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Sharp Gårding inequality on compact Lie groups

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Abstract

We establish the sharp Gårding inequality on compact Lie groups. The positivity condition is expressed in the non-commutative phase space in terms of the full matrix symbol, which is defined using the representations of the group. Applications are given to the L^2 and Sobolev boundedness of pseudo-differential operators.

Keywords: Pseudo-differential operators; Compact Lie groups; Microlocal analysis; Gårding inequality

1. Introduction

The sharp Gårding inequality on $\mathbb{R}^n$ is one of the most important tools of the microlocal analysis with numerous applications in the theory of partial differential equations. Improving on the original Gårding inequality in [6], Hörmander [7] showed that if $p \in S^m_{1,0}(\mathbb{R}^n)$ and $p(x,\xi) \geqslant 0$, then

$$\operatorname{Re}\big(p(x,D)u,u\big)_{L^2} \geqslant -C\|u\|^2_{H^{(m-1)/2}} \tag{1}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$. The scalar case was also later extended to matrix-valued operators by Lax and Nirenberg [11], Friedrichs [5] and Vaillancourt [22]. Further improvements on the

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lower bound in the scalar case were also obtained by Beals and Fefferman [1] and Fefferman and Phong [4].

Notably, the sharp Gårding inequality (1) requires the condition $p(x,\xi) \geqslant 0$ imposed on the full symbol. This is different from the original Gårding inequality for elliptic operators which can be readily extended to manifolds. The main difficulty in obtaining (1) in the setting of manifolds is that the full symbol of an operator cannot be invariantly defined via its localisations. While the standard localisation approach still yields the principal symbol and thus the standard Gårding inequality, it cannot be extended to produce an improvement of the type in (1). Nevertheless, for pseudo-differential operators $P \in \Psi^2(M)$ on a compact manifold M, under certain geometric restrictions on the characteristic variety of the principal symbol $p_2 \geqslant 0$ and certain hypothesis on p_1, Melin [13] and Hörmander [8] obtained a lower bound known as the Hörmander–Melin inequality. See also Taylor [20].

The aim of the present paper is to establish the lower bound (1) on any compact Lie group G, with the statement given in Theorem 2.1. On compact Lie groups, the non-commutative analogue of the phase space is $G \times \widehat{G}$, where $\widehat{G}$ is the unitary dual of G. We use a global quantization of operators on G consistently developed by the authors in [18] and [16]. For a continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ it produces a full matrix-valued symbol $\sigma_A(x,\xi)$ defined for $(x,[\xi]) \in G \times \widehat{G}$. Thus, in Theorem 2.1 we will show the lower bound (1) under the assumption that the full symbol satisfies $\sigma_A \geqslant 0$, i.e. when the matrices $\sigma_A(x,\xi)$ are positive for all $(x,[\xi]) \in G \times \widehat{G}$. In general, if a full symbol is positive in the phase space, the corresponding pseudo-differential operator does not have to be positive in the operator sense. However, it still has lower bounds like the one in (1). An important example is the group $\mathrm{SU}(2) \cong \mathbb{S}^3$, with the group operation (matrix product) in SU(2) corresponding to the quaternionic product in $\mathbb{S}^3$. Details of the global quantization have been worked out in [16,18].

We note that the standard Gårding inequality on compact Lie groups was derived in [2] using Langlands' results for semigroups on Lie groups [10], but no quantization yielding full symbols is required in this case because of the ellipticity assumed on the operator. The global quantization used in [18] and [16] will be briefly reviewed in Section 3. We note that it is different from the one considered by Taylor [21] because we work directly on the group without referring to the exponential mapping and the symbol classes on the Lie algebra.

We note that one of the assumptions for the Hörmander–Melin inequality to hold is the vanishing of the principal symbol $p_2 \geqslant 0$ on the set $\{p_2 = 0\}$ to exactly second order. Thus, for example, it does not apply to operators of the form $-\partial_X^2$ plus lower order terms, where ∂_X is the derivative with respect to a vector field X, unless $\dim G = 1$. For higher order operators, again, the operator $\partial_X^4 - \mathcal{L}_G$, with the bi-invariant Laplace operator $\mathcal{L}_G$, gives an example when the Hörmander–Melin inequality does not work while the full matrix-valued symbol is positive definite, so that Theorem 2.1 applies. The relaxation of the transversal ellipticity has been analysed recently by Mughetti, Parenti and Parmeggiani, and we refer to [14] for further details on this subject.

A usual proof of (1) in $\mathbb{R}^n$ relies on the Friedrichs symmetrisation of an operator done in the frequency variables (see [7,11,5,9,20]). This does not readily work in the setting of Lie groups because the unitary dual $\widehat{G}$ forms only a lattice which does not behave well enough for this type of arguments. Thereby our construction uses mollification in x-space, more resembling those used by Calderón [3] or Nagase [15] for the proof of the sharp Gårding inequality in $\mathbb{R}^n$. Other proofs, e.g. using the anti-Wick quantisation, are also available on $\mathbb{R}^n$, see [12] and references therein. We would also like to point out that the proof of Nagase [15] can be extended to prove (1) on the torus $\mathbb{T}^n$ under the assumption that the toroidal symbol $p(x,\xi)$ of the operator $P \in \Psi^m(\mathbb{T}^n)$

satisfies $p(x,\xi) \geqslant 0$ for all $x \in \mathbb{T}^n$ and $\xi \in \mathbb{Z}^n$. The toroidal quantization necessary for this proof was developed by the authors in [17] but we will not give such a proof here because such result is now included as a special case of Theorem 2.1 which covers the non-commutative groups as well. The system as well as (ρ,δ) versions of the sharp Gårding inequality will appear elsewhere.

The proof of Theorem 2.1 consists of approximating the operator A with non-negative symbol σ_A by a positive operator P. Although this approximation has a symbol of type $(1,1/2)$ and not of type $(1,0)$, it is enough to prove Theorem 2.1 due to additional cancellations in the error terms, ensured by the construction. We note that working with symbol classes of type $(1,1/2)$ is a genuine global feature of the proof and of our construction because the operators of such type cannot be defined in local coordinates.

As usual, for a compact Lie group G we denote by $\Psi^m(G)$ the Hörmander pseudo-differential operators on G, i.e. the class of operators which in all local coordinate charts give operators in $\Psi^m(\mathbb{R}^n)$. Operators in $\Psi^m(\mathbb{R}^n)$ are characterised by the symbols satisfying

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leqslant C\left(1+|\xi|\right)^{m-|\alpha|}$$

for all multi-indices α,β and all $x,\xi \in \mathbb{R}^n$. An operator in $\Psi^m(G)$ is called elliptic if all of its localisations are locally elliptic. Here and in the sequel we use the standard notation for the multi-indices $\alpha = (\alpha_1,\dots,\alpha_\mu) \in \mathbb{N}_0^\mu$, where μ may vary throughout the paper depending on the context.

The paper is organised as follows. In Section 2 we introduce the full matrix-valued symbols and state the sharp Gårding inequality in Theorem 2.1. We apply it in Corollaries 2.2 and 2.3 to the L^2 boundedness of pseudo-differential operators. In Section 3 we collect facts necessary for the proof, and develop an expansion of amplitudes of type (ρ,δ) required for our analysis. In Section 4 we approximate operators with positive symbols by positive operators and derive the error estimates.

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2. Sharp Gårding inequality

Let G be a compact Lie group of dimension n with the neutral element e. Its Lie algebra will be denoted by $\mathfrak{g}$. We now fix the necessary notation. Let $\widehat{G}$ denote the unitary dual of G, i.e. set of all equivalence classes of (continuous) irreducible unitary representations of G and let $\mathrm{Rep}(G)$ be the set of all such representations of G.

