} - \chi_{S_i^0} \right\rangle \geq 0 \quad \textit{for any } S \in \Gamma^n,$$

$$u_j^0 \big[F_j(S^0) - \lambda_j^0 G(S^0) \big] = 0 \quad \textit{for } j \in \bar{p},$$

and

$$\sum_{j=1}^{p} v_j^0 H_j(S^0) \geq 0.$$

Further, assume that $F_i - \lambda_i^0 G$, $i \in \bar{p}$, and H_j, $j \in \overline{m}$, are convex.

Then S^0 is a properly efficient solution of (MFP).

Proof. Let S be an arbitrary feasible solution of $(\mathrm{MP}_{\lambda^0})_{u^0}$. By the convexity of $F_j - \lambda_j^0 G$, $j \in \bar{p}$, and H_j, $j \in \overline{m}$,

$$\begin{aligned}
&\sum_{j=1}^{p} u_j^0 \Big[\big(F_j(S) - \lambda_j^0 G(S) \big) - \big(F_j(S^0) - \lambda_j^0 G(S^0) \big) \Big] \\
&\quad \geq \sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j^0 \Big(f_{S^0}^{ij} - \lambda_j^0 g_{S^0}^{i} \Big), \chi_{S_i} - \chi_{S_i^0} \right\rangle \\
&\quad \geq - \sum_{i=1}^{n} \left\langle \sum_{j=1}^{m} v_j^0 h_{S^0}^{ij}, \chi_{S_i} - \chi_{S_i^0} \right\rangle \\
&\quad \geq - \sum_{j=1}^{m} v_j^0 \big[H_j(S) - H_j(S^0) \big] \geq 0.
\end{aligned}$$

Hence S^0 is an optimal solution of $(\mathrm{MP}_{\lambda^0})_{u^0}$. By Theorems 3.3 and 3.4, S^0 is a properly efficient solution of (MFP).

4. DUALITY

The dual of (MFP) is defined as

$$\text{Maximize} \left(\frac{F_1(T)}{G(T)}, \dots, \frac{F_p(T)}{G(T)} \right) \text{ subject to for any } B, \qquad (\mathrm{MFD})$$

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j \left(f_T^{ij} - \lambda_j^0 g_T^i \right) + \sum_{j=1}^{m} v_j h_T^{ij}, \chi_{B_i} - \chi_{T_i} \right\rangle \geq 0, \tag{12}$$

$$u_i \left[F_i(T) - \lambda_i^0 G(T) \right] = 0 \quad \text{for } i \in \bar{p}, \tag{13}$$

$$v^t H(T) \geq 0, \tag{14}$$

$$u \in \operatorname{int} R_+^p, \qquad \sum_{j=1}^{p} u_j = 1, \qquad \lambda^0 \in R^p, \quad \text{and} \quad v \in R_+^m. \tag{15}$$

Let $u \in \operatorname{int} R_+^p$ such that $\sum_{j=1}^p u_j = 1$ and $\lambda^0 \in R^p$.

$$\text{Maximize} \sum_{i=1}^{p} u_i \left[F_i(T) - \lambda_i^0 G(T) \right] \text{ subject to for any } B, \qquad (\mathrm{MD}_{\lambda^0})_u$$

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j \left(f_T^{ij} - \lambda_j^0 g_T^i \right) + \sum_{j=1}^{m} v_j h_T^{ij}, \chi_{B_i} - \chi_{T_i} \right\rangle \geq 0, \tag{12}$$

$$v^t H(T) \geq 0, \tag{14}$$

and

$$v \in R_+^m. \tag{15}$$

Before we prove duality between (MFP) and (MFD), we first give a sufficient condition for properly efficient solution in (MFD) in terms of solutions of $(\mathrm{MD}_{\lambda^0})_u$.

THEOREM 4.1. *If for fixed $\bar{u} \in \operatorname{int} R_+^p$, $(\bar{T}, \bar{v})$ solves the program $(MD_{\lambda^0})_{\bar{u}}$, $\lambda_j^0 = (F_j(\bar{T}))/(G(\bar{T}))$, then $(\bar{T}, \bar{u}, \bar{v})$ is a properly efficient solution of (MFD).*

Proof. First we show that $(\bar{T}, \bar{u}, \bar{v})$ is efficient for (MFD). Suppose to the contrary that $(\bar{T}, \bar{u}, \bar{v})$ is not efficient for (MFD). Then there exists a feasible (T, u, v) for (MFD) such that for some $i \in \bar{p}$,

$$\frac{F_i(T)}{G(T)} > \frac{F_i(\bar{T})}{G(\bar{T})} \quad \text{and} \quad \frac{F_j(T)}{G(T)} \geq \frac{F_j(\bar{T})}{G(\bar{T})} \quad \text{for all } j \in \bar{p}.$$

Now, $F_i(T) - \lambda_i^0 G(T) > 0$ for some $i \in \bar{p}$ and $F_j(T) - \lambda_j^0 G(T) \geq 0$ for all $j \in \bar{p}$.

Since $\bar{u}_i > 0$ for all $i \in \bar{p}$,

$$\sum_{i=1}^{p} \bar{u}_i \left[F_i(T) - \lambda_i^0 G(T) \right] > 0 = \sum_{i=1}^{p} \bar{u}_i \left[F_i(\overline{T}) - \lambda_i^0 G(\overline{T}) \right],$$

which contradicts optimality of $(\overline{T}, \bar{v})$ in $(\mathrm{MD}_{\lambda^0})_{\bar{u}}$. Hence, $(\overline{T}, \bar{u}, \bar{v})$ must be efficient for (MFD).