For $f \in C^\infty(G)$ and $\xi \in \mathrm{Rep}(G)$, let

$$\widehat{f}(\xi) = \int_G f(x)\xi(x)^*\,\mathrm{d}x$$

be the (global) Fourier transform of f, where integration is with respect to the normalised Haar measure on G. For an irreducible unitary representation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ we have the linear operator $\widehat{f}(\xi) : \mathcal{H}_\xi \to \mathcal{H}_\xi$. Denote by $\dim(\xi)$ the dimension of ξ, $\dim(\xi) = \dim \mathcal{H}_\xi$. If ξ is a matrix representation, we have $\widehat{f}(\xi) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$. Since G is compact, $\widehat{G}$ is discrete and all of its elements are finite dimensional. Consequently, by the Peter–Weyl theorem we have the Fourier inversion formula

$$f(x)=\sum_{[\xi]\in\widehat{G}}\dim(\xi)\operatorname{Tr}\big(\xi(x)\widehat{f}(\xi)\big).$$

The Parseval identity takes the form

$$\|f\|^2_{L^2(G)}=\sum_{[\xi]\in\widehat{G}}\dim(\xi)\big\|\widehat{f}(\xi)\big\|^2_{HS},$$

where $\|\widehat{f}(\xi)\|^2_{HS}=\operatorname{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^*)$, which gives the norm on $\ell^2(\widehat{G})$. For a linear continuous operator from $C^\infty(G)$ to $\mathcal{D}'(G)$ we introduce its *full matrix-valued symbol* $\sigma_A(x,\xi)\in\mathbb{C}^{\dim(\xi)\times\dim(\xi)}$ by

$$\sigma_A(x,\xi)=\xi(x)^*(A\xi)(x).$$

Then it was shown in [18] and [16] that

$$Af(x)=\sum_{[\xi]\in\widehat{G}}\dim(\xi)\operatorname{Tr}\big(\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)\big) \tag{2}$$

holds in the sense of distributions, and the sum is independent of the choice of a representation ξ from each class $[\xi]\in\widehat{G}$. Moreover, we have

$$\sigma_A(x,\xi)=\int_G R_A(x,y)\xi(y)^*\,\mathrm{d}y$$

in the sense of distributions, where R_A is the right-convolution kernel of A:

$$Af(x)=\int_G K(x,y)f(y)\,\mathrm{d}y=\int_G f(y)R_A\big(x,y^{-1}x\big)\,\mathrm{d}y.$$

Symbols σ_A can be viewed as mappings on $G\times\widehat{G}$: the symbol of a continuous linear operator $A:C^\infty(G)\to C^\infty(G)$ is a mapping

$$\sigma_A:G\times\operatorname{Rep}(G)\to\bigcup_{\xi\in\operatorname{Rep}(G)}\operatorname{End}(\mathcal{H}_\xi),$$

where $\sigma_A(x,\xi):\mathcal{H}_\xi\to\mathcal{H}_\xi$ is linear for every $x\in G$ and $\xi\in\operatorname{Rep}(G)$, see [16, Rem. 10.4.9], and $\operatorname{End}(\mathcal{H}_\xi)$ is the space of all linear mappings from $\mathcal{H}_\xi$ to $\mathcal{H}_\xi$. If $\eta\in[\xi]$, i.e. there is an intertwining isomorphism $U:\mathcal{H}_\eta\to\mathcal{H}_\xi$ such that $\eta(x)=U^{-1}\xi(x)U$, then $\sigma_A(x,\eta)=U^{-1}\sigma_A(x,\xi)U$. In this sense we may think that the symbol σ_A is defined on $G\times\widehat{G}$ instead of $G\times\operatorname{Rep}(G)$. For further details of these constructions and their properties we refer to [16].

A (possibly unbounded) linear operator P on a Hilbert space $\mathcal{H}$ is called *positive* if $\langle Pv, v\rangle_{\mathcal{H}} \geqslant 0$ for every $v \in V$ for a dense subset $V \subset \mathcal{H}$. A matrix $P \in \mathbb{C}^{n\times n}$ is called positive if the natural corresponding linear operator $\mathbb{C}^n \to \mathbb{C}^n$ is positive, where $\mathbb{C}^n$ has the standard inner product.

A matrix pseudo-differential symbol σ_A is called *positive* if the matrix $\sigma_A(x,\xi) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$ is positive for every $x \in G$ and $\xi \in \mathrm{Rep}(G)$. In this case we write $\sigma(x,\xi) \geqslant 0$. We note that for each $\xi \in \mathrm{Rep}(G)$, the condition $\sigma_A(x,\xi) \geqslant 0$ implies $\sigma_A(x,\eta) \geqslant 0$ for all $\eta \in [\xi]$. We can also note that this symbol positivity does not change if we move from left symbols to right symbols:

$$\sigma_A(x,\xi) := \xi^*(x)(A\xi)(x) = \xi(x)^*\rho_A(x,\xi)\xi(x),$$
$$\rho_A(x,\xi) := (A\xi)(x)\xi^*(x) = \xi(x)\sigma_A(x,\xi)\xi(x)^*;$$

that is, σ_A is positive if and only if ρ_A is positive. Moreover, this positivity concept is natural in the sense that a left- or right-invariant operator is positive if and only if its symbol is positive, as it can be seen from the equalities

$$\langle a * f, f\rangle_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\,\mathrm{Tr}\big(\widehat{f}(\xi)\widehat{a}(\xi)\widehat{f}(\xi)^*\big), \tag{3}$$

$$\langle f * a, f\rangle_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\,\mathrm{Tr}\big(\widehat{f}(\xi)^*\widehat{a}(\xi)\widehat{f}(\xi)\big), \tag{4}$$

which can be shown by a simple calculation which we give in Proposition 3.6. At the same time, the operator M_f of multiplication by a smooth function $f \in C^\infty(G)$ is positive if and only if the function satisfies $f(x) \geqslant 0$ for every $x \in G$. The symbol of such multiplication operator is $\sigma_{M_f}(x,\xi) = f(x)I_{\dim(\xi)}$, so that this means the positivity of the matrix symbol again.

Now we can formulate the main result of this paper:

Theorem 2.1. *Let $A \in \Psi^m(G)$ be such that its full matrix symbol σ_A satisfies $\sigma_A(x,\xi) \geqslant 0$ for all $(x,[\xi]) \in G \times \widehat{G}$. Then there exists $C < \infty$ such that*

$$\mathrm{Re}(Au,u)_{L^2(G)} \geqslant -C\|u\|^2_{H^{(m-1)/2}(G)}$$

for every $u \in C^\infty(G)$.

As a corollary of Theorem 2.1 we obtain the following statement on compact Lie groups, analogous to the corresponding result on $\mathbb{R}^n$, which is often necessary in the proofs of pseudo-differential inequalities (see e.g. Theorem 3.1 in [20]).

Corollary 2.2. *Let $A \in \Psi^1(G)$ be such that its matrix symbol σ_A satisfies*

$$\big\|\sigma_A(x,\xi)\big\|_{op} \leqslant C$$

for all $(x,[\xi]) \in G \times \widehat{G}$. Then A is bounded from $L^2(G)$ to $L^2(G)$.

Here $\|\cdot\|_{op}$ denotes the $\ell^2 \to \ell^2$ operator norm of the linear finite dimensional mapping (matrix multiplication by) $\sigma_A(x,\xi)$, i.e.

$$\left\|\sigma_A(x,\xi)\right\|_{op} = \sup\left\{\left\|\sigma_A(x,\xi)v\right\|_{\ell^2}\colon\ v \in \mathbb{C}^{\dim(\xi)},\ \|v\|_{\ell^2} \leqslant 1\right\}.$$

The weights for measuring the orders of symbols are expressed in terms of the eigenvalues of the bi-invariant Laplacian $\mathcal{L}_G$. Matrix elements of every representation class $[\xi] \in \widehat{G}$ span an eigenspace of the bi-invariant Laplace–Beltrami operator $\mathcal{L}_G$ on G with the corresponding eigenvalue $-\lambda_\xi^2$. Based on these eigenvalues we define

$$\langle\xi\rangle = \left(1 + \lambda_\xi^2\right)^{1/2}.$$

For further details and properties of these constructions we refer to [16]. In particular, for the usual Sobolev spaces, we have $f \in H^s(G)$ if and only if $\langle\xi\rangle^s \widehat{f}(\xi) \in \ell^2(\widehat{G})$. To fix the norm on $H^s(G)$ for the following statement, we can then set

$$\|f\|_{H^s(G)} := \left(\sum_{[\xi]\in\widehat{G}} \dim(\xi)\langle\xi\rangle^{2s}\operatorname{Tr}\left(\widehat{f}(\xi)^*\widehat{f}(\xi)\right)\right)^{1/2},$$

and we can write this also as $\|\langle\xi\rangle^s\widehat{f}(\xi)\|_{\ell^2(\widehat{G})}$. Also, we note that by [16, Lemma 10.9.1] (or by Theorem 3.1 below), if $A \in \Psi^m(G)$, then there is a constant $0 < M < \infty$ such that $\|\sigma_A(x,\xi)\|_{op} \leqslant M\langle\xi\rangle^m$ holds for all $x \in G$ and $[\xi] \in \widehat{G}$.

As another corollary of Theorem 2.1 we can get a norm-estimate for pseudo-differential operators on compact Lie groups:

Corollary 2.3. *Let $A \in \Psi^m(G)$ and let*

$$M = \sup_{(x,[\xi])\in G\times\widehat{G}} \left(\langle\xi\rangle^{-m}\left\|\sigma_A(x,\xi)\right\|_{op}\right).$$

Then for every $s \in \mathbb{R}$ there exists a constant $C > 0$ such that

$$\|Au\|^2_{H^s(G)} \leqslant M^2\|u\|^2_{H^{s+m}(G)} + C\|u\|^2_{H^{s+m-1/2}(G)}$$

for all $u \in C^\infty(G)$.

3. Preliminary constructions

In this section we collect and develop several ideas which will be used in the proof of Theorem 2.1. These include characterisations of the class $\Psi^m(G)$, the Leibniz formula, the amplitude operators on G, and some properties of even and odd functions.

3.1. On symbols and operators

First we collect several facts and definitions required for our proof. We now introduce the notation for the symbol classes on the group G and give a characterisation of classes $\Psi^m(G)$ in terms of the matrix-valued symbols. In this, we follow the notation of [19].

We say that Q_ξ is a *difference operator* of order k if it is given by

$$Q_\xi \widehat{f}(\xi) = \widehat{q_Q f}(\xi),$$

for a function $q = q_Q \in C^\infty(G)$ vanishing of order k at the identity $e \in G$, i.e., $(P_x q_Q)(e) = 0$ for all left-invariant differential operators $P_x \in \mathrm{Diff}^{k-1}(G)$ of order $k-1$. We denote the set of all difference operators of order k as $\mathrm{diff}^k(\widehat{G})$.

A collection of $\mu \geqslant n$ first order difference operators $\Delta_1, \ldots, \Delta_\mu \in \mathrm{diff}^1(\widehat{G})$ is called *admissible*, if the corresponding functions $q_1, \ldots, q_\mu \in C^\infty(G)$ satisfy $q_j(e) = 0$, $\mathrm{d}q_j(e) \neq 0$ for all $j = 1, \ldots, \mu$, and if $\operatorname{rank}(\mathrm{d}q_1(e), \ldots, \mathrm{d}q_\mu(e)) = n$. An admissible collection is called *strongly admissible* if $\bigcap_{j=1}^{\mu} \{x \in G\colon q_j(x) = 0\} = \{e\}$.

For a given admissible selection of difference operators on a compact Lie group G we use multi-index notation $\Delta_\xi^\alpha = \Delta_1^{\alpha_1} \cdots \Delta_\mu^{\alpha_\mu}$ and $q^\alpha(x) = q_1(x)^{\alpha_1} \cdots q_\mu(x)^{\alpha_\mu}$. Furthermore, there exist corresponding differential operators $\partial_x^{(\alpha)} \in \mathrm{Diff}^{|\alpha|}(G)$ such that Taylor's formula

$$f(x) = \sum_{|\alpha| \leqslant N-1} \frac{1}{\alpha!} q^\alpha(x) \partial_x^{(\alpha)} f(e) + \mathcal{O}\big(\mathrm{dist}(x,e)^N\big) \tag{5}$$

holds true for any smooth function $f \in C^\infty(G)$ and with $\mathrm{dist}(x,e)$ the geodesic distance from x to the identity element e. An explicit construction of operators $\partial_x^{(\alpha)}$ in terms of $q^\alpha(x)$ can be found in [16, Section 10.6]. In addition to these differential operators $\partial_x^{(\alpha)} \in \mathrm{Diff}^{|\alpha|}(G)$ we introduce operators ∂_x^α as follows. Let $\{\partial_{x_j}\}_{j=1}^n \subset \mathrm{Diff}^1(G)$ be a collection of left-invariant first order differential operators corresponding to some linearly independent family of the left-invariant vector fields on G. We denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. We note that in most estimates we can freely replace operators $\partial_x^{(\alpha)}$ by ∂_x^α and in the other way around since they can be expressed in terms of each other. For further details and properties of the introduced constructions we refer to [16].

We now record the characterisation of Hörmander's classes as it appeared in [19]:

Theorem 3.1. *Let A be a linear continuous operator from $C^\infty(G)$ to $\mathcal{D}'(G)$, and let $m \in \mathbb{R}$. Then the following statements are equivalent*:

(A) $A \in \Psi^m(G)$.

(B) *For every left-invariant differential operator $P_x \in \mathrm{Diff}^k(G)$ of order k and every difference operator $Q_\xi \in \mathrm{diff}^\ell(\widehat{G})$ of order ℓ there is the symbol estimate*

$$\big\| Q_\xi P_x \sigma_A(x,\xi) \big\|_{op} \leqslant C_{Q_\xi P_x} \langle \xi \rangle^{m-\ell}.$$

(C) *For an admissible collection $\Delta_1, \ldots, \Delta_\mu \in \mathrm{diff}^1(\widehat{G})$ we have*

$$\big\| \Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi) \big\|_{op} \leqslant C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}$$

for all multi-indices α, β. Moreover, $\operatorname{sing\,supp} R_A(x,\cdot) \subseteq \{e\}$.

(D) *For a strongly admissible collection* $\Delta_1, \ldots, \Delta_\mu \in \mathrm{diff}^1(\widehat{G})$ *we have*

$$\left\| \Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi) \right\|_{op} \leqslant C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}$$

for all multi-indices α, β.

The set of symbols σ_A satisfying either of conditions (B)–(D) will be denoted by $\mathscr{S}^m_{1,0}(G) = \mathscr{S}^m(G)$. We note that if conditions (C) or (D) hold for one admissible (strongly admissible, resp.) collection of first order difference operators, they automatically hold for all admissible (strongly admissible, resp.) collections.

For the purposes of this paper, we will also need larger classes of symbols which we now introduce. We will say that a matrix-valued symbol $\sigma_A(x,\xi)$ belongs to $\mathscr{S}^m_{\rho,\delta}(G)$ if it is smooth in x and if for a strongly admissible collection $\Delta_1, \ldots, \Delta_\mu \in \mathrm{diff}^1(\widehat{G})$ we have

$$\left\| \Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi) \right\|_{op} \leqslant C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{6}$$

for all multi-indices α, β, uniformly in $x \in G$ and $\xi \in \mathrm{Rep}(G)$.

Remark 3.2. As it was pointed out in [19], in Theorem 3.1 we still have the equivalence of conditions (B), (C), (D), also if we replace symbolic inequalities in Theorem 3.1 by inequalities of the form (6). Also in this setting, if conditions (C) or (D) hold for one admissible (strongly admissible, resp.) collection of first order difference operators, they automatically hold for all admissible (strongly admissible, resp.) collections.

We will also write $a \in \mathscr{S}^m_{\rho,\delta\#}(G)$ if for every multi-index β and for every $x_0 \in G$ we have $\partial_x^\beta a(x_0,\cdot) \in \mathscr{S}^{m+\delta|\beta|}_{\rho\#}(G)$, where for a multiplier $b = b(\xi)$ we write $b \in \mathscr{S}^\mu_{\rho\#}(G)$ if for every multi-index α there is a constant C_α such that

$$\left\| \Delta_\xi^\alpha b(\xi) \right\|_{op} \leqslant C_\alpha \langle \xi \rangle^{\mu-\rho|\alpha|}$$

holds for all $[\xi] \in \widehat{G}$. We record the following straightforward lemma that follows from the smoothness of symbols in x and the compactness of G:

Lemma 3.3. *We have* $a \in \mathscr{S}^m_{\rho,\delta}(G)$ *if and only if* $a \in \mathscr{S}^m_{\rho,\delta\#}(G)$.