Now we show that $(\overline{T}, \bar{u}, \bar{v})$ is properly efficient for (MFD). Assume that $p \geq 2$ and let $M = (p-1)\max_{i,j}\{\bar{u}_j/\bar{u}_i\}$. Suppose, to the contrary, that for some criterion i and a feasible T for (MFD) we have

$$\frac{F_i(T)}{G(T)} - \frac{F_i(\overline{T})}{G(\overline{T})} > M\left[\frac{F_j(\overline{T})}{G(\overline{T})} - \frac{F_j(T)}{G(T)} \right]$$

for all j such that $(F_j(T))/(G(T)) < (F_j(\overline{T}))/(G(\overline{T}))$.

It follows directly that

$$\frac{F_i(T)}{G(T)} - \frac{F_i(\overline{T})}{G(\overline{T})} > \frac{(p-1)}{\bar{u}_i} \bar{u}_j \left[\frac{F_j(\overline{T})}{G(\overline{T})} - \frac{F_j(T)}{G(T)} \right] \quad \text{for all } j \neq i.$$

Multiplying through by $\bar{u}_i/(p-1)$ and summing over $j \neq i$ yields

$$\bar{u}_i \left[\frac{F_i(T)}{G(T)} - \frac{F_i(\overline{T})}{G(\overline{T})} \right] > \sum_{j \neq i} \bar{u}_j \left[\frac{F_j(\overline{T})}{G(\overline{T})} - \frac{F_j(T)}{G(T)} \right].$$

Rearranging terms of this inequality, we obtain

$$\sum_{i=1}^{p} \bar{u}_i \frac{F_i(T)}{G(T)} > \sum_{i=1}^{p} \bar{u}_i \frac{F_i(\overline{T})}{G(\overline{T})},$$

that is,

$$\sum_{i=1}^{p} \bar{u}_i \left[F_i(T) - \lambda_i^0 G(T) \right] > 0 = \sum_{i=1}^{p} \bar{u}_i \left[F_i(\overline{T}) - \lambda_i^0 G(\overline{T}) \right],$$

which contradicts the optimality of $(\overline{T}, \bar{v})$ in $(\mathrm{MD}_{\lambda^0})_{\bar{u}}$.

THEOREM 4.2 (Weak duality). *Let S be feasible in (MFP) and let (T, u, v) be feasible in (MFD). If also $F_i - \lambda_i^0 G$, $i \in \bar{p}$, is convex, and H_j, $j \in \overline{m}$, is convex at T, then* $\sum_{i=1}^{p} u_i(F_i(S) - \lambda_i^0 G(S)) \geq \sum_{i=1}^{p} u_i(F_i(T) - \lambda_i^0 G(T))$.

Proof. Since S is feasible in (MFP) and (T, u, v) is feasible in (MFD), $v^t H(S) - v^t H(T) \leq 0$, and since $v^t H$ is convex at T,

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{m} v_j h_T^{ij}, \chi_{S_i} - \chi_{T_i} \right\rangle \leq 0. \tag{16}$$

Now using (12) is feasible, (16) gives

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} u_j \left(f_T^{ij} - \lambda_j^0 g_T^i \right), \chi_{S_i} - \chi_{T_i} \right\rangle \geq 0,$$

and since $\sum_{i=1}^{p} u_i (F_i - \lambda_i^0 G)$ is convex at T,

$$\sum_{i=1}^{p} u_i \big(F_i(S) - \lambda_i^0 G(S) \big) \geq \sum_{i=1}^{p} u_i \big(F_i(T) - \lambda_i^0 G(T) \big).$$

THEOREM 4.3 (Strong duality). *Let $\bar{S}$ be a regular properly efficient solution of (MFP) and assume that $F_i - \lambda_i^0 G$, $i \in \bar{p}$, and H_j, $j \in \bar{m}$, are convex. Then there exists $\bar{u} \in \operatorname{int} R_+^p$, $\sum_{j=1}^{p} \bar{u}_j = 1$, and $\bar{v} \in R_+^m$ such that $(\bar{S}, \bar{u}, \bar{v})$ is feasible in (MFD) and properly efficient for (MFD).*

Proof. From Theorem 3.5, there exists $\bar{u} \in \operatorname{int} R_+^p$, $\sum_{j=1}^{p} \bar{u}_j = 1$, and $\bar{v} \in R_+^m$ such that $(\bar{S}, \bar{u}, v)$ is feasible in (MFD).

Now since for each feasible S in (MFP) and each feasible $(T, \bar{u}, v)$ in (MFD), by the weak duality theorem,

$$\bar{u}^t (F - \lambda^0 G)(S) \geq \bar{u}^t (F - \lambda^0 G)(T),$$

and since $\bar{S}$ is feasible in (MFP) and $(\bar{S}, \bar{u}, \bar{v})$ is feasible in (MFD), we obtain

$$\sum_{i=1}^{p} \bar{u}_i \big(F_i(T) - \lambda_i^0 G(T) \big) \leq \sum_{i=1}^{p} \bar{u}_i \big(F_i(\bar{S}) - \lambda_i^0 G(\bar{S}) \big)$$

for all feasible (T, v) in $(\mathrm{MD}_{\lambda^0})_{\bar{u}}$.

Hence $(\bar{S}, \bar{v})$ solves $(\mathrm{MD}_{\lambda^0})_{\bar{u}}$. Since $\bar{u} > 0$, the result now follow from Theorem 4.1.

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