Another tool which will be required for the proof is the finite version of the Leibniz formula which appeared in [19]. Given a continuous unitary matrix representation $\xi^0 = [\xi^0_{ij}]_{1\leqslant i,j\leqslant \ell} : G \to \mathbb{C}^{\ell\times\ell}$, $\ell = \dim(\xi^0)$, let $q(x) = \xi^0(x) - I$ (i.e. $q_{ij} = \xi^0_{ij} - \delta_{ij}$ with Kronecker's deltas δ_{ij}), and define

$$\mathbb{D}_{ij} \widehat{f}(\xi) := \widehat{q_{ij} f}(\xi).$$

In the previous notation, we could also write $\mathbb{D}_{ij} = \Delta_{q_{ij}}$. For a multi-index $\gamma \in \mathbb{N}_0^{\ell^2}$, we write $|\gamma| = \sum_{i,j=1}^{\ell} |\gamma_{ij}|$, and for higher order difference operators we write $\mathbb{D}^\gamma = \mathbb{D}_{11}^{\gamma_{11}} \mathbb{D}_{12}^{\gamma_{12}} \cdots \mathbb{D}_{\ell,\ell-1}^{\gamma_{\ell,\ell-1}} \mathbb{D}_{\ell\ell}^{\gamma_{\ell\ell}}$. In contrast to the asymptotic Leibniz rule [16, Thm. 10.7.12] for arbitrary difference operators, operators $\mathbb{D}$ satisfy the finite Leibniz formula:

Proposition 3.4. *For all $\gamma \in \mathbb{N}_0^{\ell^2}$ we have*

$$\mathbb{D}^\gamma(ab) = \sum_{|\varepsilon|,|\delta| \leqslant |\gamma| \leqslant |\varepsilon|+|\delta|} C_{\gamma\varepsilon\delta}\big(\mathbb{D}^\varepsilon a\big)\big(\mathbb{D}^\delta b\big),$$

with the summation taken over all $\varepsilon, \delta \in \mathbb{N}_0^{\ell^2}$ satisfying $|\varepsilon|, |\delta| \leqslant |\gamma| \leqslant |\varepsilon| + |\delta|$. In particular, for $|\gamma| = 1$, we have

$$\mathbb{D}_{ij}(ab) = (\mathbb{D}_{ij}a)b + a(\mathbb{D}_{ij}b) + \sum_{k=1}^{\ell}(\mathbb{D}_{ik}a)(\mathbb{D}_{kj}b). \tag{7}$$

Difference operators $\mathbb{D}$ lead to strongly admissible collections (see [19]):

Lemma 3.5. *The family of difference operators associated to the family of functions $\{q_{ij} = \xi_{ij} - \delta_{ij}\}_{[\xi]\in\widehat{G},\, 1\leqslant i,j\leqslant \dim(\xi)}$ is strongly admissible. Moreover, this family has a finite subfamily associated to finitely many representations which is still strongly admissible.*

We now give a simple proof of the equalities (3) and (4).

Proposition 3.6. *We have*

$$\langle a * f, f\rangle_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\big(\widehat{f}(\xi)\widehat{a}(\xi)\widehat{f}(\xi)^*\big),$$
$$\langle f * a, f\rangle_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\big(\widehat{f}(\xi)^*\widehat{a}(\xi)\widehat{f}(\xi)\big).$$

Proof. The second claimed equality follows from the following calculation:

$$\begin{aligned}
&\langle f * a, f\rangle_{L^2(G)} \\
&\quad= \int_G (f*a)(x)\overline{f(x)}\,\mathrm{d}x \\
&\quad= \int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\big(\xi(x)\widehat{a}(\xi)\widehat{f}(\xi)\big)\overline{\sum_{[\eta]\in\widehat{G}} \dim(\eta)\operatorname{Tr}\big(\eta(x)\widehat{f}(\eta)\big)}\,\mathrm{d}x \\
&\quad= \int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\sum_{k,l,m=1}^{\dim(\xi)} \xi(x)_{kl}\widehat{a}(\xi)_{lm}\widehat{f}(\xi)_{mk}\overline{\sum_{[\eta]\in\widehat{G}} \dim(\eta)\sum_{p,q=1}^{\dim(\eta)} \eta(x)_{pq}\widehat{f}(\eta)_{qp}}\,\mathrm{d}x \\
&\quad= \sum_{[\xi]\in\widehat{G}} \dim(\xi)\sum_{k,l,m=1}^{\dim(\xi)} \widehat{a}(\xi)_{lm}\widehat{f}(\xi)_{mk}\overline{\widehat{f}(\xi)_{lk}} \\
&\quad= \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\big(\widehat{a}(\xi)\widehat{f}(\xi)\widehat{f}(\xi)^*\big),
\end{aligned}$$

where we used the orthogonality of the matrix elements of the representations. The first claimed equality can be proven in an analogous way. □

We also record the Sobolev boundedness result that was Theorem 10.8.1 in [16]:

Theorem 3.7. *Let G be a compact Lie group. Let A be a continuous linear operator from $C^\infty(G)$ to $C^\infty(G)$ and let σ_A be its symbol. Assume that there exist constants $m, C_\alpha \in \mathbb{R}$ such that*

$$\left\| \partial_x^\alpha \sigma_A(x,\xi) \right\|_{op} \leqslant C_\alpha \langle \xi \rangle^m$$

holds for all $x \in G$, $\xi \in \mathrm{Rep}(G)$, and all multi-indices α. Then A extends to a bounded operator from $H^s(G)$ to $H^{s-m}(G)$ for all $s \in \mathbb{R}$.

3.2. Amplitudes on G

Let $0 \leqslant \delta, \rho \leqslant 1$. An amplitude $a \in \mathcal{A}^m_{\rho,\delta}(G)$ is a mapping defined on $G \times G \times \mathrm{Rep}(G)$, smooth in x and y, such that for an irreducible unitary representation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ we have[2] linear operators

$$a(x,y,\xi) : \mathcal{H}_\xi \to \mathcal{H}_\xi,$$

and for a strongly admissible collection of difference operators Δ_ξ^α the amplitude satisfies the *amplitude inequalities*

$$\left\| \Delta_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi) \right\|_{op} \leqslant C_{\alpha\beta\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta+\gamma|},$$

for all multi-indices α, β, γ and for all $(x, y, [\xi]) \in G \times G \times \widehat{G}$. For an amplitude a, the *amplitude operator* $\mathrm{Op}(a) : C^\infty(G) \to \mathcal{D}'(G)$ is defined by

$$\mathrm{Op}(a)u(x) := \sum_{[\eta] \in \widehat{G}} \dim(\eta)\, \mathrm{Tr}\left(\eta(x) \int_G a(x,y,\eta) u(y) \eta(y)^* \,\mathrm{d}y \right). \tag{8}$$

Notice that if here $a(x,y,\eta) = \sigma_A(x,\eta)$ then $\mathrm{Op}(a) = A$ as in (2). This definition can be justified as follows:

Proposition 3.8. *Let $0 \leqslant \delta < 1$ and $0 \leqslant \rho \leqslant 1$, and let $a \in \mathcal{A}^m_{\rho,\delta}(G)$. Then $\mathrm{Op}(a)$ is a continuous linear operator from $C^\infty(G)$ to $C^\infty(G)$.*

Proof. By the definition of $\langle \eta \rangle$ we have $(1 - \mathcal{L}_G)\eta(y) = \langle \eta \rangle^2 \eta(y)$. On the other hand, the Weyl spectral asymptotics formula for the Laplace operator $\mathcal{L}_G$ implies that $\langle \eta \rangle^{-1} \leqslant C \dim(\eta)^{-2/\dim(G)}$ (see Proposition 10.3.19 in [16]). Consequently, integrating by parts in the $\mathrm{d}y$-integral in (8) with operator $\langle \eta \rangle^{-2}(I - \mathcal{L}_G)$ arbitrarily many times, we see that the η-series in (8) converges, so that $\mathrm{Op}(a)u \in C^\infty(G)$ provided that $u \in C^\infty(G)$. The continuity of $\mathrm{Op}(a)$ on $C^\infty(G)$ follows by a similar argument. □

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[2] Especially, if ξ is a unitary matrix representation of dimension d, then $a(x,y,\xi) \in \mathbb{C}^{d \times d}$.

Remark 3.9. In the proof we used the inequality $\dim(\eta) \leqslant C\langle\eta\rangle^{n/2}$, $n = \dim G$, which easily follows from the Weyl spectral asymptotic formula (see Proposition 10.3.19 in [16]), and which is enough for the purposes of the proof. However, a stronger inequality $\dim(\eta) \leqslant C\langle\eta\rangle^{(n-l)/2}$ can be obtained from the Weyl character formula, with $l = \operatorname{rank} G$. For the details of this, see e.g. [23, (11), (12)].

Proposition 3.10. *Let $0 \leqslant \delta < \rho \leqslant 1$ and let $a \in \mathcal{A}^m_{\rho,\delta}(G)$. Then $A = \mathrm{Op}(a)$ is a pseudo-differential operator on G with a matrix symbol $\sigma_A \in \mathscr{S}^m_{\rho,\delta}(G)$. Moreover, σ_A has the asymptotic expansion*

$$\sigma_A(x,\xi) \sim \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \partial_y^{(\alpha)} \Delta_\xi^\alpha a(x,y,\xi)|_{y=x}.$$

Proof. If σ_A is the matrix symbol of the continuous linear operator $A = \mathrm{Op}(a) : C^\infty(G) \to C^\infty(G)$, we can find it from the formula $\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x)$. By fixing some basis in the representation spaces, we have

$$\begin{aligned}
\sigma_A(x,\xi)_{mn} &= \sum_{l=1}^{\dim(\xi)} \xi\big(x^{-1}\big)_{ml} (A\xi_{ln})(x) \\
&= \sum_{l=1}^{\dim(\xi)} \xi\big(x^{-1}\big)_{ml} \int_G \sum_{[\eta]\in\widehat{G}} \dim(\eta) \operatorname{Tr}\big(\eta(x) a(x,y,\eta) \xi(y)_{ln} \eta(y)^*\big)\, dy \\
&= \int_G \xi\big(x^{-1}y\big)_{mn} \sum_{[\eta]\in\widehat{G}} \dim(\eta) \operatorname{Tr}\big(\eta\big(y^{-1}x\big) a(x,y,\eta)\big)\, dy \\
&= \int_G \xi\big(x^{-1}y\big)_{mn} \sum_{[\eta]\in\widehat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\eta)} \eta\big(y^{-1}x\big)_{jk} a(x,y,\eta)_{kj}\, dy \\
&= \int_G \xi\big(z^{-1}\big)_{mn} \sum_{[\eta]\in\widehat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\eta)} \eta(z)_{jk} a\big(x, xz^{-1}, \eta\big)_{kj}\, dz \\
&\sim \sum_{\alpha\geqslant 0} \frac{1}{\alpha!} \partial_u^{(\alpha)} \sum_{[\eta]\in\widehat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\eta)} a(x,u,\eta)_{kj}|_{u=x} \int_G \xi\big(z^{-1}\big)_{mn} \eta(z)_{jk} q_\alpha(z)\, dz,
\end{aligned}$$

by the Taylor expansion (5). Using difference operators $\Delta_\xi^\alpha \widehat{s}(\xi) := \widehat{q_\alpha s}(\xi)$, we find

$$\begin{aligned}
&\sum_{[\eta]\in\widehat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\eta)} a(x,u,\eta)_{kj} \int_G \xi\big(z^{-1}\big) \eta(z)_{jk} q_\alpha(z)\, dz \\
&\quad = \int_G \xi(z)^* q_\alpha(z) \sum_{[\eta]\in\widehat{G}} \dim(\eta) \operatorname{Tr}\big(\eta(z) a(x,u,\eta)\big)\, dz = \Delta_\xi^\alpha a(x,u,\xi).
\end{aligned}$$

Thus

$$\sigma_A(x,\xi) \sim \sum_{\alpha\geqslant 0} \frac{1}{\alpha!} \partial_u^{(\alpha)} \int_G \xi(z)^* q_\alpha(z) \sum_{[\eta]\in\widehat{G}} \dim(\eta)\operatorname{Tr}\big(\eta(z)a(x,u,\eta)\big)\,\mathrm{d}z\big|_{u=x}$$
$$= \sum_{\alpha\geqslant 0} \frac{1}{\alpha!} \partial_u^{(\alpha)} \Delta_\xi^\alpha a(x,u,\xi)|_{u=x}.$$

The remainder in this asymptotic expansion can be dealt with in a way similar to the argument for the composition formulae (see [16]), so we omit the proof. □

3.3. Properties of even and odd functions

On a group G, function $f : G \to \mathbb{C}$ is called *even* if it is inversion-invariant, i.e. if $f(x^{-1}) = f(x)$ for every $x \in G$. Function $f : G \to \mathbb{C}$ is called *odd* if $f(x^{-1}) = -f(x)$ for every $x \in G$. Recall that $f : G \to \mathbb{C}$ is *central* if $f(xy) = f(yx)$ for all $x, y \in G$. Linear combinations of characters $\chi_\xi = (x \mapsto \operatorname{Tr}(\xi(x)))$ of irreducible unitary representations ξ of a compact group G are central, and such linear combinations are dense among the central functions of $C(G)$. When G is a compact Lie group, for $Y \in \mathfrak{g}$ and $f \in C^\infty(G)$ we define

$$L_Y f(x) := \frac{\mathrm{d}}{\mathrm{d}t} f\big(x\exp(tY)\big)\big|_{t=0}, \qquad R_Y f(x) := \frac{\mathrm{d}}{\mathrm{d}t} f\big(\exp(tY)x\big)\big|_{t=0},$$

so that L_Y, R_Y are the first order differential operators, L_Y being left-invariant and R_Y right-invariant. For a central function f we have $L_Y f = R_Y f$, which would not be true for an arbitrary smooth function f. Moreover, if f is even and central then

$$L_Y f\big(x^{-1}\big) = -L_Y f(x),$$

i.e. $L_Y f$ is odd in this case. Similarly $L_Y f$ is even for odd central functions f, but $L_Y f$ does not have to be central. More precisely, for central $f \in C^\infty(G)$ we obtain

$$L_Y f\big(u^{-1}xu\big) = L_{uYu^{-1}} f(x),$$

where $u \in G$. For higher order derivatives of even and odd functions, taking the differential of

$$f\big(x\exp(t_1X_1)\cdots\exp(t_kX_k)\big) = \pm f\big(x^{-1}\exp\big((-t_k)X_k\big)\cdots\exp\big((-t_1)X_1\big)\big)$$

at $t_1 = \cdots = t_k = 0$, we obtain:

Proposition 3.11. *Let $f \in C^\infty(G)$ be even and central, and $X_1, \ldots, X_k \in \mathfrak{g}$. Then*

$$L_{X_1}L_{X_2}\cdots L_{X_{k-1}}L_{X_k} f\big(x^{-1}\big) = (-1)^k L_{X_k}L_{X_{k-1}}\cdots L_{X_2}L_{X_1} f(x).$$

Similarly, if $f \in C^\infty(G)$ is an odd central function, then we have the equality

$$L_{X_1}L_{X_2}\cdots L_{X_{k-1}}L_{X_k} f\big(x^{-1}\big) = (-1)^{k+1} L_{X_k}L_{X_{k-1}}\cdots L_{X_2}L_{X_1} f(x).$$

4. Proof of the sharp Gårding inequality

We notice that if a linear operator $Q : H^{(m-1)/2}(G) \to H^{-(m-1)/2}(G)$ is bounded then

$$\begin{aligned}\operatorname{Re}(Qu,u)_{L^2} &\geqslant -\left|(Qu,u)_{L^2}\right| \\ &\geqslant -\|Qu\|_{H^{-(m-1)/2}}\|u\|_{H^{(m-1)/2}} \\ &\geqslant -\|Q\|_{\mathcal{L}(H^{(m-1)/2},H^{-(m-1)/2})}\|u\|^2_{H^{(m-1)/2}}.\end{aligned}$$

Hence Theorem 2.1 would follow if we could show that $A = P + Q$, where P is positive (on $C^\infty(G) \subset L^2(G)$) and $Q : H^{(m-1)/2}(G) \to H^{-(m-1)/2}(G)$ is bounded. The proof of this decomposition will be done in several steps.

4.1. Construction of w_ξ

First, we construct an auxiliary function w_ξ which will play a crucial role for our proof.

We can treat G as a closed subgroup of $\mathrm{GL}(N,\mathbb{R}) \subset \mathbb{R}^{N\times N}$ for some $N \in \mathbb{N}$. Then its Lie algebra $\mathfrak{g} \subset \mathbb{R}^{N\times N}$ is an n-dimensional vector subspace (hence identifiable with $\mathbb{R}^n$) such that $[A,B] := AB - BA \in \mathfrak{g}$ for every $A,B \in \mathfrak{g}$. Let $U \subset G$ be a neighbourhood of the neutral element $e \in G$, and let $V \subset \mathfrak{g}$ be a neighbourhood of $0 \in \mathfrak{g} \cong \mathbb{R}^n$, such that the matrix exponential mapping is a diffeomorphism $\exp : V \to U$.

For the construction and for the notation only in Section 4.1, we define the central norm $|\cdot|$ on $\mathfrak{g}$ as follows.[3] Take the Euclidean norm $|\cdot|_0$ on $\mathfrak{g}$ and define

$$|X| = \int_G \left|uXu^{-1}\right|_0 \mathrm{d}u, \tag{9}$$

where we may view the product under the integral as the product of matrices in $\mathbb{R}^{N\times N}$. Then by definition the norm (9) is invariant by the adjoint representation, and we have, in particular $|\exp^{-1}(xy)| = |\exp^{-1}(yx)|$, etc.

We may assume that V is the open ball $V = \mathbb{B}(0,r) = \{Z \in \mathbb{R}^n\colon |Z| < r\}$ of radius $r > 0$. Let $\phi : [0,r) \to [0,\infty)$ be a smooth function such that $(Z \mapsto \phi(|Z|)) : \mathfrak{g} \to \mathbb{R}$ is supported in V and $\phi(s) = 1$ for small $s > 0$. For every $\xi \in \mathrm{Rep}(G)$ we define

$$w_\xi(x) := \phi\left(\left|\exp^{-1}(x)\right|\langle\xi\rangle^{1/2}\right)\psi\left(\exp^{-1}(x)\right)\langle\xi\rangle^{n/4}, \tag{10}$$

where

$$\psi(Y) = C_0\left|\det D\exp(Y)\right|^{-1/2} f(Y)^{-1/2},$$

$D\exp$ is the Jacobi matrix of exp, $f(Y)$ is the density with respect to the Lebesgue measure of the Haar measure on G pulled back to $\mathfrak{g} \cong \mathbb{R}^n$ by the exponential mapping, and with constant $C_0 = (\int_{\mathbb{R}^n} \phi(|Z|)^2\,\mathrm{d}Z)^{-1/2}$. By $I_{\dim(\xi)}$ we denote the identity mapping on $\mathbb{C}^{\dim(\xi)}$. For $x, y \in G$ close to each other, $\mathrm{dist}(x,y)$ is the geodesic distance between x and y.

[3] In fact, any central norm $|\cdot|$ on $\mathfrak{g}$ will work.

Lemma 4.1. *We have $w_\xi \in C^\infty(G)$, $w_\xi(e) = C_0\langle\xi\rangle^{n/4}$, w_ξ is central and inversion-invariant, i.e. $w_\xi(xy) = w_\xi(yx)$ and $w_\xi(x^{-1}) = w_\xi(x)$ for every $x, y \in G$. Also,* $\operatorname{dist}(x,e) \approx |\exp^{-1}(x)| \leqslant r\langle\xi\rangle^{-1/2}$ *on the support of w_ξ. Moreover, $\|w_\xi\|_{L^2(G)} = 1$ for all $\xi \in \operatorname{Rep}(G)$. Finally, we have $((x,\xi) \mapsto w_\xi(x) I_{\dim(\xi)}) \in \mathscr{S}^{n/4}_{1,1/2}(G)$.*

Proof. It is easy to see that $w_\xi \in C^\infty(G)$, $w_\xi(e) = C_0\langle\xi\rangle^{n/4}$, and that w_ξ is inversion-invariant. Clearly $\operatorname{dist}(x,e) \approx |\exp^{-1}(x)| \leqslant r\langle\xi\rangle^{-1/2}$ on the support of w_ξ in view of properties of the function ϕ. In particular, (10) is well defined and $\operatorname{supp} w_\xi \subset U$. From (9) it also follows that w_ξ is central since f is invariant under adjoint representation as a density of two bi-invariant measures.

Let us now show that $\|w_\xi\|_{L^2(G)} = 1$ for all $\xi \in \operatorname{Rep}(G)$. Indeed,

$$\int_G \bigl|w_\xi(x)\bigr|^2\,\mathrm{d}x = \langle\xi\rangle^{n/2}\int_{\mathbb{R}^n} \phi\bigl(|Y|\langle\xi\rangle^{1/2}\bigr)^2\bigl|\psi(Y)\bigr|^2\bigl|\det D\exp(Y)\bigr| f(Y)\,\mathrm{d}Y$$

$$= C_0^2\int_{\mathbb{R}^n}\phi\bigl(|Z|\bigr)^2\,\mathrm{d}Z,$$

so that $\|w_\xi\|_{L^2(G)} = 1$ in view of the choice of the constant C_0. Thus, the main thing is to check that $w_\xi I_{\dim(\xi)} \in \mathscr{S}^{n/4}_{1,1/2}(G)$. By Lemma 3.3, we need to check that for every multi-index β and every $x_0 \in G$ we have $\partial_x^\beta w_\xi(x_0) \in \mathscr{S}^{n/4+|\beta|/2}_{1\#}(G)$. We observe that the x-derivatives of w_ξ are sums of terms of the form

$$\chi\bigl(\exp^{-1}(x)\bigr)\widetilde{\phi}\bigl(\bigl|\exp^{-1}(x)\bigr|\langle\xi\rangle^{1/2}\bigr)\langle\xi\rangle^{n/4+l/2} I_{\dim(\xi)}, \tag{11}$$

where $\chi \in C_0^\infty(V)$, $\widetilde{\phi} \in C_0^\infty(\mathbb{R})$, $\widetilde{\phi}$ is constant near the origin, and l is an integer such that $0 \leqslant l \leqslant |\beta|$. We note that $\langle\xi\rangle^{n/4+l/2} I_{\dim(\xi)}$ is the symbol of the pseudo-differential operator $(1-\mathcal{L}_G)^{n/8+l/4}$, and hence $\langle\xi\rangle^{n/4+l/2} I_{\dim(\xi)} \in \mathscr{S}^{n/4+l/2}_{1\#} \subset \mathscr{S}^{n/4+|\beta|/2}_{1\#}$. Moreover, we can eliminate it from the formulae by the composition formulae for the matrix-valued symbols (see [16, Thm. 10.7.9]). Thus we have to check that for every $x_0 \in G$, the other terms in (11) fixed at $x = x_0$ are in $\mathscr{S}^0_{1\#}(G)$, i.e. that

$$\widetilde{\phi}\bigl(\bigl|\exp^{-1}(x_0)\bigr|\langle\xi\rangle^{1/2}\bigr) I_{\dim(\xi)} \in \mathscr{S}^0_{1\#}(G). \tag{12}$$

If $\exp^{-1}(x_0) = 0$, then this symbol is a constant times the identity $I_{\dim(\xi)}$ and hence it is in $\mathscr{S}^0_{1\#}(G)$. On the other hand, if $\exp^{-1}(x_0) \neq 0$, then the symbol (12) is compactly supported in ξ, and hence defines a smoothing operator. Indeed, in this case it has decay of any order in $\langle\xi\rangle$, together with all difference operators applied to it, with constants depending on x_0, so it is smoothing by Theorem 3.1.

Let us also give an alternative argument relating this operator to a corresponding operators on $\mathfrak{g}$. Writing $\varphi_v(t) := \widetilde{\phi}(|v|t)$ and using the characterisation of pseudo-differential operators in Theorem 3.1, we notice that (12) holds if for all $x_0 \in G$, the operators $\varphi_{\exp^{-1}(x_0)}((I-\mathcal{L}_G)^{1/4})$ belong to $\Psi^0(G)$. Looking at these operators locally near every point $x \in G$ and introducing

$\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\theta \circ \exp_x^{-1}$ is supported in a small neighbourhood near x, with $\exp_x := (Z \mapsto x\exp(Z)) : \mathfrak{g} \to G$ the exponential mapping centred at x, we have to show that

$$\theta(y)\varphi_v(B) \in \Psi^0\big(\mathbb{R}^n\big) \tag{13}$$

holds locally on the support of θ, for all $v = \exp^{-1}(x_0)$, where operator B is the pullback by $\exp_x$ of the operator $(I - \mathcal{L}_G)^{1/4}$ near x. In particular, we have $B \in \Psi_{1,0}^{1/2}(\mathbb{R}^n)$, B is elliptic on the support of θ, and its symbol is real-valued.

We now observe that if $v = 0$, then the operator in (13) is the multiplication operator by a smooth function, so that (13) is true in this case. If $v \neq 0$, we can show that the operator in (13) is actually a smoothing operator, so that (13) is also true. Here $\varphi_v \in C_0^\infty(\mathbb{R})$ since $v \neq 0$. We denote $D_t = \frac{1}{\mathrm{i}2\pi}\partial_t$. Let $f \in L^2(\mathbb{R}^n)$ be compactly supported, and let $u = u(t,x)$ be the solution to the Cauchy problem

$$D_t u = Bu, \qquad u(0,\cdot) = f.$$

We can write $u(t,\cdot) = \mathrm{e}^{\mathrm{i}2\pi tB} f$ and we have $u(t,\cdot) \in L^2(\mathbb{R}^n)$. Consequently,

$$\varphi_v(B)f = \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}2\pi tB} f\big)\widehat{\varphi_v}(t)\,\mathrm{d}t = \int_{\mathbb{R}} B^{-k}u(t,\cdot)D_t^k\widehat{\varphi_v}(t)\,\mathrm{d}t,$$

where we integrated by parts k times using the relation $u = B^{-1}D_t u$, and where we can localise to a neighbourhood of a point x at each step. Consequently, we obtain that $\varphi_v(B)f \in H_{loc}^{k/2}(\mathbb{R}^n)$ for all $k \in \mathbb{Z}^+$, so that actually $\varphi_v(B)f \in C^\infty(\mathbb{R}^n)$. Thus, the operator $\varphi_v(B)$ is smoothing and (13) holds also for $v \neq 0$. □

4.2. Auxiliary positive operator P

We now introduce a positive operator P which will be important for the proof of the sharp Gårding inequality. This operator P will give a positive approximation to our operator A.

Proposition 4.2. *Let $\sigma_A \in \mathscr{S}_{1,0}^m(G)$. Let us define an amplitude p by*

$$p(x,y,\xi) := \int_G w_\xi\big(xz^{-1}\big)w_\xi\big(yz^{-1}\big)\sigma_A(z,\xi)\,\mathrm{d}z, \tag{14}$$

where $w_\xi \in C^\infty(G)$ is as in (10)*. Let the amplitude operator $P = \mathrm{Op}(p)$ be given by*

$$Pu(x) = \int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\,\mathrm{Tr}\big(\xi\big(y^{-1}x\big)p(x,y,\xi)\big)u(y)\,\mathrm{d}y.$$

Then $p \in \mathcal{A}_{1,1/2}^m(G)$ and the operator P is positive.

Proof. We observe that

$$\left\|p(x,y,\xi)\right\|_{op} \leqslant \int_G \left|w_\xi\left(xz^{-1}\right)w_\xi\left(yz^{-1}\right)\right| \mathrm{d}z \Big(\sup_{z\in G}\left\|\sigma_A(z,\xi)\right\|_{op}\Big) \leqslant C\langle\xi\rangle^m$$

because $\|w_\xi\|^2_{L^2(G)} = 1$ by Lemma 4.1. Then $p \in \mathcal{A}^m_{1,1/2}(G)$ follows from Lemma 4.1 and the Leibniz formula in Proposition 3.4 by an argument similar to the one which will be given in the proof of Lemma 4.4, so we omit it. Let $(e_k)_{k=1}^\ell$ be an orthonormal basis for $\mathbb{C}^\ell$. For matrices $M, Q \in \mathbb{C}^{\ell\times\ell}$, where Q is positive, we have

$$\operatorname{Tr}\left(M^*QM\right) = \sum_{k=1}^{\ell}\left\langle M^*QMe_k, e_k\right\rangle_{\mathbb{C}^\ell} = \sum_{k=1}^{\ell}\langle QMe_k, Me_k\rangle_{\mathbb{C}^\ell} \geqslant 0. \tag{15}$$

Let us denote

$$M(z,\xi) := \int_G w_\xi\left(yz^{-1}\right)\xi\left(yz^{-1}\right)^* u(y)\,\mathrm{d}y.$$

We can now show that the operator P is positive:

$$\begin{aligned}
&\langle Pu, u\rangle_{L^2(G)} \\
&\quad = \int_G Pu(x)\overline{u(x)}\,\mathrm{d}x \\
&\quad = \int_G\int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\left(\xi(x)p(x,y,\xi)u(y)\xi(y)^*\right)\mathrm{d}y\,\overline{u(x)}\,\mathrm{d}x \\
&\quad = \int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\int_G \operatorname{Tr}\left(\xi(x)\int_G w_\xi\left(xz^{-1}\right)w_\xi\left(yz^{-1}\right)\sigma_A(z,\xi)\,\mathrm{d}z\,u(y)\xi(y)^*\,\mathrm{d}y\right)\overline{u(x)}\,\mathrm{d}x \\
&\quad = \int_G \sum_{[\xi]\in\widehat{G}} \dim(\xi)\operatorname{Tr}\left(M(z,\xi)^*\sigma_A(z,\xi)M(z,\xi)\right)\mathrm{d}z,
\end{aligned}$$

which is non-negative because of (15). □

4.3. *The difference* $p(x,x,\xi) - \sigma_A(x,\xi)$

In the earlier notation, we show here that $p(x,x,\xi) - \sigma_A(x,\xi)$ is a symbol of a bounded operator from $H^s(G)$ to $H^{s-(m-1)}(G)$.

Lemma 4.3. *Let $s \in \mathbb{R}$. Then the pseudo-differential operator with the symbol $p(x,x,\xi) - \sigma_A(x,\xi)$ is bounded from $H^s(G)$ to $H^{s-(m-1)}(G)$.*

Proof. By Theorem 3.7 it is enough to show that

$$\left\| \partial_x^\beta \big(p(x,x,\xi) - \sigma_A(x,\xi) \big) \right\|_{op} \leqslant C_\beta \langle \xi \rangle^{m-1}$$

holds for every multi-index β. By Lemma 4.1 we have

$$\partial_x^\beta \big(p(x,x,\xi) - \sigma_A(x,\xi) \big) = \int_G w_\xi(z)^2 \big(\partial_x^\beta \sigma_A\big(xz^{-1},\xi\big) - \partial_x^\beta \sigma_A(x,\xi) \big)\, \mathrm{d}z.$$

We notice that $\mathrm{dist}(z,e) \leqslant C\langle \xi \rangle^{-1/2}$ on the support of w_ξ, and we can use the Taylor expansion of $\partial_x^\beta \sigma_A(xz^{-1},\xi)$ at x to get

$$\partial_x^\beta \sigma_A\big(xz^{-1},\xi\big) = \partial_x^\beta \sigma_A(x,\xi) + \sum_{|\gamma|=1} \partial_x^{(\gamma)} \partial_x^\beta \sigma_A(x,\xi) q_\gamma(z) + O\big(\mathrm{dist}(z,e)^2\big). \tag{16}$$

Taking the Taylor polynomials q_γ to be odd, $q_\gamma(z) = -q_\gamma(z^{-1})$, and using the evenness of w_ξ from Lemma 4.1, we can conclude that $\int_G w_\xi(z)^2 q_\gamma(z)\, \mathrm{d}z = 0$. Since for all β and γ we have $\|\partial_x^{(\gamma)} \partial_x^\beta \sigma_A(x,\xi)\|_{op} \leqslant C\langle \xi \rangle^m$, we can estimate

$$\left\| \partial_x^\beta \big(p(x,x,\xi) - \sigma_A(x,\xi) \big) \right\|_{op} \leqslant C\langle \xi \rangle^m \sum_{|\gamma|=2} \int_G w_\xi(z)^2 \big| q_\gamma(z) \big| \, \mathrm{d}z \leqslant C\langle \xi \rangle^{m-1}$$

because $|q_\gamma(z)| \leqslant C\langle \xi \rangle^{-1}$ on the support of w_ξ, for $|\gamma| = 2$. □

4.4. The difference $\sigma_P(x,\xi) - p(x,x,\xi)$

Let σ_P be the matrix symbol of the operator P from Proposition 4.2.

Lemma 4.4. *Let $s \in \mathbb{R}$. Then the pseudo-differential operator with the symbol $\sigma_P(x,\xi) - p(x,x,\xi)$ is bounded from $H^s(G)$ to $H^{s-(m-1)}(G)$.*

Proof. Observe that for a fixed $s \in \mathbb{R}$, it is enough to take sufficiently many derivatives (and not infinitely many) for the Sobolev boundedness in Theorem 3.7. Thus it is enough to prove that for sufficiently many $\beta \in \mathbb{N}_0^n$ it holds that

$$\left\| \partial_x^\beta \big(\sigma_P(x,\xi) - p(x,x,\xi) \big) \right\|_{op} \leqslant C_\beta \langle \xi \rangle^{m-1}.$$

By an argument in the proof of Proposition 3.10 we have the expansion

$$\sigma_P(x,\xi) \sim \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \Delta_\xi^\alpha \partial_y^{(\alpha)} p(x,y,\xi)|_{y=x},$$

whose asymptotic properties we will discuss below. Instead of studying the terms $\partial_x^\beta(\Delta_\xi^\alpha \partial_y^{(\alpha)} p(x,y,\xi)|_{y=x})$, we may study $\partial_x^\beta(\Delta_\xi^\alpha \partial_y^\alpha p(x,y,\xi)|_{y=x})$ as well. Moreover, abusing the notation slightly, without loss of generality we can look only at the right-invariant derivatives ∂_y^α and left-invariant derivatives ∂_x^β. Recalling that

$$p(x,y,\xi)=\int_G w_\xi\big(xz^{-1}\big)w_\xi\big(yz^{-1}\big)\sigma_A(z,\xi)\,\mathrm{d}z,$$

we notice that

$$\partial_x^\beta\big(\Delta_\xi^\alpha\partial_y^\alpha p(x,y,\xi)|_{y=x}\big)=\Delta_\xi^\alpha\int_G w_\xi(z)\big(\partial_z^\alpha w_\xi\big)(z)\partial_x^\beta\sigma_A\big(z^{-1}x,\xi\big)\,\mathrm{d}x. \tag{17}$$

We notice also that by Remark 3.2 we can replace differences Δ_ξ by $\mathbb{D}_\xi$ with a suitable correction for multi-indices. The application of $\mathbb{D}^\alpha$ here introduces (due to the Leibniz formula in Proposition 3.4) a finite sum of terms of the type

$$\int_G \big(\mathbb{D}_\xi^\kappa w_\xi(z)\big)\big(\mathbb{D}_\xi^\lambda\partial_z^\alpha w_\xi(z)\big)\big(\mathbb{D}_\xi^\mu\sigma_A\big(z^{-1}x,\xi\big)\big)\,\mathrm{d}z, \tag{18}$$

where $|\kappa+\lambda+\mu|\geqslant|\alpha|$. Recalling that $w_\xi\in\mathscr{S}^{n/4}_{1,1/2}$ by Lemma 4.1, we get that

$$\big|\big(\mathbb{D}_\xi^\kappa w_\xi(z)\big)\big(\mathbb{D}_\xi^\lambda\partial_z^\alpha w_\xi(z)\big)\big(\mathbb{D}_\xi^\mu\sigma_A\big(z^{-1}x,\xi\big)\big)\big|\leqslant C\langle\xi\rangle^{m+n/2-|\alpha|/2}.$$

Taking into account that the support of $z\mapsto w_\xi(z)$ is contained in the set of measure $C\langle\xi\rangle^{-n/2}$ by Lemma 4.1, and that taking differences in ξ does not increase the support in z, we get that the integral in (18) can be estimated by $C\langle\xi\rangle^{m-|\alpha|/2}$. Thus, we get

$$\big\|\partial_x^\beta\big(\Delta_\xi^\alpha\partial_y^\alpha p(x,y,\xi)|_{y=x}\big)\big\|_{op}\leqslant C\langle\xi\rangle^{m-|\alpha|/2}. \tag{19}$$

For $|\alpha|\geqslant 2$ this implies the desired bound by $C\langle\xi\rangle^{m-1}$ for the Sobolev boundedness of the corresponding operator. Now, assume that $|\alpha|=1$. Taking the Taylor expansion of $\sigma_A(z^{-1}x,\xi)$ at x similar to the one in (16) we see that the first term vanishes:

$$\int_G w_\xi(z)\big(\partial_z^\alpha w_\xi\big)(z)\,\mathrm{d}z=0$$

for $|\alpha|=1$ because functions w_ξ and $\partial_z^\alpha w_\xi$ are even and odd, respectively, see Proposition 3.11. Consequently, for $|\gamma|\geqslant 1$, we can estimate

$$\big|w_\xi(z)\big(\partial_z^\alpha w_\xi\big)(z)q_\gamma(z)\big|\leqslant C\langle\xi\rangle^{n/2+|\alpha|/2-|\gamma|/2},$$

which together with (17) gives

$$\big\|\partial_x^\beta\big(\Delta_\xi^\alpha\partial_y^\alpha p(x,y,\xi)|_{y=x}\big)\big\|_{op}\leqslant C\langle\xi\rangle^{m-|\alpha|/2-|\gamma|/2}\leqslant C\langle\xi\rangle^{m-1}$$

because $|\alpha| = 1$ and $|\gamma| \geqslant 1$. Finally, let us look at the remainder

$$\sigma_{R_N}(x,\xi) = \sigma_P(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \Delta_\xi^\alpha \partial_y^{(\alpha)} p(x,y,\xi)|_{y=x}.$$

By the arguments similar to the above we can see that

$$\left\| \partial_x^\beta \sigma_{R_N}(x,\xi) \right\|_{op} \leqslant C_\beta \langle\xi\rangle^{m+n/2+|\beta|/2-N/2},$$

so that for every $s, t \in \mathbb{R}$ there exists a sufficiently large N_{st} such that R_N is bounded from $H^s(G)$ to $H^t(G)$ whenever $N \geqslant N_{st}$. This concludes the proof. □

4.5. *Proof of Theorem 2.1*

Let $Q = A - P$ with operator P as in Proposition 4.2. Let $u \in C^\infty(G)$. Then $A = P + Q$ and the positivity of P implies

$$\operatorname{Re}(Au,u)_{L^2(G)} = \operatorname{Re}(Pu,u)_{L^2(G)} + \operatorname{Re}(Qu,u)_{L^2(G)} \geqslant \operatorname{Re}(Qu,u)_{L^2(G)}.$$

Let now $P_0 = \operatorname{Op}(p(x,x,\xi))$. Writing $Q = (A - P_0) + (P_0 - P)$, we have

$$\sigma_{A-P_0}(x,\xi) = \sigma_A(x,\xi) - p(x,x,\xi) \quad \text{and} \quad \sigma_{P_0-P}(x,\xi) = p(x,x,\xi) - \sigma_P(x,\xi).$$

Consequently, $A - P_0$ and $P_0 - P$ are bounded from $H^{(m-1)/2}(G)$ to $H^{-(m-1)/2}(G)$ by Lemma 4.3 and Lemma 4.4, respectively. Hence Q is bounded from $H^{(m-1)/2}(G)$ to $H^{-(m-1)/2}(G)$, so that

$$\left|\operatorname{Re}(Qu,u)_{L^2(G)}\right| \leqslant \|Qu\|_{H^{-(m-1)/2}(G)} \|u\|_{H^{(m-1)/2}(G)} \leqslant C\|u\|^2_{H^{(m-1)/2}(G)},$$

completing the proof of Theorem 2.1.

4.6. *Proof of Corollary 2.2*

We note that the assumption $\|\sigma_A(x,\xi)\|_{op} \leqslant C$ implies that for any $\theta \in \mathbb{R}$ be have the inequality $\operatorname{Re}(C - e^{i\theta}\sigma_A(x,\xi)) \geqslant 0$. Consequently, the sharp Gårding inequality in Theorem 2.1 implies that we have

$$\operatorname{Re}\big((C - e^{i\theta}A)u,u\big)_{L^2(G)} \geqslant -C'\|u\|^2_{L^2(G)}$$

for all $u \in L^2(G)$. From this it follows that $\operatorname{Re}(e^{i\theta}(Au,u)_{L^2(G)}) \leqslant C''\|u\|^2_{L^2(G)}$, so that $|(Au,u)_{L^2(G)}| \leqslant C''\|u\|^2_{L^2(G)}$, completing the proof of Corollary 2.2.

4.7. *Proof of Corollary 2.3*

Let us define

$$B(x,\xi) = M^2\langle\xi\rangle^{2m+2s} I_{\dim\xi} - \sigma_A(x,\xi)^*\sigma_A(x,\xi)\langle\xi\rangle^{2s}.$$

By the Leibniz formula, $B \in \mathscr{S}^{2m+2s}(G)$, and $B(x,\xi) \geqslant 0$ due to the definition of M. Consequently, by Theorem 2.1, we have

$$\operatorname{Re}\big(\operatorname{Op}(B)u, u\big)_{L^2(G)} \geqslant -C\|u\|_{H^{m+s-1/2}(G)}.$$

Recall that for the bi-invariant Laplace–Beltrami operator $\mathcal{L}_G$ on G, the symbol of $I - \mathcal{L}_G$ is $\langle\xi\rangle^2$, so that $\|Au\|^2_{H^s(G)} = (A^*(I-\mathcal{L}_G)^s Au, u)_{L^2(G)}$. On the other hand,

$$\operatorname{Op}(B) + A^*(I-\mathcal{L}_G)^s A - M^2(I-\mathcal{L}_G)^{m+s} \in \Psi^{2m+2s-1}(G)$$

because its symbol is in $\mathscr{S}^{2m+2s-1}(G)$ by the composition formula for pseudo-differential operators (see [16, Thm. 10.7.9]) combined with the formula for the adjoint operator (see [16, Thm. 10.7.10]). Combining these facts we obtain the statement of Corollary 2.3 by Theorem 3.7.

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